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Corrigendum

Corrigendum to "Numerical approximation of the data-rate limit for state estimation under communication constraints" [J. Math. Anal. Appl. 473 (2) (2019) 1280–1304]

Sigurdur Hafstein [a,*], Christoph Kawan [b]

[a] Faculty of Physical Sciences, University of Iceland, Dunhagi 5, IS-107 Reykjavik, Iceland
[b] Institute of Informatics, LMU Munich, Oettingenstraße 67, 80538 München, Germany

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ABSTRACT

In a recent publication, the authors developed an algorithm for the computation of upper bounds on the restoration entropy for nonlinear systems. In a computational example for the Lorenz system, there was an error in the code that resulted in too low upper bounds. Indeed, we computed an upper bound lower than the theoretical value, published contemporaneous with our paper. We corrected the error and performed the computations again. Furthermore, we additionally used our method to compute an optimal Lyapunov-like function for the matrix from the theoretical derivation and present the results.

1. Correction

The numerical example considered in [2] is the Lorenz system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\sigma x + \sigma y \\ rx - y - xz \\ -bz + xy \end{pmatrix} =: g(x, y, z), \tag{1}$$

with parameters $\sigma = 10$, $r = 28$, and $b = 8/3$. To simplify the computations, the system is scaled such that its attractors are contained in a smaller set. For this purpose, the matrix $S := \operatorname{diag}(s_x, s_y, s_z)$, for constants $s_x, s_y, s_z > 0$, is defined and the system $\dot{\mathbf{x}} = f(\mathbf{x})$ with $f(\mathbf{x}) = S^{-1}g(S\mathbf{x})$ is considered ($\mathbf{x} := [x, y, z]^\top$). The formulas for the scaled system are

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* Corresponding author.
E-mail addresses: shafstein@hi.is (S. Hafstein), christoph.kawan@lmu.de (C. Kawan).

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$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\sigma x + \sigma \frac{s_y}{s_x} y \\ r\frac{s_x}{s_y} x - y - \frac{s_x s_z}{s_y} xz \\ -bz + \frac{s_x s_y}{s_z} xy \end{pmatrix}. \tag{2}$$

Unfortunately, there was an error in the code used in the computations in [2] for an upper bound on the restoration entropy of the scaled Lorenz system. More precisely, in the implementation of Optimization Problem 4.2 in [2], the Constraints 4, i.e.

For each simplex $\mathfrak{S}_\xi = \mathrm{co}(x_0, \ldots, x_n) \in \mathcal{T}^*$ and each vertex x_k of $\mathfrak{S}_\xi$:

$$\nabla V_\xi \cdot f(x_k) + h_\xi^2 \cdot nB_\xi^* D_\xi^V + \widetilde{m}\mu(x_k) \leq Q$$

were not added to the linear programming problem whenever $x_k = 0$. This is very unfortunate, because the maximum value for Q, that is being minimized, is obtained exactly at $x = 0$. It was later shown in [4, Thm. 15] that

$$R := \frac{1}{2\ln(2)}\left(\sqrt{(\sigma-1)^2 + 4r\sigma} - (\sigma+1)\right) \tag{3}$$

from [5] for an upper bound on the restoration entropy, actually delivers the exact value for the restoration entropy for most interesting parameter values. In particular, for the parameter values we used in our computations, $R \approx 17.0638$ is the restoration entropy.

An explanation for this programming error, but not an excuse, is that the implementation of Optimization Problem 4.2 was done by adapting code for the computation of CPA Lyapunov functions for nonlinear systems; see, e.g., [1] or [3]. There, the corresponding constraint is

$$\nabla V_\xi \cdot f(x_k) + h_\xi^2 \cdot nB_\xi^* D_\xi^V \leq -\|x_k\|_2,$$

which is automatically fulfilled for $x_k = 0$ when the upper bound $h_\xi^2 \cdot nB_\xi^*$ is replaced by a closer bound that depends on x_0 and x_k in $\mathfrak{S}_\xi$. This closer bound is chosen, such that the bound is zero for $x_k = 0$. However, for our computation it is simply wrong and leads to incorrect results to drop these constraints for $x_k = 0$. In [2], these closer bounds were also used, cf.

Because we are using such a simple axially parallel triangulation, one can use somewhat less conservative bounds in the LP problems. That is, the term $nh_\xi^2 B_\xi^*$ in Constraints 4 in Optimization Problem 4.2 can be replaced with a smaller number and Theorem 4.12 still holds true. For these less conservative bounds we refer to [3, Lem. 4.16].

The updated results of the computations are shown in Table 1 (corresponds to Table 1 in [2]). We performed the computations on AMD Threadripper 3990X (64 cores, 256 GB RAM) using Gurobi 9.01 for solving the linear programming problems.

2. Addendum

It is revealing to perform the computations from [2] with the optimal matrix computed in [4, Thm. 15]. In the computations presented in Table 1, first the metric

$$P_c := \begin{pmatrix} 0.1008469737786 & -0.01415360101927 & 0 \\ -0.01415360101927 & 0.3361537095909 & 0 \\ 0 & 0 & 0.3139832543019 \end{pmatrix}$$

Table 1
The results of our computations. N_x, N_y, N_z are the parameters for the computational grid as explained in the original paper, 'time' is the total time in seconds needed to write and solve the problem, 'impr. bounds' states whether the improved bounds discussed in the text are used (Yes) or not (No), Q is the objective that is minimized in Optimization Problem 4.2, and 'u.b.' is the associated upper bound $Q/(2\ln(2))$ on the topological/restoration entropy. For reference, the value of the restoration entropy is 17.0638 [4, Thm. 15].

N_x	N_y	N_z	time [s]	impr. bounds	Q	u.b.
30	14	28	53	No	25.7566	18.5795
30	14	28	44	Yes	24.8473	17.9236
42	14	28	88	No	25.7567	18.5795
42	14	28	81	Yes	24.8473	17.9236
50	18	32	140	Yes	24.8473	17.9236
70	22	40	342	Yes	24.8473	17.9236

Table 2
The results of our computations using the analytically computed metric M. The optimal value is obtained if the triangulation $\mathcal{T}^*$ consists of sufficiently small simplices. The parameters are as in Table 1.

N_x	N_y	N_z	time [s]	impr. bounds	Q	u.b.
15	7	7	5	No	26.1038	18.8299
15	7	7	16	Yes	23.9644	17.2867
30	14	28	53	No	24.8447	17.9217
30	14	28	44	Yes	23.6554	17.0638

was computed using semidefinite optimization. As stated in [2], if we only use P_c in the formula in [5, Thm. 3.2] for the upper bound on the topological/restoration entropy, i.e. if we set $V(x) = \text{const.}$, we obtain the upper bound $27/(2\ln(2)) \approx 19.4764$ on the positively invariant set $\mathcal{K} := [-1, 1] \times [-0.29, 0.29] \times [0, 0.57]$, which contains all attractors of the system. However, by using Optimization Problem 4.2 to compute subsequently an optimal Lyapunov-like function V, we can improve these upper bounds to 17.9236, using P_c and V, as shown in Table 1. Note that we compute V on a slightly larger set than $\mathcal{K}$ in order to have $\mathcal{K}^\circ \cap \{z = 0\} \neq \emptyset$, because we want the equilibrium at the origin to be in the interior of $\mathcal{K}$.

In the proof of [5, Thm. 4.3], the symmetric matrix

$$P = \begin{pmatrix} \frac{r\sigma+(b-1)(\sigma-1)}{\sigma^2} & -\frac{b-1}{\sigma} & 0 \\ -\frac{b-1}{\sigma} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

that is positive definite if $r\sigma + (b-1)(\sigma-1) > 0$, is used to obtain the upper bound (3) on the restoration entropy of the Lorenz system (1). It turns out that the upper bound (3) is the exact value of the restoration entropy [4, Thm. 15]. In Remark 2.1, we show that the matrix $M = S^\top PS$ is the corresponding matrix for the scaled Lorenz system (2). If we use M in the formula in [5, Thm. 3.2] for the upper bound on the restoration entropy, we obtain the value 20.6977 on $\mathcal{K}$, i.e. a worse estimate than by using the numerically computed metric P_c. However, when we subsequently compute a Lyapunov-like function using Optimization Problem 4.2, we obtain the results in Table 2. As can be seen, the theoretical value for the restoration entropy is obtained with the matrix M and the corresponding Lyapunov-like function computed with Optimization Problem 4.2 in [2]. Note that these results are not a priori obvious, because the Lyapunov-like function computed by the optimization problem is a continuous, piecewise affine function, but the Lyapunov-like function computed analytically in [5] is not.

Remark 2.1. Let us show that for the scaled Lorenz system (2) the matrix $M = S^\top PS$ corresponds to the optimal matrix P from (4) for the unscaled Lorenz system (1). Recall that $S := \text{diag}(s_x, s_y, s_z)$ is the scaling matrix and for $v\colon \mathbb{R}^n \to \mathbb{R}$, the gradient ∇v is a row vector.

In the proof of [5, Thm. 4.3], one first factorizes $P = B^\top B$ (B is denoted S in [5]) and observes that for the Jacobian matrix $J := J(x, y, z)$ of g in (1) and $\lambda \in \mathbb{C}$ we have

$$PJ + J^\top P - \lambda P = B^\top \left[BJB^{-1} + (BJB^{-1})^\top - \lambda I\right] B. \tag{5}$$

Note that the matrix function J is also denoted by A in the proof of [5, Thm. 4.3].

The Lorenz system (1) is given by $\dot{\mathbf{x}} = g(\mathbf{x})$ and the scaled Lorenz system (2) by $\dot{\mathbf{x}} = f(\mathbf{x})$ with $f(\mathbf{x}) = S^{-1}g(S\mathbf{x})$, see above. Hence, the Jacobian matrix of f is $A := Df = S^{-1}JS$.

Since for $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} MA + A^\top M - \lambda M &= S^\top B^\top BS \cdot S^{-1}JS + S^\top J^\top (S^{-1})^\top \cdot S^\top B^\top BS - \lambda S^\top B^\top BS \\ &= S^\top B^\top \left[BJB^{-1} + (B^\top)^{-1}J^\top B^\top - \lambda I\right] BS \\ &= (BS)^\top \left[BJB^{-1} + (BJB^{-1})^\top - \lambda I\right] BS, \end{aligned}$$

similarly to (5), we have

$$\det(MA + A^\top M - \lambda M) = 0, \quad \text{if and only if} \quad \det(PJ + J^\top P - \lambda P) = 0. \tag{6}$$

Hence, the proof of [5, Thm. 4.3] for the unscaled Lorenz system (1) can be used, with modest modifications, for the scaled Lorenz system (2) as well. In detail, the function $v(\mathbf{x})$ in [5] with orbital derivative $\dot{v}(\mathbf{x}) = [\nabla v(\mathbf{x})]g(\mathbf{x})$ along solution trajectories of (1), is replaced by $\tilde{v}(\mathbf{x}) := v(S\mathbf{x})$ with orbital derivative

$$\dot{\tilde{v}}(\mathbf{x}) = [\nabla v(S\mathbf{x})]SS^{-1}g(S\mathbf{x}) = [\nabla v(S\mathbf{x})]g(S\mathbf{x}) = \dot{v}(S\mathbf{x})$$

along the solution trajectories of (2). It then follows from $\dot{v}(\mathbf{x}) + w(\mathbf{x}) \leq 0$, i.e. (19) in [5], that with $\tilde{w}(\mathbf{x}) := w(S\mathbf{x})$ we have $\dot{\tilde{v}}(\mathbf{x}) + \tilde{w}(\mathbf{x}) \leq 0$ and the propositions of [5, Thm. 4.3] for the unscaled Lorenz system (1) apply to the scaled Lorenz system (2) as well.

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