Simplicial complex with approximate rotational symmetry: A general class of simplicial complexes

Sigurdur Albertsson *
Sigurdur Hafstein†
The Science Institute
University of Iceland
Dunhagi 5
107 Reykjavik
Iceland

Peter Giesl‡
Department of Mathematics
University of Sussex
Falmer BN1 9QH
United Kingdom

Skuli Gudmundsson§
Svensk Exportkredit
Klarabergsviadukten 61-63
11164 Stockholm
Sweden

June 6, 2019

Abstract

We study the transformation of the vertices of a certain simple simplicial complex in n-dimensional Euclidian space and prove that the resulting set of simplices is a simplicial complex with an approximate rotational symmetry. Such simplicial complexes have applications in computing Lyapunov function for nonlinear dynamical systems using linear optimization and are also of interest for other applications.

1 Introduction

Triangulations of $\mathbb{R}^n$ and its subsets have numerous applications in image processing [27, 1], mesh construction in numerical analysis [4, 6, 26], and other fields. The construction of a triangulation, often referred to as mesh generation or grid generation, is thus an important topic in various different disciplines. In a shape-regular triangulation the triangles, or simplices in dimensions $n \geq 3$, have to intersect in a certain way. Such sets of simplices are frequently referred to as simplicial complexes. The so-called standard triangulation is a simplicial complex with vertices in $\mathbb{Z}^n$ and has a number of nice properties, cf. e.g. [10, Def. 4.8]. However, when refining the mesh and adjusting it to a certain geometry, one would like to obtain other, more appropriate triangulations in a constructive and simple way.

One specific application of simplicial complexes is in the computation of continuous and piecewise affine (CPA) Lyapunov functions for nonlinear dynamical systems given by an autonomous ordinary differential equation [20, 19, 23, 14, 9, 22] or an iteration [8, 21, 16]. Furthermore, they are used for CPA contraction metrics [7, 18]. On a given triangulation of a compact subset of $\mathbb{R}^n$, the function is determined by its values at the vertices and is interpolated affinely on the simplices. For a nonlinear system with a hyperbolic, asymptotically stable equilibrium, one can easily construct a quadratic Lyapunov function for the linearization around the equilibrium, and this function is locally also a Lyapunov function for the nonlinear system. To extend the domain of this Lyapunov function in the framework of CPA functions, one is particularly interested in triangulations that can mimic the level sets of the quadratic Lyapunov functions, namely hyper-ellipsoids, with a reasonably small number of simplices. The main idea for such a construction is to generate a general class of triangulations by starting from the standard triangulation, a triangulation that is very simple to generate and such that its vertices are $\mathbb{Z}^n$. Then we map the vertices of the standard triangulation by the map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, where $\mathbf{F}(\mathbf{x}) = \frac{\|\mathbf{x}\|_\infty}{\|\mathbf{x}\|_2}\mathbf{x}$, and consider the set of simplices

*email sja12@hi.is
†email shafstein@hi.is
‡email p.a.giesl@sussex.ac.uk
§email skgu@sek.se

$\mathrm{co}\{\mathbf{F}(\mathbf{x}_0), \mathbf{F}(\mathbf{x}_1), \dots, \mathbf{F}(\mathbf{x}_n)\}$, where $\mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ runs over the simplices of the standard triangulation. Note that $\mathbf{F}$ maps hyper-cubes to hyper-spheres. After that, one can map the vertices with a radial function, namely $\Phi(\mathbf{x}) = \rho(\|\mathbf{x}\|_\infty)\cdot\mathbf{F}(\mathbf{x})$ with a suitable function ρ, to reduce the size of the simplices and subsequently, using a nondegenerate symmetric matrix $A = A^{\mathrm{T}}$, map the hyper-spheres to hyper-ellipsoids $\Phi(\mathbf{x}) \mapsto A\Phi(\mathbf{x})$, see Figures 1 and 2 below. The nontrivial question is: *when mapping the vertices in this way, is the resulting set of simplices a triangulation?* A positive answer to this question is the main result of this paper.

The strategy to this aim is to characterize a shape-regular triangulation/simplicial complex by the property that each point is an *inner point* of a unique subsimplex of dimension $0 \le k \le n$. An inner point is a point such that all coefficients in the convex combination of the vertices are nonzero. Then we define a continuous transformation, parameterized by $t \in [0,1]$, from the standard triangulation to the final one by moving the vertices continuously. We will prove that for each fixed t the resulting set of simplices is a triangulation. This result will be useful to construct a general class of triangulations in many applications.

Note that although one can always use a Delaunay triangulation to triangulate a given set of points in general position in n-dimensions [2, 13, 5], a Delaunay triangulation is not necessary the optimal one for our [12] or other applications. In particular, our triangulation allows for efficient algorithms to locate simplices containing a given point [15, 17], which is of great advantage or even essential when using CPA functions.

After introducing notations in Section 1.1, we prove our main result, Theorem 2.17, through a series of lemmas in Section 2 before we conclude in Section 3.

1.1 Prerequisites and notation

We utilize a bold-face font for vectors $\mathbf{x} \in \mathbb{R}^n$ and denote its components either by x_i or $[\mathbf{x}]_i$. A vector $\mathbf{x} \in \mathbb{R}^n$ is considered to be a column vector, i.e. $\mathbf{x} \in \mathbb{R}^{n\times 1}$, and $\mathbf{x}^{\mathrm{T}} \in \mathbb{R}^{1\times n}$ is its transpose. For a vector $\mathbf{x} \in \mathbb{R}^n$ and $p \ge 1$ we define the norm $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. We also define $\|\mathbf{x}\|_\infty = \max_{i\in\{1:n\}} |x_i|$, where $\{1:n\} := \{1, 2, \dots, n\}$.

For a matrix $A \in \mathbb{R}^{m\times n}$ we write A^{T} for its transpose. A diagonal matrix with entries $\mathbf{a} = (a_1, a_2, \dots, a_n)^{\mathrm{T}}$ on its diagonal is denoted by $\mathrm{diag}(\mathbf{a})$. We denote by $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ the standard orthonormal basis of $\mathbb{R}^n$ and use the Kronecker delta symbol defined by $\delta_{ij} = \mathbf{e}_i^{\mathrm{T}}\mathbf{e}_j$. Also, we denote by $\mathbb{I}_n$ the identity matrix in $\mathbb{R}^{n\times n}$.

The *convex combination* of vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n$ is denoted by

$$\mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m\} := \left\{ \mathbf{y} \in \mathbb{R}^n \;\middle|\; \mathbf{y} = \sum_{i=0}^m \lambda_i \mathbf{x}_i,\ 0 \le \lambda_i \le 1,\ \sum_{i=0}^m \lambda_i = 1 \right\}.$$

The vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n$ are said to be *affinely independent* if

$$\sum_{i=0}^m \lambda_i \mathbf{x}_i = \mathbf{0} \quad\text{and}\quad \sum_{i=0}^m \lambda_i = 0 \quad\text{implies}\quad \lambda_0 = \lambda_1 = \cdots = \lambda_m = 0.$$

Equivalent characterizations for affine independence are that each $\mathbf{x} \in \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m\}$ has a unique representation as a convex combination of the $\mathbf{x}_i$, that for any (and then all) $j \in \{0:m\}$ the vectors $\mathbf{x}_i - \mathbf{x}_j$, $i \in \{0:m\} \setminus \{j\}$, are linearly independent, or that the vectors $\mathbf{x}_0^a, \mathbf{x}_1^a, \dots, \mathbf{x}_m^a \in \mathbb{R}^{n+1}$, $\mathbf{x}_i^a := [\mathbf{x}_i^{\mathrm{T}}\ 1]^{\mathrm{T}}$ for $i \in \{0:m\}$, are linearly independent. We call the set $S := \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m\}$ a *simplex*. If the vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n$ are affinely independent, then we call the simplex S *proper* and we refer to the vectors in the set $\mathrm{ve}\, S := \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m\}$ as its *vertices*. Otherwise we say that the simplex S is *degenerate*. If we want to emphasize that a proper simplex S has positive m-dimensional volume, or equivalently that it has $m+1$ vertices, we call it a *proper m-simplex*. Note that in the literature the term simplex is often reserved for proper simplices.

A *face* of a proper $m-$simplex $S = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m\}$ is the convex hull $\mathrm{co}\, A$ of a nonempty proper subset A of its vertices, i.e. $\emptyset \neq A \subsetneq \mathrm{ve}\, S$. Clearly, a face F of S is a subset of the boundary of S, i.e. $F \subset \partial S$, and F is a proper k-simplex, $0 \le k < m$, when $\mathrm{ve}\, F$ has $k+1$ elements. We use the term *subsimplex* of S, for a subset $F \subset S$ that is either a face of S or $F = S$.

We denote by $\mathrm{Sym}(n)$ the set of permutations of the set $\{1:n\}$ and by $|A|$ the cardinality of a set A. Finally, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{a} \neq \mathbf{b}$, we define the line segment $[\mathbf{a}, \mathbf{b}]$, the open line segment $(\mathbf{a}, \mathbf{b})$ and the ray $[\mathbf{a}, \mathbf{b}\rangle$ as the point set $\{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{x} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})\}$ with $t \in [0,1]$, $t \in (0,1)$ and $t \in [0,\infty)$ respectively.

2 Triangulations

We start with a few definition before we state and prove a useful characterization of simplicial complexes in Lemma 2.5. Note that a triangulation is a set of n-simplices, while a simplicial complex is a set of k-simplices with $0 \le k \le n$. Although we are essentially interested in triangulations, it is often more convenient to work with the associated simplicial complexes.

Definition 2.1 (Triangulation)**.** *A* triangulation $\mathcal{T}$ *is a set of proper n-simplices* $\{S_\nu\}_{\nu\in I} = \{\operatorname{co} C_\nu\}_{\nu\in I}$*, with vertices* $C_\nu := \operatorname{ve} S_\nu = \{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\} \subset \mathbb{R}^n$*, where the pairwise intersection of simplices in $\mathcal{T}$ satisfies*

$$S_\mu \cap S_\nu = \operatorname{co} C_\mu \cap \operatorname{co} C_\nu = \operatorname{co}(C_\mu \cap C_\nu). \tag{2.1}$$

Note that the condition (2.1) *means two different simplices in $\mathcal{T}$ intersect in a common face or not at all.*

The domain *and the* complete set of vertices *of a triangulation $\mathcal{T}$ are denoted by*

$$\mathcal{D}_\mathcal{T} := \bigcup_{\nu\in I} S_\nu \quad \textit{and} \quad \mathcal{V}_\mathcal{T} := \bigcup_{\nu\in I} C_\nu$$

respectively. We also write $\mathcal{D}_\mathcal{S}$ for the union of any set $\mathcal{S}$ of simplices. We say that $\mathcal{T}$ is a triangulation of $\mathbb{R}^n$ if $\mathcal{D}_\mathcal{T} = \mathbb{R}^n$.

While we defined a triangulation to be a set of proper n-simplices that intersect in a certain way, a simplicial complex, see below, is a set of proper k-simplices, $0 \le k \le n$.

Definition 2.2. *A* simplicial complex $\mathcal{K}$ *is a set of proper k-simplices, $0 \le k \le n$, having the property that:*

- *If S is a simplex from $\mathcal{K}$ then each subsimplex of S is also in $\mathcal{K}$.*
- *If S_1 and S_2 are two simplices from $\mathcal{K}$ which intersect, then their intersection is a subsimplex of both of them.*

Thus, for a triangulation $\mathcal{T} = \{S_\nu\}_{\nu\in I}$ and with $C_\nu := \operatorname{ve} S_\nu$ for $\nu \in I$, it follows easily from Lemma 2.5 below that we have the associated simplicial complex

$$\mathcal{K}_\mathcal{T} := \{\operatorname{co} C \,|\, \emptyset \ne C \subset C_\nu \text{ for } \nu \in I\}.$$

On the other hand, we can start with a simplicial complex and define a triangulation by throwing out all k-simplices with $k < n$. Note that in the framework of convex polytopes, a simplicial complex just the set of all face lattices of all included n-simplices, such that the nonempty intersection of two faces in the set is also a face in set [11]. In the following definition the sets C_ν will later be the set of vertices of proper n-simplices that are mapped in a certain way. The question is then if the resulting set of simplices is a triangulation.

Definition 2.3. *Let $\mathcal{C} := \{C_\nu \subset \mathbb{R}^n \,|\, \nu \in I\}$, where each $|C_\nu| = n+1$. We define the* complex

$$C[\mathcal{C}] := \{\operatorname{co} C \,|\, \emptyset \ne C \subset C_\nu \textit{ for } \nu \in I\}.$$

Note that some or all of the simplices $\operatorname{co} C$ in the set $C[\mathcal{C}]$ might be degenerate. If, however, the simplices $\operatorname{co} C_\nu$ are proper n-simplices, then all the elements in $C[\mathcal{C}]$ are proper simplices because subsets of a set of affinely independent vectors are also sets of affinely independent vectors. Further, the set of simplices $\{\operatorname{co} C_\nu\}_{\nu\in I}$ is a triangulation, if and only if the complex $C[\mathcal{C}]$ is a simplicial complex.

The following definition is needed for the characterization of a triangulation in the next lemma.

Definition 2.4. *Let $S = \operatorname{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a proper k-simplex in $\mathbb{R}^n$, $0 \le k \le n$. We say that a point $\mathbf{p} \in \mathbb{R}^n$ is an* inner point *of S if the representation of $\mathbf{p}$ as the convex sum of the vertices of S has strictly positive coefficients, i.e. $\mathbf{p}$ has the representation*

$$\mathbf{p} = \sum_{i=0}^k \lambda_i \mathbf{x}_i \quad \textit{with} \quad \sum_{i=0}^k \lambda_i = 1 \quad \textit{and} \quad \lambda_i > 0 \quad \textit{for } i = 0:k.$$

Note that for any point $\mathbf{p} \in S$, where $S = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a proper k-simplex in $\mathbb{R}^n$, $\mathbf{p}$ is an inner point of exactly one subsimplex S^* of S. Namely

$$S^* = \mathrm{co}\{\mathbf{x}_{i_j}\} \text{ where the } \lambda_{i_j} \neq 0 \text{ in } \sum_{i=0}^{k} \lambda_i \mathbf{x}_i = \mathbf{p}. \tag{2.2}$$

In particular, a singleton simplex $\{\mathbf{x}\} = \mathrm{co}\{\mathbf{x}\}$ has exactly one inner point.

Lemma 2.5. *A set $\mathcal{S} = \{S_\nu\}_{\nu \in I}$ of proper n-simplices in $\mathbb{R}^n$ is a triangulation, if and only if for every $\mathbf{p} \in \mathcal{D}_\mathcal{S}$ there exists a unique k-simplex S, $0 \leq k \leq n$, with $S \in C[\{\mathrm{ve}\, S_\nu\}_{\nu \in I}]$ such that $\mathbf{p}$ is an inner point of S.*

PROOF: Assume $\mathcal{S}$ is a triangulation and let $\mathbf{p} \in \mathcal{D}_\mathcal{S}$ be arbitrary. Clearly there exists an $S \in \mathcal{S}$ such that $\mathbf{p} \in S$ and then a unique subsimplex T of S such that $\mathbf{p}$ is an inner point of T, corresponding to the non-zero coefficients λ. We must show that $\mathbf{p} \in S^* \in \mathcal{S}$ implies that T is the unique subsimplex of S^* such that $\mathbf{p}$ is an inner point of T. With $C = \mathrm{ve}\, S$ and $C^* = \mathrm{ve}\, S^*$ and because $\mathcal{S}$ is a triangulation, we have that

$$\mathbf{p} \in S \cap S^* = \mathrm{co}\, C \cap \mathrm{co}\, C^* = \mathrm{co}(C \cap C^*)$$

and as in (2.2) we get the existence of a unique subsimplex of the simplex $\mathrm{co}(C \cap C^*)$ with $\mathbf{p}$ as an inner point. Evidently this subsimplex must be T, which concludes the **if** part of the proof.

Now suppose that for any point $\mathbf{p} \in \mathcal{D}_\mathcal{S}$ there exists a unique k-simplex S, $0 \leq k \leq n$ with $S \in C[\{\mathrm{ve}\, S_\nu\}_{\nu \in I}]$ which has $\mathbf{p}$ as an inner point. We show that $\mathcal{S}$ is necessarily a triangulation. Fix arbitrary $S, S^* \in \mathcal{S}$ such that $S \cap S^* \neq \emptyset$ and set $C = \mathrm{ve}\, S$ and $C^* = \mathrm{ve}\, S^*$. Since clearly $\mathrm{co}\,(C \cap C^*) \subset \mathrm{co}\, C \cap \mathrm{co}\, C^*$, it suffices to show that $\mathrm{co}\, C \cap \mathrm{co}\, C^* \subset \mathrm{co}(C \cap C^*)$ to prove $\mathrm{co}\, C \cap \mathrm{co}\, C^* = \mathrm{co}(C \cap C^*)$. To show this, let $\mathbf{p} \in \mathrm{co}\, C \cap \mathrm{co}\, C^*$ be arbitrary. Then $\mathbf{p}$ is an inner point of subsimplices T and T^* of S and S^* respectively. From the hypothesis, we have $T = T^*$ and hence $\mathbf{p} \in T \subset \mathrm{co}\,(C \cap C^*)$, which finishes the proof. □

The basis for our construction of a general class of triangulations is the standard triangulation, see the following definition.

Definition 2.6 (The standard triangulation of $\mathbb{R}^n$). *The* standard triangulation *is the triangulation $\mathcal{T}_{\mathrm{std}} = \{S_\nu\}_{\nu \in I}$ with indices $\nu = (\mathbf{z}, \sigma, \mathbf{J}) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{Sym}(n) \times \{-1, +1\}^n =: I$ and vertices $\mathrm{ve}\, S_\nu = C_\nu = \{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\}$ given by:*

$$\mathbf{x}_k^\nu = R_\mathbf{J} \left(\mathbf{z} + \sum_{l=1}^{k} \mathbf{e}_{\sigma(l)} \right) = R_\mathbf{J} \mathbf{z} + R_\mathbf{J}\, \mathbf{u}_k^\sigma. \tag{2.3}$$

Here, $R_\mathbf{J} = \mathrm{diag}(J_1, J_2, \ldots, J_n) \in \mathbb{R}^{n \times n}$ with $J_i \in \{-1, +1\}$ for $i \in \{1 : n\}$ and $\mathbf{u}_k^\sigma = \sum_{l=1}^{k} \mathbf{e}_{\sigma(l)}$. We abbreviate the associated simplicial complex $\mathcal{K}_{\mathcal{T}_{\mathrm{std}}}$ by $\mathcal{K}_{\mathrm{std}}$.

Notice that for the standard triangulation $\mathcal{T}_{\mathrm{std}}$ we have $\mathcal{V}_{\mathcal{T}_{\mathrm{std}}} = \mathbb{Z}^n$, i.e. the vertex-set is just the integer lattice of $\mathbb{R}^n$, and $\mathcal{D}_{\mathcal{T}_{\mathrm{std}}} = \mathbb{R}^n$.

Remark 2.7. *We can also define the permutation matrix $P_\sigma = \left(\mathbf{e}_{\sigma(1)}, \mathbf{e}_{\sigma(2)}, \ldots, \mathbf{e}_{\sigma(n)}\right) \in \mathbb{R}^{n \times n}$ corresponding to the permutation specified by $\sigma \in \mathrm{Sym}(n)$ and then write:*

$$\mathbf{x}_k^\nu = R_\mathbf{J} \mathbf{z} + R_\mathbf{J} \mathbf{u}_k^\sigma = R_\mathbf{J} \mathbf{z} + R_\mathbf{J}\, P_\sigma \mathbf{v}_k$$

where $\mathbf{v}_k = \sum_{l=1}^{k} \mathbf{e}_l \in \mathbb{R}^n$ has the first k components equal to 1 and the remaining $n - k$ equal to 0.

The vectors $\mathbf{u}_k^\sigma$ depend on both k and σ. However, for $k = n$ it is clear that $\mathbf{u}_n^\sigma = (1, 1, \ldots, 1)^{\mathrm{T}}$ for all $\sigma \in \mathrm{Sym}(n)$. In fact, with k fixed, the vector $\mathbf{u}_k^\sigma$ takes on exactly $\binom{n}{k}$ distinct values while σ runs over $\mathrm{Sym}(n)$. This matches exactly because $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, the number of integer coordinates for the cube $R_\mathbf{J}(\mathbf{z} + [0, 1]^n)$ which are the vertices of the simplices in that cube. There is a more detailed account of the construction and the various properties of the standard triangulation in [24].

2.1 Mapping the standard triangulation

We will now generate new triangulations by rearranging the vertices of $\mathcal{T}_{\text{std}}$ but retaining the triangulation structure through the specification of the vertex-sets $\{C_\nu\}_{\nu\in I}$. To be more precise, with $\mathcal{T}_{\text{std}} = \{\operatorname{co} C_\nu\}_{\nu\in I}$ we will consider the set of simplices given by $\mathcal{T}_\psi := \{\operatorname{co}\psi(C_\nu)\}_{\nu\in I}$ where the mapping $\psi : \mathbb{R}^n \to \mathbb{R}^n$ performs the rearrangement of the vertices. For a linear $\psi : \mathbb{R}^n \to \mathbb{R}^n$, $\psi(\mathbf{x}) = A\mathbf{x}$ with a nonsingular matrix $A \in \mathbb{R}^{n\times n}$, the set $\mathcal{T}_\psi$ is clearly a triangulation because $A \operatorname{co} C_\nu = \operatorname{co}(AC_\nu)$ and $AC_\nu = \{A\mathbf{x}_0, A\mathbf{x}_1, \ldots, A\mathbf{x}_n\}$ is a set of affinely independent vectors because the vectors $C_\nu = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ are affinely independent. For nonlinear ψ in general $\operatorname{co}\psi(C_\nu) \neq \psi(\operatorname{co} C_\nu)$ and then the interesting question arises: when is the set of simplices $\mathcal{T}_\psi = \{\operatorname{co}\psi(C_\nu)\}_{\nu\in I}$ in fact a triangulation?

We will study a general class of transformations Φ in the next definition. The key element is the map $\mathbf{F}$ mapping hyper-cubes to hyper-spheres, and then Φ is constructed by multiplication with a rescaling function ρ.

Definition 2.8. *Consider the mapping*

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f(\mathbf{0}) = 1 \quad \textit{and} \quad f(\mathbf{x}) = \frac{\|\mathbf{x}\|_\infty}{\|\mathbf{x}\|_2} \quad \textit{for } \mathbf{x} \neq \mathbf{0},$$

and the following transformations:

$$\mathbf{F}, \Phi : \mathbb{R}^n \to \mathbb{R}^n, \qquad \mathbf{F}(\mathbf{x}) = f(\mathbf{x})\mathbf{x} \quad \textit{and} \quad \Phi(\mathbf{x}) = \rho(\|\mathbf{x}\|_\infty) \cdot \mathbf{F}(\mathbf{x}), \tag{2.4}$$

with $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous and non-decreasing with $\rho(x) > 0$ if $x > 0$. We refer to ρ as a rescaling-function*. With $\mathcal{T}_{\text{std}} = \{\operatorname{co} C_\nu\}_{\nu\in I}$ and $C_\nu = \{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\}$ as before, define the following set of simplices:*

$$\mathcal{T}_\Phi = \{\operatorname{co}\Phi(C_\nu)\}_{\nu\in I} = \{\operatorname{co}\{\Phi(\mathbf{x}_0^\nu), \Phi(\mathbf{x}_1^\nu), \ldots, \Phi(\mathbf{x}_n^\nu)\}\}_{\nu\in I}.$$

If $\emptyset \neq A \subset C_\nu$ we refer to $\operatorname{co}\Phi(A)$ as the (sub)simplex in $\mathcal{T}_\Phi$ corresponding to the (sub)simplex $\operatorname{co} A$ in $\mathcal{T}_{\text{std}}$.

Remark 2.9. *Note that $\mathbf{F}$ and Φ are radial transformations, i.e. $\Phi(\mathbf{x}) = c(\mathbf{x})\mathbf{x}$ where $c(\mathbf{x}) = \rho(\|\mathbf{x}\|_\infty) \cdot \|\mathbf{x}\|_\infty/\|\mathbf{x}\|_2 \in \mathbb{R}$ for $\mathbf{x} \neq \mathbf{0}$. $\mathbf{F}$ maps level sets of $\|\cdot\|_\infty$ (hyper-cubes) to $\|\cdot\|_2$ level sets (hyper-spheres). Because of $\|\Phi(\mathbf{x})\|_2 = \rho(\|\mathbf{x}\|_\infty)\|\mathbf{x}\|_\infty$ the effect of the transformation Φ is to map the n-cube: $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_\infty = M\}$ to the n-sphere: $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_2 = \rho(M)M\}$. Hence, if Φ acts on the set of vertices of the standard triangulation $\mathcal{V}_{\mathcal{T}_{\text{std}}}$, then $\mathcal{V}_{\mathcal{T}_\Phi}$ is a vertex distribution which is approximately rotationally symmetric and is radially scaled by ρ, cf. Figure 1.*

In the reminder of this paper we will prove that $\mathcal{T}_\Phi$ is a triangulation, a fact that seems quite evident, but is surprisingly difficult to prove. We achieve this through a series of lemmas, before we come to the main theorem of this paper, Theorem 2.17, and its corollary.

For the proof we create a continuously parameterized set of transformations which starts from the identity mapping $\mathrm{Id}_{\mathbb{R}^n}$ and ends with Φ. This parameterized set of transformations corresponds to the intuitive notion of rearranging the vertices in a continuous or gradual fashion. For some $\mathbf{x} \in \mathbb{R}^n$ consider the line segment between $\mathbf{x}$ and $\Phi(\mathbf{x})$. Because Φ is a radial transformation, this line lies on the straight line from $\mathbf{0}$ to $\mathbf{x}$, and our set of transformed vertices will be on that straight line, too. Let us be more precise:

Definition 2.10. *For each $t \in [0,1]$ and for any rescaling function ρ we define $h_t^\rho : \mathbb{R}^n \to \mathbb{R}_{>0}$ and $\mathbf{H}_t^\rho : \mathbb{R}^n \to \mathbb{R}^n$ as follows: $h_t^\rho(\mathbf{0}) = 1$ and for $\mathbf{x} \neq \mathbf{0}$ we set*

$$h_t^\rho(\mathbf{x}) := \frac{\rho(\|\mathbf{x}\|_\infty)\|\mathbf{x}\|_\infty}{t\|\mathbf{x}\|_2 + (1-t)\rho(\|\mathbf{x}\|_\infty)\|\mathbf{x}\|_\infty} \quad \textit{and} \quad \mathbf{H}_t^\rho(\mathbf{x}) := h_t^\rho(\mathbf{x})\mathbf{x}. \tag{2.5}$$

We emphasize that $\mathbf{H}_0^\rho = \mathrm{Id}_{\mathbb{R}^n}$ and $\mathbf{H}_1^\rho = \Phi$ with Φ from (2.4). For a fixed $\mathbf{x} \in \mathbb{R}^n$ the path $[0,1] \to \mathbb{R}^n; t \mapsto \mathbf{H}_t^\rho(\mathbf{x})$ continuously parameterizes the straight radial line segment connecting $\mathbf{x}$ and $\Phi(\mathbf{x})$. In terms of the functions $f, h_t^\rho : \mathbb{R}^n \to \mathbb{R}_{>0}$ this means $h_t^\rho(\mathbf{0}) = 1$ for all $t \in [0,1]$ and for $\mathbf{x} \neq \mathbf{0}$ we have:

$$\begin{aligned} h_0^\rho(\mathbf{x}) &= 1 \\ h_1^\rho(\mathbf{x}) &= \rho(\|\mathbf{x}\|_\infty) \cdot f(\mathbf{x}) \\ \frac{1}{h_t^\rho(\mathbf{x})} &= \frac{t}{\rho(\|\mathbf{x}\|_\infty)} \frac{1}{f(\mathbf{x})} + 1 - t. \end{aligned} \tag{2.6}$$

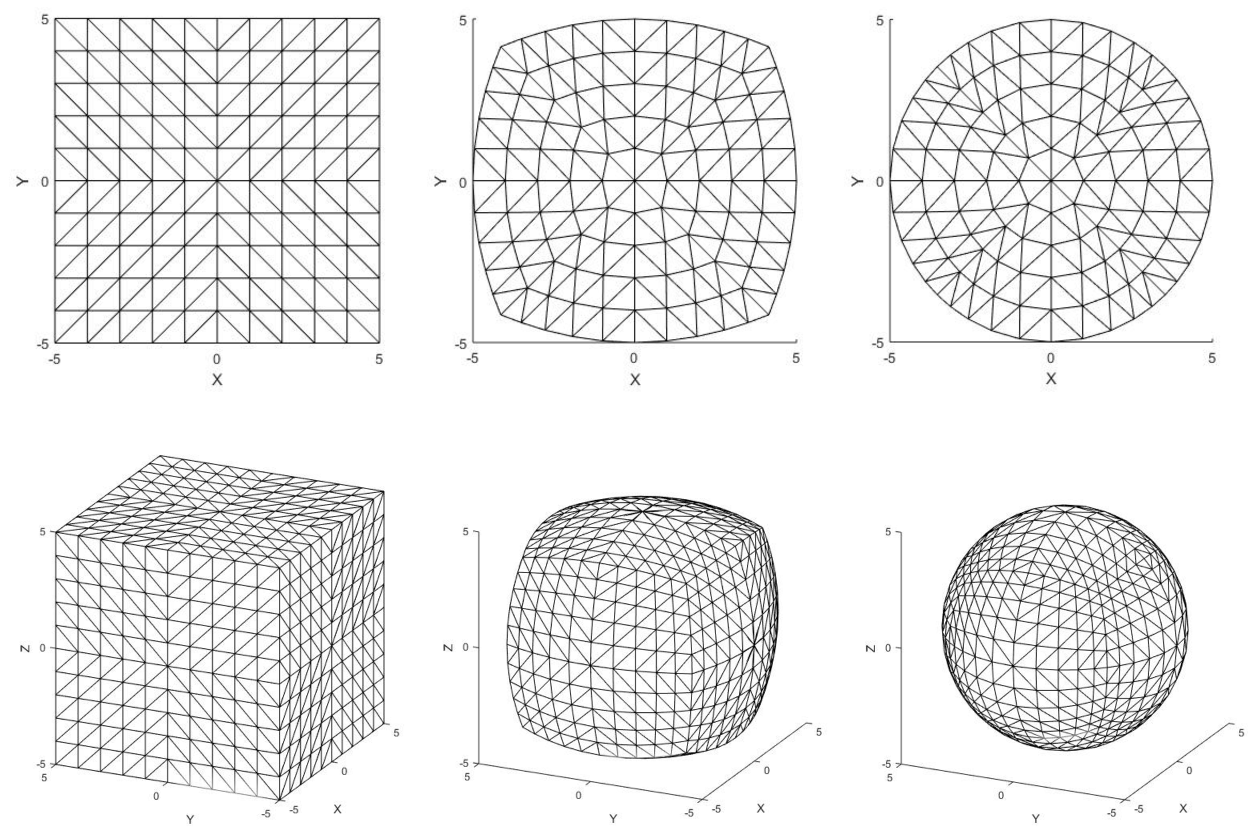

Figure 1: The images of the standard triangulation in $\mathbb{R}^2$ (upper row) and $\mathbb{R}^3$ (lower row) restricted to $[-5,5]^2$ and $[-5,5]^3$ and the same restrictions under the mappings $\mathbf{H}_0^1 = \mathrm{Id}_{\mathbb{R}^n}$, $\mathbf{H}_{1/2}^1$, and $\mathbf{H}_1^1 = \Phi$ from left to right ($\rho(x) = 1$), see Definition 2.10 for $\mathbf{H}_t^1$.

The following lemma shows that an n-simplex of the original triangulation is mapped to a proper n-simplex for each t.

Lemma 2.11. *For an arbitrary n-simplex $S = \operatorname{co} C = \operatorname{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\} \in \mathcal{T}_{\mathrm{std}}$ and for any choice of fixed $t \in [0,1]$ and rescaling function ρ, the transformed vertices*

$$\mathbf{H}_t^\rho(C) = \{\mathbf{H}_t^\rho(\mathbf{x}_0), \mathbf{H}_t^\rho(\mathbf{x}_1), \ldots, \mathbf{H}_t^\rho(\mathbf{x}_n)\}$$

are affinely-independent. As a result, the convex combination

$$\operatorname{co} \mathbf{H}_t^\rho(C) = \operatorname{co}\{\mathbf{H}_t^\rho(\mathbf{x}_0), \mathbf{H}_t^\rho(\mathbf{x}_1), \ldots, \mathbf{H}_t^\rho(\mathbf{x}_n)\}$$

is a proper n-simplex.

Proof: Let $S = \operatorname{co} C = \operatorname{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\} \in \mathcal{T}_{\mathrm{std}}$ be an arbitrary simplex of the original triangulation determined by some value of $(\mathbf{z}, \sigma, \mathbf{J}) \in \mathbb{Z}_{\geq 0}^n \times \mathrm{Sym}(n) \times \{-1,+1\}^n$. The form (2.3) of a general vertex of $S \in \mathcal{T}_{\mathrm{std}}$ reveals that we can write $\mathbf{x}_k = R_{\mathbf{J}}\mathbf{x}_k^*$ with $\mathbf{x}_k^* \in \mathbb{R}_{\geq 0}^n$ for all $k \in \{0:n\}$ and so with $C^* = \{\mathbf{x}_0^*, \mathbf{x}_1^*, \ldots, \mathbf{x}_n^*\} \subset \mathbb{N}_0^n$ we can apply the linearity of $R_{\mathbf{J}}$ and write:

$$S = \operatorname{co} C = \operatorname{co} R_{\mathbf{J}} C^* = R_{\mathbf{J}} \operatorname{co} C^* = R_{\mathbf{J}} S^*.$$

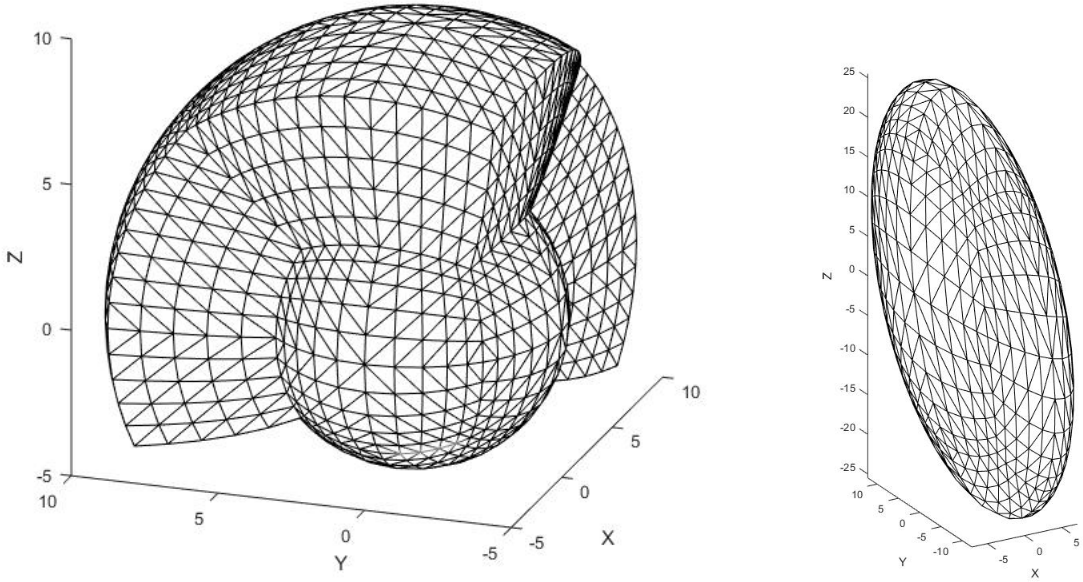

Figure 2: Left: The image of the restriction of the standard triangulation in $\mathbb{R}^3$ to $[-5,10]^3$ under the mapping $\mathbf{H}_1^1 = \Phi$. Right: The image of the restriction of the standard triangulation in $\mathbb{R}^3$ to $[-5,5]^3$ under the mapping $\mathbf{x} \mapsto A\Phi(\mathbf{x})$ with the matrix $A = ((1,-1,0)^{\mathrm{T}}, (-1,2,1)^{\mathrm{T}}, (0,1,5)^{\mathrm{T}})$.

Therefore we can focus our proof on $S^* = \operatorname{co} C^* \subset \mathbb{R}^n_{\geq 0}$ rather than S without loss of generality. We drop the asterisk and continue with (2.3) replaced by :

$$\mathbf{x}_k = \mathbf{z} + \mathbf{u}_k^\sigma.$$

To prove that the vectors $\mathbf{H}_t^\rho(\mathbf{x}_0)$, $\mathbf{H}_t^\rho(\mathbf{x}_1), \dots, \mathbf{H}_t^\rho(\mathbf{x}_n)$ are affinely independent in $\mathbb{R}^n$ is equivalent to proving that the vectors $\mathbf{H}_0, \mathbf{H}_1, \dots, \mathbf{H}_n$ are linearly independent in $\mathbb{R}^{n+1}$, where

$$\mathbf{H}_k := \left[\begin{array}{c} \uparrow \\ \mathbf{H}_t^\rho(\mathbf{x}_k) \\ \downarrow \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} \uparrow \\ h_t^\rho(\mathbf{x}_k)\mathbf{x}_k \\ \downarrow \\ \hline 1 \end{array}\right] \in \mathbb{R}^{n+1}.$$

1 We abbreviate $c_k = h_t^\rho(\mathbf{x}_k)$ and write $\mathbf{c} = (c_0, c_1, \dots, c_n)^{\mathrm{T}}$. We will show that the matrix

$$X_{\mathbf{H}} = \left[\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \cdots & \uparrow \\ c_0\mathbf{x}_0 & c_1\mathbf{x}_1 & c_2\mathbf{x}_2 & \cdots & c_n\mathbf{x}_n \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ \hline 1 & 1 & 1 & \cdots & 1 \end{array}\right] \in \mathbb{R}^{(n+1)\times(n+1)} \tag{2.7}$$

is invertible, and thereby prove our lemma. Prior to the action of $\mathbf{H}_t^\rho$ we have:

$$X = \left[\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \cdots & \uparrow \\ \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ \hline 1 & 1 & 1 & \cdots & 1 \end{array}\right] \in \mathbb{R}^{(n+1)\times(n+1)}.$$

Because S is a proper n-simplex, it follows that the columns of X are linearly independent and therefore X is invertible. In fact, we will see that $\det X = \pm 1$. With $C = \mathrm{diag}(\mathbf{c}) \in \mathbb{R}^{(n+1)\times(n+1)}$ the product $X\,C$ differs from $X_{\mathbf{H}}$ only in that it has $\mathbf{c}^{\mathrm{T}}$ in the last row rather than 1s. We write:

$$X_{\mathbf{H}} = X\,C + \sum_{k=1}^{n+1} (1 - c_{k-1}) \bar{\mathbf{e}}_{n+1} \bar{\mathbf{e}}_k^{\mathrm{T}}$$

where $\bar{\mathbf{e}}_k$ denotes the k-th unit vector in $\mathbb{R}^{n+1}$ ($\mathbf{e}_k$ is reserved for the k-th unit vector of $\mathbb{R}^n$). Recall that the rank 1 product of unit vectors $\bar{\mathbf{e}}_i \bar{\mathbf{e}}_j^{\mathrm{T}} \in \mathbb{R}^{(n+1)\times(n+1)}$ is an all-zero matrix with 1 in row i and column j and hence the second term represents the correction in the last row of $X\,C$. Expressing $X_{\mathbf{H}}$ as a rank 1 correction of $X\,C$ allows us to use the following identity from [3], related to the Sherman-Morrison lemma [25]: $\det(A + \mathbf{u}\mathbf{v}^{\mathrm{T}}) = (1 + \mathbf{v}^{\mathrm{T}} A^{-1} \mathbf{u}) \det(A)$ for an invertible square matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{n+1}$. Hence

$$\begin{aligned}
\det X_{\mathbf{H}} &= \left(1 + \left(\sum_{k=1}^{n+1} (1 - c_{k-1}) \bar{\mathbf{e}}_k \right)^{\mathrm{T}} \left(X\,C \right)^{-1} \bar{\mathbf{e}}_{n+1} \right) \det(XC) \\
&= \left(1 + \sum_{k=1}^{n+1} \left(\frac{1 - c_{k-1}}{c_{k-1}} \right) \bar{\mathbf{e}}_k^{\mathrm{T}} X^{-1} \bar{\mathbf{e}}_{n+1} \right) \det(XC) \\
&= \left(1 + \mathbf{b}^{\mathrm{T}} X^{-1} \bar{\mathbf{e}}_{n+1} \right) \det(XC),
\end{aligned} \tag{2.8}$$

where $\mathbf{b} = \left(\frac{1-c_0}{c_0}, \frac{1-c_1}{c_1}, \ldots, \frac{1-c_n}{c_n} \right)^{\mathrm{T}}$. We will now obtain an expression for X^{-1}. Let us define the following $(n+1) \times (n+1)$ matrices:

$$P_\sigma = \left[\begin{array}{cccc|c} \uparrow & \uparrow & \cdots & \uparrow & 0 \\ \mathbf{e}_{\sigma(1)} & \mathbf{e}_{\sigma(2)} & \cdots & \mathbf{e}_{\sigma(n)} & \vdots \\ \downarrow & \downarrow & \cdots & \downarrow & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array} \right] \tag{2.9}$$

$$P_z = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \hline & & & 0 \\ & \mathbb{I}_n & & \vdots \\ & & & 0 \end{array} \right] \tag{2.10}$$

$$\Delta = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{2.11}$$

Notice that these matrices are Gauss-Jordan style manipulation matrices, which we will apply to X in a particular way. First, we multiply X by Δ from the right, which subtracts from each column the previous column as follows:

$$X \cdot \Delta = \left[\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \cdots & \uparrow \\ \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ \hline 1 & 1 & 1 & \cdots & 1 \end{array} \right] \cdot \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \left[\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \cdots & \uparrow \\ \mathbf{z} & \mathbf{e}_{\sigma(1)} & \mathbf{e}_{\sigma(2)} & \cdots & \mathbf{e}_{\sigma(n)} \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ \hline 1 & 0 & 0 & \cdots & 0 \end{array} \right]$$

using $\mathbf{x}_k - \mathbf{x}_{k-1} = \mathbf{e}_{\sigma(k)}$ in the last identity. P_σ and P_z are permutation matrices and by multiplying with

P_z from the right, we put the first column last:

$$
\begin{aligned}
X \cdot \Delta \cdot P_z &= \left[\begin{array}{ccccc}
\uparrow & \uparrow & \uparrow & \cdots & \uparrow \\
\mathbf{z} & \mathbf{e}_{\sigma(1)} & \mathbf{e}_{\sigma(2)} & \cdots & \mathbf{e}_{\sigma(n)} \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
\hline
1 & 0 & 0 & \cdots & 0
\end{array}\right] \cdot \left[\begin{array}{ccc|c}
0 & \cdots & 0 & 1 \\
\hline
 & & & 0 \\
 & \mathbb{I}_n & & \vdots \\
 & & & 0
\end{array}\right] \\
&= \left[\begin{array}{cccc|c}
\uparrow & \uparrow & \cdots & \uparrow & \uparrow \\
\mathbf{e}_{\sigma(1)} & \mathbf{e}_{\sigma(2)} & \cdots & \mathbf{e}_{\sigma(n)} & \mathbf{z} \\
\downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
\hline
0 & 0 & \cdots & 0 & 1
\end{array}\right]
\end{aligned}
$$

Finally, we multiply with P_σ^{T} from the left, which is a row rearrangement corresponding to σ^{-1}:

$$
\begin{aligned}
P_\sigma^{\mathrm{T}} \cdot X \cdot \Delta \cdot P_z &= \left[\begin{array}{ccc|c}
\leftarrow & \mathbf{e}_{\sigma(1)} & \rightarrow & 0 \\
\leftarrow & \mathbf{e}_{\sigma(2)} & \rightarrow & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\leftarrow & \mathbf{e}_{\sigma(n)} & \rightarrow & 0 \\
\hline
0 & \cdots & 0 & 1
\end{array}\right] \left[\begin{array}{cccc|c}
\uparrow & \uparrow & \cdots & \uparrow & \uparrow \\
\mathbf{e}_{\sigma(1)} & \mathbf{e}_{\sigma(2)} & \cdots & \mathbf{e}_{\sigma(n)} & \mathbf{z} \\
\downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
\hline
0 & 0 & \cdots & 0 & 1
\end{array}\right] \\
&= \left[\begin{array}{ccc|c}
 & & & \uparrow \\
 & \mathbb{I}_n & & \mathbf{z}_\sigma \\
 & & & \downarrow \\
\hline
0 & \cdots & 0 & 1
\end{array}\right] =: I(\mathbf{z}_\sigma)
\end{aligned}
$$

where $\mathbf{z}_\sigma = (z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(n)})^{\mathrm{T}}$, i.e. $\mathbf{z}_\sigma \in \mathbb{R}^n$ is the vector $\mathbf{z}$ with rearranged components according to the permutation σ. The last reduced matrix $I(\mathbf{z}_\sigma)$ has a simple inverse, namely:

$$
\Big(P_\sigma^{\mathrm{T}} \cdot X \cdot \Delta \cdot P_z\Big)^{-1} = \Big(I(\mathbf{z}_\sigma)\Big)^{-1} = I(-\mathbf{z}_\sigma) := \left[\begin{array}{ccc|c}
 & & & \uparrow \\
 & \mathbb{I}_n & & -\mathbf{z}_\sigma \\
 & & & \downarrow \\
\hline
0 & \cdots & 0 & 1
\end{array}\right]
$$

and therefore

$$
X^{-1} = \Delta \cdot P_z \cdot I(-\mathbf{z}_\sigma) \cdot P_\sigma^{\mathrm{T}},
$$

from which it easy to see that $\det X = \pm 1$. With the following simplifications:

$$
\begin{aligned}
\mathbf{b}^{\mathrm{T}} X^{-1} \bar{\mathbf{e}}_{n+1} &= \mathbf{b}^{\mathrm{T}} \Delta \, P_z \, I(-\mathbf{z}_\sigma) \, P_\sigma^{\mathrm{T}} \, \bar{\mathbf{e}}_{n+1} \\
&= \left[P_z^{\mathrm{T}} \, \Delta^{\mathrm{T}} \, \mathbf{b}\right]^{\mathrm{T}} \cdot \left[I(-\mathbf{z}_\sigma) \, P_\sigma^{\mathrm{T}} \bar{\mathbf{e}}_{n+1}\right] \\
&= \begin{bmatrix} b_1 - b_0 \\ b_2 - b_1 \\ \vdots \\ b_n - b_{n-1} \\ b_0 \end{bmatrix}^{\mathrm{T}} \cdot \left[\begin{array}{c} \uparrow \\ -\mathbf{z}_\sigma \\ \downarrow \\ \hline 1 \end{array}\right]
\end{aligned}
$$

we can rewrite expression (2.8) as follows:

$$
\begin{aligned}
\det X_{\mathbf{H}} &= \big(1 + \mathbf{b}^{\mathrm{T}} X^{-1} \bar{\mathbf{e}}_{n+1}\big) \det(X\,C) \\
&= \left(1 + b_0 - \sum_{k=1}^{n} (b_k - b_{k-1}) \, z_{\sigma(k)}\right) \det(X\,C) \\
&= \left(\frac{1}{c_0} - \sum_{k=1}^{n} \left(\frac{1}{c_k} - \frac{1}{c_{k-1}}\right) z_{\sigma(k)}\right) \det(X\,C). \qquad (2.12)
\end{aligned}
$$

1 We have $|\det(XC)| = |\det X| \cdot |\det C| = |\det C| = \det C = \prod_{k=0}^{n} c_k > 0$

At this juncture, let us make the following observations: With $\|\mathbf{z}\|_\infty =: M$ and $\mathbf{x}_k = \mathbf{z} + \mathbf{u}_k^\sigma$ it is clear that there exists a $k_0 \in \{1 : n\}$ such that:

$$\|\mathbf{x}_k\|_\infty = M_k = \begin{cases} M & \text{if } k < k_0 \\ M+1 & \text{if } k \geq k_0. \end{cases}$$

In fact, consider $\left(z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(n)}\right)$ and let the k_0-th term be the first one which is M or equivalently, define k_0 through $z_{\sigma(k_0)} = M$ and $0 \leq z_{\sigma(k)} < M$ for all $k < k_0$. With k_0 determined in this way, we further let:

$$\rho\left(\|\mathbf{x}_k\|_\infty\right) = \rho_k = \begin{cases} \rho(M) =: r & \text{if } k < k_0 \\ \rho(M+1) =: R & \text{if } k \geq k_0. \end{cases}$$

Let us first consider the case $\mathbf{z} = \mathbf{0}$. Then

$$|\det X_{\mathbf{H}}| = \frac{1}{c_0} |\det(X\,C)| = \det(C) > 0$$

Let us now consider the case $\mathbf{z} \neq \mathbf{0}$, so $M \in \mathbb{N}$. Using the above notation and expression (2.6), we can write:

$$\frac{1}{c_k} = t\frac{\|\mathbf{x}_k\|_2}{\rho_k M_k} + 1 - t.$$

2 Since $\mathbf{x}_k - \mathbf{x}_{k-1} = \mathbf{e}_{\sigma(k)}$ we know that $\mathbf{x}_k$ matches $\mathbf{x}_{k-1}$ in all components except for $\sigma(k)$, where $[\mathbf{x}_k]_{\sigma(k)} =$
3 $z_{\sigma(k)} + 1$ and $[\mathbf{x}_{k-1}]_{\sigma(k)} = z_{\sigma(k)}$. We therefore have

$$\|\mathbf{x}_k\|_2^2 - \|\mathbf{x}_{k-1}\|_2^2 \quad = \quad \sum_{i=1}^{n}([\mathbf{x}_k]_i)^2 - \sum_{i=1}^{n}([\mathbf{x}_{k-1}]_i)^2 = (z_{\sigma(k)}+1)^2 - (z_{\sigma(k)})^2 = 2\,z_{\sigma(k)} + 1. \tag{2.13}$$

4 This means that for $k \neq k_0$ we have

$$\frac{1}{c_k} - \frac{1}{c_{k-1}} = \frac{t}{\rho_k M_k}\left(\|\mathbf{x}_k\|_2 - \|\mathbf{x}_{k-1}\|_2\right) = \frac{t}{\rho_k M_k}\frac{2z_{\sigma(k)}+1}{\|\mathbf{x}_k\|_2 + \|\mathbf{x}_{k-1}\|_2} \tag{2.14}$$

For $k = k_0$ however, we have:

$$\begin{aligned}
\frac{1}{c_{k_0}} - \frac{1}{c_{k_0-1}} &= t\left(\frac{\|\mathbf{x}_{k_0}\|_2}{R(M+1)} - \frac{\|\mathbf{x}_{k_0-1}\|_2}{rM}\right) \\
&= t\left(\frac{\|\mathbf{x}_{k_0}\|_2}{R(M+1)} - \frac{\|\mathbf{x}_{k_0-1}\|_2}{rM}\right) - t\frac{\|\mathbf{x}_{k_0-1}\|_2}{R(M+1)} + t\frac{\|\mathbf{x}_{k_0-1}\|_2}{R(M+1)} \\
&= \frac{t}{\rho_{k_0}M_{k_0}}\left(\|\mathbf{x}_{k_0}\|_2 - \|\mathbf{x}_{k_0-1}\|_2\right) - t\|\mathbf{x}_{k_0-1}\|_2\left(\frac{1}{rM} - \frac{1}{R(M+1)}\right) \\
&= \frac{t}{\rho_{k_0}M_{k_0}}\frac{2z_{\sigma(k_0)}+1}{\|\mathbf{x}_{k_0}\|_2 + \|\mathbf{x}_{k_0-1}\|_2} - t\|\mathbf{x}_{k_0-1}\|_2\left(\frac{1}{rM} - \frac{1}{R(M+1)}\right)
\end{aligned}$$

by (2.13). Consider the sum in expression (2.12): since $\frac{1}{rM} - \frac{1}{R(M+1)} > 0$ we have

$$\begin{aligned}
&\sum_{k=1}^{n}\left(\frac{1}{c_k} - \frac{1}{c_{k-1}}\right) z_{\sigma(k)} \\
&= \sum_{k=1}^{n}\frac{t}{\rho_k M_k}\frac{2z_{\sigma(k)}+1}{\|\mathbf{x}_k\|_2 + \|\mathbf{x}_{k-1}\|_2} z_{\sigma(k)} - t\|\mathbf{x}_{k_0-1}\|_2\left(\frac{1}{rM} - \frac{1}{R(M+1)}\right) z_{\sigma(k_0)} \\
&< \frac{t}{rM\|\mathbf{z}\|_2}\sum_{k=1}^{n}\left((z_{\sigma(k)})^2 + \frac{1}{2}\left|z_{\sigma(k)}\right|\right) - \frac{t\|\mathbf{z}\|_2}{rM}\left(1 - \frac{rM}{R(M+1)}\right) M \\
&\leq \frac{t\|\mathbf{z}\|_2}{rM} + \frac{t\|\mathbf{z}\|_1}{2rM\|\mathbf{z}\|_2} - \frac{t\|\mathbf{z}\|_2}{r(M+1)}\left(1 + \frac{R-r}{R}M\right).
\end{aligned}$$

Further,

$$|\det X_{\mathbf{H}}| = \left| \frac{1}{c_0} - \sum_{k=1}^{n} \left(\frac{1}{c_k} - \frac{1}{c_{k-1}} \right) z_{\sigma(k)} \right| \; |\det(X\,C)|$$
$$= \left| \frac{1}{c_0} - \sum_{k=1}^{n} \left(\frac{1}{c_k} - \frac{1}{c_{k-1}} \right) z_{\sigma(k)} \right| \; \det(C).$$

Let us consider the first term:

$$\frac{1}{c_0} - \sum_{k=1}^{n} \left(\frac{1}{c_k} - \frac{1}{c_{k-1}} \right) z_{\sigma(k)} > \frac{t\|\mathbf{z}\|_2}{rM} + (1-t) - \frac{t\|\mathbf{z}\|_2}{rM} - \frac{t\|\mathbf{z}\|_1}{2rM\|\mathbf{z}\|_2} + \frac{t\|\mathbf{z}\|_2}{r(M+1)} \left(1 + \frac{R-r}{R} M\right)$$
$$\geq 1 - t - \frac{t\|\mathbf{z}\|_1}{2rM\|\mathbf{z}\|_2} + \frac{t\|\mathbf{z}\|_2}{r(M+1)}$$
$$\geq \frac{t}{r} \left(\frac{\|\mathbf{z}\|_2}{M+1} - \frac{\|\mathbf{z}\|_1}{2M\|\mathbf{z}\|_2} \right) \geq 0$$

because

$$\|\mathbf{z}\|_1 \leq \frac{2M}{M+1} \|\mathbf{z}\|_2^2, \quad \text{i.e.} \quad \frac{\|\mathbf{z}\|_2}{M+1} - \frac{\|\mathbf{z}\|_1}{2M\|\mathbf{z}\|_2} \geq 0,$$

holds true for any $\mathbf{z} \in \mathbb{N}_0^n$ with $\|\mathbf{z}\|_\infty = M \in \mathbb{N}$ as $2M/(M+1) \geq 1$ and $\|\mathbf{z}\|_1 = \sum_{i=1}^n |z_i| \leq \sum_{i=1}^n |z_i|^2 = \|\mathbf{z}\|_2^2$. Hence, $|\det X_{\mathbf{H}}| > 0$, see (2.12), which concludes our proof. □

After having established that the simplices in $\mathcal{T}_{H_t^\rho} = \{\mathrm{co}(H_t^\rho(\mathrm{ve}\, S_\nu))\}_{S_\nu \in \mathcal{T}_{\mathrm{std}}}$ are proper for all $t \in [0,1]$, in particular $\mathcal{T}_\Phi = \mathcal{T}_{H_1^\rho}$, we now proceed to prove that they intersect in the correct way for $\mathcal{T}_\Phi$ to be a triangulation. We start with a few lemmas that simplify the proof of the main theorem.

Lemma 2.12. *Let $\mathbf{a}, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k, \mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and assume that $\mathbf{q}_i \in [\mathbf{0}, \mathbf{p}_i\rangle$ for $i = 1:k$. Set $P = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k\}$ and $Q = \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k\}$. Then*

$$[\mathbf{0}, \mathbf{a}\rangle \cap \mathrm{co}\, P = \emptyset, \quad \textit{if and only if} \quad [\mathbf{0}, \mathbf{a}\rangle \cap \mathrm{co}\, Q = \emptyset. \tag{2.15}$$

If the vectors in P and the vectors in Q are affinely independent, then additionally

$$|[\mathbf{0}, \mathbf{a}\rangle \cap \mathrm{co}\, P| = 1, \quad \textit{if and only if} \quad |[\mathbf{0}, \mathbf{a}\rangle \cap \mathrm{co}\, Q| = 1. \tag{2.16}$$

In this case, denoting $[\mathbf{0}, \mathbf{a}\rangle \cap \mathrm{co}\, P =: \{\mathbf{b}\}$ and $[\mathbf{0}, \mathbf{a}\rangle \cap \mathrm{co}\, Q =: \{\mathbf{c}\}$, $\mathbf{b}$ is an inner point of $\mathrm{co}\, P$, if and only if $\mathbf{c}$ is an inner point of $\mathrm{co}\, Q$.

Proof: First note that there exist constants $s_i > 0$ such that $\mathbf{q}_i = s_i \mathbf{p}_i$ for $i = 1:k$. To prove claim (2.15) assume that $\mathbf{c} \in [\mathbf{0}, \mathbf{a}\rangle \cap \mathrm{co}\, Q$. Then we can write

$$\mathbf{c} = \sum_{i=1}^{k} \lambda_i \mathbf{q}_i, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{k} \lambda_i = 1. \tag{2.17}$$

Set

$$c := \sum_{i=1}^{k} \lambda_i s_i > 0 \;\text{ and }\; \mu_i := \frac{\lambda_i s_i}{c} \;\text{ for }\; i = 1:k.$$

Then

$$\mathbf{b} := c^{-1}\mathbf{c} = \sum_{i=1}^{k} \frac{\lambda_i}{c} \mathbf{q}_i = \sum_{i=1}^{k} \mu_i \mathbf{p}_i, \quad \mu_i \geq 0, \;\text{ and }\; \sum_{i=1}^{k} \mu_i = \frac{1}{c} \sum_{i=1}^{k} \lambda_i s_i = 1. \tag{2.18}$$

Thus, for every representation of $\mathbf{c} \in \mathrm{co}\, Q$ as in (2.17) there is a corresponding $\mathbf{b} = c^{-1}\mathbf{c} \in \mathrm{co}\, P$ as in (2.18). As $\mathbf{b} \in [\mathbf{0}, \mathbf{c}\rangle = [\mathbf{0}, \mathbf{a}\rangle$ it follows that $[\mathbf{0}, \mathbf{a}\rangle$ and $\mathrm{co}\, P$ intersect. The "only if" part of claim (2.15) follows by symmetry.

For proving claim (2.16) assume that the vectors in P are affinely independent and suppose $\mathbf{c}, \mathbf{c}^* \in [\mathbf{0}, \mathbf{a}\rangle \cap \operatorname{co} Q$, $\mathbf{c} \neq \mathbf{c}^*$. We can write

$$\mathbf{c} = \sum_{i=1}^{k} \lambda_i \mathbf{q}_i, \quad \mathbf{c}^* = \sum_{i=1}^{k} \lambda_i^* \mathbf{q}_i, \quad \lambda_i, \lambda_i^* \geq 0, \quad \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \lambda_i^* = 1.$$

Set

$$c := \sum_{i=1}^{k} \lambda_i s_i > 0, \quad c^* := \sum_{i=1}^{k} \lambda_i^* s_i > 0, \quad \text{and} \quad \mu_i := \frac{\lambda_i s_i}{c} \quad \text{and} \quad \mu_i^* := \frac{\lambda_i^* s_i}{c^*} \quad \text{for} \quad i = 1 : k,$$

and consider the vectors $\mathbf{b}, \mathbf{b}^* \in \operatorname{co} P$:

$$\mathbf{b} := c^{-1}\mathbf{c} = \sum_{i=1}^{k} \mu_i \mathbf{p}_i \quad \text{and} \quad \mathbf{b}^* := (c^*)^{-1}\mathbf{c}^* = \sum_{i=1}^{k} \mu_i^* \mathbf{p}_i.$$

We show that $\mathbf{b} \neq \mathbf{b}^*$. Assume for contradiction that $\mathbf{b} = \mathbf{b}^*$. Then

$$\mathbf{b} = \sum_{i=1}^{k} \mu_i \mathbf{p}_i = \sum_{i=1}^{k} \mu_i^* \mathbf{p}_i = \mathbf{b}^*$$

and because the vectors in P are affinely independent we have

$$\mu_i = \frac{\lambda_i s_i}{c} = \frac{\lambda_i^* s_i}{c^*} = \mu_i^*, \quad \text{i.e.} \quad \frac{\lambda_i}{c} = \frac{\lambda_i^*}{c^*}, \quad \text{for } i = 1 : k.$$

Because $c^{-1} = \sum_{i=1}^{k} \frac{\lambda_i}{c} = \sum_{i=1}^{k} \frac{\lambda_i^*}{c^*} = (c^*)^{-1}$ we have $c = c^*$ and from $c^{-1}\mathbf{c} = \mathbf{b} = \mathbf{b}^* = (c^*)^{-1}\mathbf{c}^*$ we get $\mathbf{c} = \mathbf{c}^*$ contradictory to assumption. The claim (2.16) now follows by symmetry. The last statement follows from the proof above by observing that, since $\mu_i = \frac{\lambda_i s_i}{c}$, we have $\lambda_i > 0$ for all i, if and only if $\mu_i > 0$ for all i. □

Remark 2.13. *Consider $P = \{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2\}$, $Q = \{\mathbf{e}_1, (\mathbf{e}_1 + \mathbf{e}_2)/2, \mathbf{e}_2\}$, $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2$, and note that the vectors in P are affinely independent and the vectors in Q are not. Now $[\mathbf{0}, \mathbf{a}\rangle \cap \operatorname{co} P = \{t(\mathbf{e}_1 + \mathbf{e}_2) \mid t \in [1/2, 1]\}$ but $[\mathbf{0}, \mathbf{a}\rangle \cap \operatorname{co} Q = \{(\mathbf{e}_1 + \mathbf{e}_2)/2\}$. We thus need to assume that both the vectors in P and the vectors in Q are affinely independent for claim* (2.16).

The following lemma is a simple consequence of the last lemma and its proof.

Lemma 2.14. *Let $P = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k\} \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ be a set of affinely independent vectors and $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ be such that $[\mathbf{0}, \mathbf{a}\rangle \cap \operatorname{co} P = \{\mathbf{b}\}$. Let $\mathbf{q}_i(t) = s_i(t)\mathbf{p}_i$ for $i = 1 : k$, where for an interval $J \subset \mathbb{R}$ the $s_i : J \to \mathbb{R}$ are continuous functions and $s_i(t) > 0$ for all $t \in J$. Assume that $Q(t) = \{\mathbf{q}_1(t), \mathbf{q}_2(t), \ldots, \mathbf{q}_k(t)\}$ is a set of affinely independent vectors for all $t \in J$. Then there is a continuous function $c : J \to \mathbb{R}$ such that $\mathbf{c} : J \to \mathbb{R}^n$ defined through*

$$\{\mathbf{c}(t)\} := [\mathbf{0}, \mathbf{a}\rangle \cap \operatorname{co} Q(t)$$

can be written as $\mathbf{c}(t) = c(t)\mathbf{b}$ for all $t \in J$.

Proof: That $\mathbf{c} : J \to \mathbb{R}^n$ is a properly defined function follows by the fact shown in the proof of Lemma 2.12, that (2.17) and (2.18) define a bijection between the elements of $\operatorname{co} P$ and $\operatorname{co} Q(t)$ for every $t \in J$. Further, with $\mathbf{b} = \sum_{i=1}^{k} \mu_i \mathbf{p}_i$ we have as in (2.17) and (2.18) with

$$\mathbf{c}(t) = \sum_{i=1}^{k} \lambda_i(t)\mathbf{q}_i(t), \quad \sum_{i=1}^{k} \lambda_i(t) = 1, \; c(t) := \sum_{i=1}^{k} \lambda_i(t) s_i(t) > 0 \quad \text{and} \quad \mu_i := \frac{\lambda_i(t) s_i(t)}{c(t)},$$

that

$$\sum_{i=1}^{k} \frac{\mu_i}{s_i(t)} = \sum_{i=1}^{k} \frac{\lambda_i(t)}{c(t)} = \frac{1}{c(t)}, \quad \text{i.e.} \quad c(t) = \left(\sum_{i=1}^{k} \frac{\mu_i}{s_i(t)} \right)^{-1} \quad \text{is continuous}$$

and $\mathbf{c}(t) = c(t)\mathbf{b}$ since

$$\mathbf{b} = \sum_{i=1}^{k} \mu_i \mathbf{p}_i = \frac{1}{c(t)} \sum_{i=1}^{k} \lambda_i s_i(t) \mathbf{p}_i = \frac{1}{c(t)} \sum_{i=1}^{k} \lambda_i \mathbf{q}_i(t) = \frac{\mathbf{c}(t)}{c(t)}$$

□

We prove two more lemmas before we state and prove the main theorem.

Lemma 2.15. *Let $S = \operatorname{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a proper k-simplex in $\mathbb{R}^n$ and $[\mathbf{a}, \mathbf{b}\rangle$ be a ray in $\mathbb{R}^n$, $\mathbf{a} \neq \mathbf{b}$, such that $\mathbf{a}$ is not an inner point of S. Assume $[\mathbf{a}, \mathbf{b}\rangle \cap S \supset \{\mathbf{c}, \mathbf{d}\}$, $\mathbf{c} \neq \mathbf{d}$ and $\mathbf{c}$ is an inner point of S.*

Then there is an open line segment $(\mathbf{e}, \mathbf{f}) \ni \mathbf{c}$ with $\mathbf{e}, \mathbf{f} \in [\mathbf{a}, \mathbf{b})$ such that all points $\mathbf{q} \in (\mathbf{e}, \mathbf{f})$ are inner points of S.

Furthermore, there are two different subsimplices T and T^* of S, which $[\mathbf{a}, \mathbf{b}\rangle$ intersects in unique inner points of T and T^*, respectively and $\operatorname{ve} S = \operatorname{ve} T \cup \operatorname{ve} T^*$.

Proof: Note that the line parameterized by $\mathbf{p}(t) = t\mathbf{a} + (1-t)\mathbf{b}$, $t \in \mathbb{R}$, lies in the affine space $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \sum_{i=0}^{k} \lambda_i \mathbf{x}_i,\ \sum_{i=0}^{k} \lambda_i = 1\}$. Thus

$$\mathbf{p}(t) = \sum_{i=0}^{k} \lambda_i(t) \mathbf{x}_i, \quad \text{where} \quad \sum_{i=0}^{k} \lambda_i(t) = 1 \quad \text{for all } t \in \mathbb{R},$$

and it is easily seen that the $\lambda_i : \mathbb{R} \to \mathbb{R}$ are affine functions of t, i.e. $\lambda_i(t) = a_i + b_i t$ for some constants $a_i, b_i \in \mathbb{R}$. Indeed, the line can also be parameterized by $\mathbf{p}(s) = s\mathbf{c} + (1-s)\mathbf{d}$, $s \in \mathbb{R}$, since $\{\mathbf{c}, \mathbf{d}\} \subset [\mathbf{a}, \mathbf{b}\rangle$, and $\mathbf{c} \neq \mathbf{d}$ and $\mathbf{c} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i$ and $\mathbf{d} = \sum_{i=1}^{k} \mu_i \mathbf{x}_i$ with $\sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \mu_i = 1$, since $\mathbf{c}, \mathbf{d} \in S$. Hence, $\mathbf{p}(s) = \sum_{i=1}^{k} (s\lambda_i + (1-s)\mu_i)\mathbf{x}_i$ with $\sum_{i=1}^{k} (s\lambda_i + (1-s)\mu_i) = 1$.

Let $t_{\mathbf{c}} \in \mathbb{R}$ be such that $\mathbf{p}(t_{\mathbf{c}}) = \mathbf{c}$. Since $\mathbf{c}$ is an inner point of S we have $\lambda_l(t_{\mathbf{c}}) > 0$ for $l \in \{0:k\}$ and, since S is compact and $[\mathbf{a}, \mathbf{b}\rangle$ is unbounded and $\mathbf{a}$ is not an inner point of S, there are $t_{\mathbf{e}} < t_{\mathbf{c}} < t_{\mathbf{f}}$ and indices $i, j \in \{0:k\}$ such that

$$\lambda_l(t) > 0 \quad \text{for all } l \in \{0:k\} \text{ and } t \in (t_{\mathbf{e}}, t_{\mathbf{f}}), \quad \lambda_i(t_{\mathbf{e}}) = 0 \quad \text{and} \quad \lambda_j(t_{\mathbf{f}}) = 0.$$

Clearly $i \neq j$ and $\mathbf{e} := \mathbf{p}(t_{\mathbf{e}})$ and $\mathbf{f} := \mathbf{p}(t_{\mathbf{f}})$ are on the ray $[\mathbf{a}, \mathbf{b}\rangle$, noting again that $\mathbf{a}$ is not an inner point of S. This shows that all points on $(\mathbf{e}, \mathbf{f})$ are inner points of S.

Now denote by T and T^* the sub-simplices of S having $\mathbf{e}$ and $\mathbf{f}$ as inner points, respectively. In particular, $\mathbf{x}_i \notin \operatorname{ve} T$ and $\mathbf{x}_j \notin \operatorname{ve} T^*$.

We finish the proof by showing that $T \cap [\mathbf{a}, \mathbf{b}\rangle = \{\mathbf{e}\}$, $T^* \cap [\mathbf{a}, \mathbf{b}\rangle = \{\mathbf{f}\}$, and $\operatorname{ve} S = \operatorname{ve} T \cup \operatorname{ve} T^*$. Note that $\lambda_i(t) < 0$ for $t < t_{\mathbf{e}}$ and $\lambda_i(t) > 0$ for $t > t_{\mathbf{e}}$ and therefore $\mathbf{p}(t) \notin T$ for $t \neq t_{\mathbf{e}}$ as $\mathbf{x}_i \notin \operatorname{ve} T$, i.e. $T \cap [\mathbf{a}, \mathbf{b}\rangle = \{\mathbf{e}\}$. Similarly $T^* \cap [\mathbf{a}, \mathbf{b}\rangle = \{\mathbf{f}\}$. To show that $\operatorname{ve} S = \operatorname{ve} T \cup \operatorname{ve} T^*$, let us assume in contradiction to the statement that $\mathbf{x}_l \in \operatorname{ve} S$ and $\mathbf{x}_l \notin \operatorname{ve} T \cup \operatorname{ve} T^*$ for $l \in \{0:k\}$. The latter statement implies that $\lambda_l(t_{\mathbf{e}}) = 0$ and $\lambda_l(t_{\mathbf{f}}) = 0$, and thus $\lambda_l(t) = 0$ for all $t \in \mathbb{R}$ since λ_l is an affine function of t. This is a contradiction to $\lambda_l(t_{\mathbf{c}}) > 0$, since $\mathbf{c}$ is an inner point of S, which shows $\operatorname{ve} S = \operatorname{ve} T \cup \operatorname{ve} T^*$. □

Lemma 2.16. *Assume the ray $[\mathbf{0}, \mathbf{p}\rangle$, $\mathbf{p} \in \mathbb{R}^n$, intersects two different subsimplices T and T^* of a simplex $S \in \mathcal{K}_{\text{std}}$ in unique inner points $\mathbf{a} = s\mathbf{p} \in T$ and $\mathbf{b} = s^*\mathbf{p} \in T^*$, $s < s^*$. Then for every $t \in [0,1]$ the ray intersects the subsimplices $\operatorname{co} \mathbf{H}_t^{\rho}(\operatorname{ve} T)$ and $\operatorname{co} \mathbf{H}_t^{\rho}(\operatorname{ve} T^*)$ of $S^t := \operatorname{co} \mathbf{H}_t^{\rho}(\operatorname{ve} S)$ in unique inner points $\mathbf{a}_t = s_t \mathbf{p}$ and $\mathbf{b}_t = s_t^* \mathbf{p}$, respectively, where $s_t < s_t^*$.*

Proof: By Lemma 2.12 the ray $[\mathbf{0}, \mathbf{p}\rangle$ intersects the simplices $\operatorname{co} \mathbf{H}_t^{\rho}(\operatorname{ve} T)$ and $\operatorname{co} \mathbf{H}_t^{\rho}(\operatorname{ve} T^*)$ in unique points for every $t \in [0,1]$. Assume for a contradiction that there exists a $t \in [0,1]$ such that $s_t \geq s_t^*$. By Lemma 2.14 the functions $t \mapsto s_t$ and $t \mapsto s_t^*$ are continuous and thus there exists an $r \in [0,1]$ such that $s_r = s_r^*$. It follows that S^r is not a proper simplex, because $\mathbf{a}_r = s_r \mathbf{p} = s_r^* \mathbf{p} = \mathbf{b}_r$ is an inner point of both $\operatorname{co} \mathbf{H}_r^{\rho}(\operatorname{ve} T)$ and $\operatorname{co} \mathbf{H}_r^{\rho}(\operatorname{ve} T^*)$ and thus can be written in two different ways as a convex combination of the vectors in $\mathbf{H}_r^{\rho}(\operatorname{ve} S)$. This is a contradiction to Lemma 2.11 and we conclude $s_t < s_t^*$ for all $t \in [0,1]$. □

Theorem 2.17. *For every $t \in [0,1]$ the set of simplices $\mathcal{S}^t := \{\mathrm{co}(\mathbf{H}_t^\rho(\mathrm{ve}\, S_\nu))\}_{S_\nu \in \mathcal{T}_{\mathrm{std}}}$ is a triangulation of $\mathbb{R}^n$.*

PROOF: Let $\mathbf{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ be arbitrary but fixed throughout the proof. Consider the ray $[\mathbf{0}, \mathbf{p}\rangle$, namely $\{s\mathbf{p}\}$ with $s \geq 0$. The ray intersects infinitely many proper n-simplices of the standard triangulation. For each point $s\mathbf{p}$, $s \geq 0$, there exists a unique k-simplex $S_s \in \mathcal{K}_{\mathrm{std}}$, $k \in \{0 : n\}$ such that $s\mathbf{p}$ is an inner point of S_s. For different s, these simplices S_s may or may not be equal. We are interested in boundaries of intervals (s_i, s_{i+1}) such that the simplices are equal for all $s \in (s_i, s_{i+1})$

In particular, we can define numbers $0 = s_0 < s_1 < s_2 < \dots$ with $s_i \to \infty$ as $i \to \infty$ such that $S_s = S_t =: S_{i,i+1}$ for all $s, t \in (s_i, s_{i+1})$ and the interval (s_i, s_{i+1}) is maximal with this property. Moreover, we define $S_i := S_{s_i} \in \mathcal{K}_{\mathrm{std}}$. We have that $\{s_i\mathbf{p}\}$ is an inner point of S_i and $\{s_i\mathbf{p}\} = S_i \cap [\mathbf{0}, \mathbf{p}\rangle$ for $i = 0, 1, 2, \dots$, since if $S_i \cap [\mathbf{0}, \mathbf{p}\rangle$ consisted of the inner point $\mathbf{c} = s_i\mathbf{p}$ and at least one further point $\mathbf{d} \neq \mathbf{c}$, then Lemma 2.15 shows that there is an interval $(t_1, t_2) \ni s_i$ such that $t\mathbf{p}$ are inner points of S_i for all $t \in (t_1, t_2)$ in contradiction to the definition of S_i.

For $s_i < s < s_{i+1}$ we have $\{s\mathbf{p}\} \neq S \cap [\mathbf{0}, \mathbf{p}\rangle$ for any $S \in \mathcal{K}_{\mathrm{std}}$; we have, however $\{s\mathbf{p}\} \subsetneq S_{i,i+1}$. Lemma 2.15 shows the existence of two different sub-simplices T, T^* of $S_{i,i+1}$ that the ray $[\mathbf{0}, \mathbf{p}\rangle$ intersects in unique inner points. Further, $\mathrm{ve}\, S_{i,i+1} = \mathrm{ve}\, T \cup \mathrm{ve}\, T^*$. Clearly we can assume $S_i = T$ and $S_{i+1} = T^*$.

Now fix a $t \in [0,1]$ and define in a similar way, using Lemma 2.12 and 2.11, the numbers $0 = s_0^t < s_1^t < s_2^t < \dots$ and $S_i^t \in \mathcal{C}[\mathcal{S}^t]$, such that $\{s_i^t\mathbf{p}\} = S_i^t \cap [\mathbf{0}, \mathbf{p}\rangle$ for $i = 0, 1, 2, \dots$, and if $s_i^t < s^t < s_{i+1}^t$ then $\{s^t\mathbf{p}\} \neq S^t \cap [\mathbf{0}, \mathbf{p}\rangle$ for any $S^t \in \mathcal{C}[\mathcal{S}^t]$. Moreover, $s_i^t\mathbf{p}$ are inner points of S_i^t. By Lemma 2.16 we have that $\mathrm{ve}\, S_i^t = \mathrm{ve}\, \mathbf{H}_t^\rho(S_i)$ for $i = 0, 1, 2, \dots$. Define $S_{i,i+1}^t := \mathrm{co}(\mathrm{ve}\, S_i^t \cup \mathrm{ve}\, S_{i+1}^t) = \mathrm{co}(\mathbf{H}_t^\rho(\mathrm{ve}\, S_i) \cup \mathbf{H}_t^\rho(\mathrm{ve}\, S_i))$ and note that we have $[s_i^t, s_{i+1}^t]\mathbf{p} \subset S_{i,i+1}^t$ because $s_i^t\mathbf{p}, s_{i+1}^t\mathbf{p} \in S_{i,i+1}^t$. Further and with identical arguments, if for some $s^t > 0$ the point $s^t\mathbf{p}$ is an inner point of an $S_*^t \in \mathcal{C}[\mathcal{S}^t]$ and $\{s^t\mathbf{p}\} \neq S_*^t \cap [\mathbf{0}, \mathbf{p}\rangle$, then necessarily with i such that $s_i^t < s^t < s_{i+1}^t$ we have $S_*^t = S_{i,i+1}^t$. If $s_t = 0$ then $S^t = \{\mathbf{0}\}$ is the unique simplex in $\mathcal{C}[\mathcal{S}^t]$, of which $0\mathbf{p} = \mathbf{0}$ is an inner point and if $\{s^t\mathbf{p}\} = S_*^t \cap [\mathbf{0}, \mathbf{p}\rangle$ then $s^t = s_i^t$ for some $i = 1, 2, \dots$ and $S_*^t = S_i^t$ by Lemma 2.12. Since $\mathbf{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ was arbitrary and $\mathcal{S}^t$ is a set of proper n-simplices by Lemma 2.11, the fact that $s^t\mathbf{p}$ is the inner point of a unique $S^t \in \mathcal{C}[\mathcal{S}^t]$ concludes our proof of that $\mathcal{S}^t$ is a triangulation by Lemma 2.5.

It remains to be shown that $\mathcal{D}_{\mathcal{S}^t} = \mathbb{R}^n$. Choose a point $\mathbf{p} \in \mathbb{R}^n$. We will show that $s_i^t \to \infty$ as $i \to \infty$, from which the statement follows. Let $M > 1$ be arbitrary and let $N \in \mathbb{N}$ be such that $s_N\|\mathbf{p}\|_\infty > M + 1$; note that $s_i \to \infty$ as $i \to \infty$. We have $s_N\mathbf{p} \in S_N$ for a simplex $S_N = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k\} \in \mathcal{K}_{\mathrm{std}}$. Thus, $\|\mathbf{x}_i\|_\infty = \lfloor M+1 \rfloor$ or $\|\mathbf{x}_i\|_\infty = \lfloor M+1 \rfloor + 1$ for all $i \in \{0 : k\}$, in particular $\|\mathbf{x}\|_\infty \geq M > 1$. It follows with $\|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$ that

$$h_t^\rho(\mathbf{x}_i) = \frac{\rho(\|\mathbf{x}_i\|_\infty)\|\mathbf{x}_i\|_\infty}{t\|\mathbf{x}_i\|_2 + (1-t)\rho(\|\mathbf{x}_i\|_\infty)\|\mathbf{x}_i\|_\infty} \geq \frac{\rho(\|\mathbf{x}_i\|_\infty)}{t\sqrt{n} + (1-t)\rho(\|\mathbf{x}_i\|_\infty)} \geq \frac{\rho(1)}{t\sqrt{n} + (1-t)\rho(1)} =: \rho^* > 0,$$

and thus, since $s_N^t \in \mathrm{co}(\mathbf{H}_t^\rho\, \mathrm{ve}\, S_N)$ and $\rho(\|\mathbf{x}_i\|_\infty) \geq \rho(1)$

$$s_N^t\|\mathbf{p}\|_\infty \geq \min_{i \in \{0:k\}} h_t^\rho(\mathbf{x}_i)\|\mathbf{x}_i\|_\infty \geq \rho^* M,$$

which concludes our proof. $\square$

An obvious corollary is:

Corollary 2.18. *$\mathcal{T}_\Phi = \{\mathrm{co}(\Phi(\mathrm{ve}\, S_\nu))\}_{S_\nu \in \mathcal{T}_{\mathrm{std}}}$ is a triangulation of $\mathbb{R}^n$.*

It is worth noting that a subset $\mathcal{T} \subset \mathcal{T}_{\mathrm{std}}$ is a triangulation and so is the set $\mathcal{T}^* := \{\mathrm{co}(\mathbf{H}_t^\rho(\mathrm{ve}\, S_\nu))\}_{S_\nu \in \mathcal{T}}$. Further, $\mathcal{D}_\mathcal{T}$ is connected, if and only if $\mathcal{D}_{\mathcal{T}^*}$ is connected. This is easily seen from Definition 2.1 and the proof of Theorem 2.17. However, the convexity of $\mathcal{D}_\mathcal{T}$ does not imply the convexity of $\mathcal{D}_{\mathcal{T}^*}$, as can be seen in Figure 2 (left).

3 Conclusions

We presented a method to generate a set of simplices in $\mathbb{R}^n$ from a very simple simplicial complex (standard triangulation) and proved that the resulting set is also a simplicial complex. This new simplicial complex has an approximate rotational symmetry, cf. Figure 1, and has applications when computing continuous and piecewise affine Lyapunov functions and contraction metrics for nonlinear systems [14, 9, 8, 7]. In particular, it can be easily transformed to a simplicial complex that matches the level sets of quadratic Lyapunov functions, that is, hyper-ellipsoids, cf. Figure 2 (right), and allows for efficient algorithms to locate simplices [15, 17].

Acknowledgement: This research was supported by the Icelandic Research Fund (Rannís) in grants number 163074-052 and 152429-051, *Complete Lyapunov functions: Efficient numerical computation* and *Lyapunov Methods and Stochastic Stability* respectively.

References

[1] F. Davoine, M. Aritonini, J. Chassery, and M. Barlaud. Fractal image compression based on Delaunay triangulation and vector quantization. *IEEE Trans. Image Process.*, 5(2):338–346, 1996.

[2] B. Delaunay. Sur la sphère vide. a la mémoire de Georges Voronoï. *Bulletin de l'Académie des Sciences de l'URSS. Classe des sciences mathématiques et na*, (6):793–800, 1934.

[3] J. Ding and A. Zhou. Eigenvalues of rank-one updated matrices with some applications. *Appl. Math. Lett.*, 20(12):1223–1226, 2007.

[4] H. Edelsbrunner. *Geometry and Topology for Mesh Generation*. Cambridge University Press, 2001.

[5] H. Edelsbrunner, T. Tan, and R. Waupotitsch. An $o(n^2 \log n)$ time algorithm for the minmax angle triangulation. *SIAM Journal on Scientific and Statistical Computing*, 13(4):994–1008, 1992.

[6] P. Frey and P. George. *Mesh Generation: Application to Finite Elements*. Hermes Science, 2000.

[7] P. Giesl and S. Hafstein. Construction of a CPA contraction metric for periodic orbits using semidefinite optimization. *Nonlinear Anal.*, 86:114–134, 2013.

[8] P. Giesl and S. Hafstein. Computation of Lyapunov functions for nonlinear discrete systems by linear programming. *J. Difference Equ. Appl.*, 20:610–640, 2014.

[9] P. Giesl and S. Hafstein. Revised CPA method to compute Lyapunov functions for nonlinear systems. *J. Math. Anal. Appl.*, 410:292–306, 2014.

[10] P. Giesl and S. Hafstein. Computation and verification of Lyapunov functions. *SIAM Journal on Applied Dynamical Systems*, 14(4):1663–1698, 2015.

[11] B. Grünbaum, V. Kaibel, V. Klee, and G. Ziegler. *Convex Polytopes*. Graduate Texts in Mathematics 221. Springer, 2003.

[12] S. Gudmundsson and S. Hafstein. Lyapunov function verification: MATLAB implementation. In *Proceedings of the 1st Conference on Modelling, Identification and Control of Nonlinear Systems (MICNON)*, number 0235, pages 806–811, 2015.

[13] L.. Guibas, D. Knuth, and M. Sharir. Randomized incremental construction of Delaunay and Voronoi diagrams. *Algorithmica*, 7(1-6):381–413, 1992.

[14] S. Hafstein. A constructive converse Lyapunov theorem on exponential stability. *Discrete Contin. Dyn. Syst. - Series A*, 10(3):657–678, 2004.

[15] S. Hafstein. Efficient algorithms for simplicial complexes used in the computation of Lyapunov functions for nonlinear systems. In *Proceedings of the 7th International Conference on Simulation and Modeling Methodologies, Technologies and Applications (SIMULTECH)*, pages 398–409, 2017.

[16] S. Hafstein. Common Lyapunov function computation for discrete-time systems. In *Proceedings of 57rd IEEE Conference on Decision and Control (CDC)*, pages 3211–3216, 2018.

[17] S. Hafstein. *Simulation and Modeling Methodologies, Technologies and Applications*, volume 873 of *Advances in Intelligent Systems and Computing*, chapter Fast Algorithms for Computing Continuous Piecewise Affine Lyapunov Functions, pages 274–299. Springer, 2019.

[18] S. Hafstein and C. Kawan. Numerical approximation of the data-rate limit for state estimation under communication constraints. *J. Math. Anal. Appl.*, 473(2):1280–1304, 2019.

[19] P. Julian. *A High Level Canonical Piecewise Linear Representation: Theory and Applications*. PhD thesis: Universidad Nacional del Sur, Bahia Blanca, Argentina, 1999.

[20] P. Julian, J. Guivant, and A. Desages. A parametrization of piecewise linear Lyapunov functions via linear programming. *Int. J. Control*, 72(7-8):702–715, 1999.

[21] H. Li, S. Hafstein, and C. Kellett. Computation of continuous and piecewise affine Lyapunov functions for discrete-time systems. *J. Difference Equ. Appl.*, 21(6):486–511, 2015.

[22] H. Li and A. Liu. Computation of non-monotonic Lyapunov functions for continuous-time systems. *Commun. Nonlinear Sci. Numer. Simulat.*, 50:35–50, 2017.

[23] S. Marinósson. Lyapunov function construction for ordinary differential equations with linear programming. *Dynamical Systems: An International Journal*, 17:137–150, 2002.

[24] S. Marinósson. *Stability Analysis of Nonlinear Systems with Linear Programming: A Lyapunov Functions Based Approach*. PhD thesis: Gerhard-Mercator-University Duisburg, Duisburg, Germany, 2002.

[25] J. Sherman and W. Morrison. Adjustment of an inverse matrix corresponding to a change in one element of a given matrix. *Ann. Math. Statist.*, 21(1):124–127, 1950.

[26] J. Thompson, Z. Warsi, and C. Mastin. *Numerical Grid Generation: Foundations and Applications*. Elsevier, 1985.

[27] D. Watson and G. Philip. Systematic triangulations. *Computer Vision, Graphics, and Image Processing*, 26(2):217–223, 1984.