Common Lyapunov functions for switched linear systems: Linear Programming based approach

Stefania Andersen, Peter Giesl, and Sigurdur Hafstein

***Abstract*— We study the stability of an equilibrium of arbitrarily switched, autonomous, continuous-time systems through the computation of a common Lyapunov function (CLF). The switching occurs between a finite number of individual subsystems, each of which is assumed to be linear. We present a linear programming (LP) based approach to compute a continuous and piecewise affine (CPA) CLF and compare this approach with different methods in the literature. In particular we compare it with the prevalent use of linear matrix inequalities (LMIs) and semidefinite optimization to parameterize a quadratic common Lyapunov function (QCLF) for the linear subsystems.**

***Index Terms*— Common Lyapunov function, Linear Matrix Inequalities, Linear Programming, Linear systems, Switched Systems.**

I. Introduction

SWITCHED systems often arise in applications, e.g. in hybrid systems where the switching is due to the interaction between continuous-time dynamics, i.e. the individual subsystems, and discrete-time dynamics, i.e. the switching. Another common source of switched systems is uncertainty quantification in continuous-time systems and the associated differential inclusions. Some general references for switched systems and stability are [15], [27], [41], [43].

A linear system has an exponentially stable equilibrium at the origin, if and only if there exists a quadratic Lyapunov function for the system, and a Lyapunov function can be efficiently computed [6] by solving the continuous-time Lyapunov equation. This is discussed in essentially all textbooks on stability theory, e.g. [24], [26], [40], [45]. Because of this simple characterization for linear systems, a considerable effort has been devoted to characterize when an arbitrarily switched system, where each subsystem is linear, possesses a quadratic common Lyapunov function (QCLF). For planar systems with two subsystems, there is a characterization in terms of the negativity of the real parts of the eigenvalues of all pairwise convex combinations of the system matrices and their inverses, see [14] and [27] for some generalizations. In the literature several sufficient conditions have been derived for the existence of a QCLF for more general systems, see e.g. [2], [27], and if such a Lyapunov function exists then one can be computed by numerically solving a linear matrix inequality (LMI) problem [10]; see also [28]. Note, however, that it is possible that an arbitrarily switched system has an exponentially stable equilibrium at the origin, but there does not exist a QCLF for the system. For this reason several methods computing norms (Minkowskii, weighted), that can serve as Lyapunov functions, have been proposed. They usually use linear programming (LP) to parameterize or verify the conditions of a Lyapunov function, see e.g. [7]–[9], [11], [12], [35]–[38], [47], [48]; the converse theorems from [33], [34] have been important for many of these approaches. For more methods see, e.g., the review [20].

We will present a somewhat different approach to parameterize continuous and piecewise affine (CPA) Lyapunov functions for arbitrarily switched systems. It is an adaptation of the CPA method to compute Lyapunov functions, see e.g. [5], [19], [23], [30], to the arbitrarily switched linear case. Note that in the CPA method often an arbitrary small neighbourhood of the equilibrium at the origin is left out of the domain of the computed Lyapunov function, while in the linear case any such neighbourhood defines the Lyapunov function on the whole state-space. Suitable classes of Lyapunov functions for linear switched systems have been thoroughly studied in a recent preprint [32], see also [22], [31], where it is shown that the class of piecewise linear functions is large enough to contain Lyapunov functions whenever the origin is exponentially stable. Note, however, that these results are not directly applicable to our method, as we use an a priori fixed triangulation. Nevertheless, considering our numerical results, we are optimistic that the results from [32] can be adapted to prove that our method is constructive.

II. Preliminaries

In this paper we consider an arbitrarily switched system of the following type: Let $A_i \in \mathbb{R}^{n\times n}$, $i = 1, 2, \ldots, N$, and consider the system

$$\dot{\mathbf{x}} = A_\sigma \mathbf{x} \tag{1}$$

for all $\sigma \in \mathcal{P}$. Here, $\mathcal{P}$ denotes the set of all switching signals $\sigma\colon [0,\infty) \to \{1, 2, \ldots, N\}$ such that σ is right-continuous and has only a finite number of discontinuity points on any finite interval. A solution to the system (1) with fixed $\sigma \in \mathcal{P}$ and with initial value $\boldsymbol{\xi} \in \mathbb{R}^n$ at time $t = 0$, is an absolutely continuous function $t \mapsto \boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi})$ that fulfills $\boldsymbol{\phi}_\sigma(0, \boldsymbol{\xi}) = \boldsymbol{\xi}$ and $\dot{\boldsymbol{\phi}}_\sigma(t, \boldsymbol{\xi}) = A_{\sigma(t)}\boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi})$

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S. Andersen and S. Hafstein are with University of Iceland, Faculty of Physical Sciences, Dunhagi 5, 107 Reykjavik, Iceland (email: saa20@hi.is and shafstein@hi.is).

P. Giesl is with University of Sussex, Department of Mathematics, Falmer BN1 9QH, U.K. (email: p.a.giesl@sussex.ac.uk).

at every $t \geq 0$ that is not a discontinuity point of σ. In essence, the solution trajectory of (1) is obtained by gluing together solution trajectory segments of the systems $\dot{\mathbf{x}} = A_i\mathbf{x}$, $i = 1, 2, \ldots, N$, using $i = \sigma([t_j, t_{j+1}))$, where $t_0 = 0$ and $t_1 < t_2 < \ldots$ are the discontinuity-points of σ. Arbitrary switching refers to the fact that we are interested in the family of all solutions as σ varies over all possible switching signals.

The exponential stability of the origin is equivalent to the existence of a common Lyapunov function (CLF) for the system (1) [41], i.e. a locally Lipschitz continuous function $V\colon \mathbb{R}^n \to \mathbb{R}$ that is a Lyapunov function for all the individual subsystems of (1). This means that there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, where $\mathcal{K}_\infty$ is the set of all strictly increasing continuous functions $\alpha\colon [0,\infty) \to [0,\infty)$ fulfilling $\alpha(0) = 0$ and $\lim_{x\to\infty} \alpha(x) = \infty$, such that for all $\mathbf{x} \in \mathbb{R}^n$ and all $\sigma \in \mathcal{P}$ the inequalities

$$\alpha_1(\|\mathbf{x}\|_2) \leq V(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|_2) \quad \text{and} \tag{2}$$

$$D_\sigma^+ V(\mathbf{x}) := \limsup_{h\to 0+} \frac{V(\boldsymbol{\phi}_\sigma(h,\mathbf{x})) - V(\mathbf{x})}{h} \leq -\alpha_3(\|\mathbf{x}\|_2)$$

hold true. The Dini-derivative $D_\sigma^+ V(\mathbf{x})$ is a generalization of the orbital derivative; for a differentiable V we have

$$D_\sigma^+ V(\mathbf{x}) = \nabla V(\mathbf{x}) \bullet A_{\sigma(0)}\mathbf{x}.$$

Further, since V is locally Lipschitz, we have

$$D_\sigma^+ V(\mathbf{x}) = \limsup_{h\to 0+} \frac{V(\mathbf{x} + hA_{\sigma(0)}\mathbf{x}) - V(\mathbf{x})}{h},$$

see e.g. [17, Lem. 3.3]; note that $\sigma(h) = \sigma(0)$ for all small enough $h > 0$. The second inequality in the condition (2) is equivalent to

$$D^+ V(\mathbf{x}) := \max_{\sigma\in\mathcal{P}} D_\sigma^+ V(\mathbf{x}) \leq -\alpha_3(\|\mathbf{x}\|_2),$$

or

$$\limsup_{h\to 0+} \frac{V(\mathbf{x} + hA_i\mathbf{x}) - V(\mathbf{x})}{h} \leq -\alpha_3(\|\mathbf{x}\|_2)$$

for $i = 1, 2, \ldots, N$.

Denote by $\mathcal{S}_n$ the set of all symmetric $\mathbb{R}^{n\times n}$ matrices and denote the positive (semi)definiteness of P by ($P \succeq 0$) $P \succ 0$. Negative (semi)definiteness is analogously denoted ($P \preceq 0$) $P \prec 0$. It is well known that the existence of a QCLF for (1) is equivalent to the existence of a matrix $P \in \mathcal{S}_n$, such that

$$P \succ 0 \quad \text{and} \quad A_i^T P + PA_i \prec 0 \quad \text{for } i = 1, 2, \ldots, N. \tag{3}$$

III. CPA Lyapunov functions for arbitrarily switched linear systems

THE CPA method to compute Lyapunov functions attempts to parameterize a Lyapunov function using LP on a compact domain $\mathcal{D} \subset \mathbb{R}^n$ of the state-space of the system in question. Thus, first a triangulation $\mathcal{T}$ of the domain $\mathcal{D}$ is needed, i.e. a subdivision of $\mathcal{D}$ into simplices

$$\begin{aligned}\mathfrak{S}_\nu &= \mathrm{co}\{\mathbf{x}_0^\nu, \mathbf{x}_1, \ldots, \mathbf{x}_n^\nu\} \\ &:= \left\{\sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu \colon \sum_{i=0}^n \lambda_i = 1 \text{ and all } \lambda_i \geq 0\right\}.\end{aligned}$$

We write $\mathcal{D}_\mathcal{T}$ for the set-theoretic union of the simplices in $\mathcal{T}$ and say that $\mathcal{T}$ triangulates $\mathcal{D}_\mathcal{T} = \mathcal{D} \subset \mathbb{R}^n$. The triangulation must be shape-regular in the sense that two different simplices

$$\mathfrak{S}_\gamma := \mathrm{co}\{\mathbf{x}_0^\gamma, \mathbf{x}_1^\gamma, \ldots, \mathbf{x}_n^\gamma\}, \quad \gamma \in \{\nu, \mu\},$$

of the triangulation intersect in a common face

$$\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \mathrm{co}\{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_k\}, \quad \mathbf{y}_j = \mathbf{x}_{\ell_j^\nu}^\nu = \mathbf{x}_{\ell_j^\mu}^\mu,$$

where $j = 0, 1, \ldots, k < n$, $\ell_j^\nu, \ell_j^\mu \in \{0, 1, \ldots, n\}$, and $\ell_j^\gamma \neq \ell_m^\gamma$ if $j \neq m$, $\gamma \in \{\nu, \mu\}$. We are only interested in non-degenerated simplices, i.e. $\mathfrak{S}_\nu \in \mathcal{T}$ has an n-dimensional volume strictly larger than zero or equivalently, the vertices $\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu$ are affinely independent; see [18] for details.

Particular to our problem of parameterizing CPA Lyapunov functions for (1), where each subsystem is linear, is that we want a triangulation $\mathcal{T}$, such that $\mathcal{D}_\mathcal{T}$ is a neighbourhood of the origin and each simplex $\mathfrak{S}_\nu \in \mathcal{T}$ has the origin as a vertex. The advantage is that for linear systems a CPA Lyapunov function, defined on a neighbourhood of the origin, can easily be extended to a Lyapunov function on the entire $\mathbb{R}^n$. Concrete instances of such triangulations will be constructed in the next section.

A. The Triangulation $\mathcal{T}_K^{\mathbf{F}}$

A suitable concrete triangulation for our aim of parameterizing common Lyapunov functions for the system (1) is the triangular-fan of the triangulation in [18], where its efficient implementation is also discussed. In its definition we use the functions $\mathbf{R}^\mathcal{J} : \mathbb{R}^n \to \mathbb{R}^n$, defined for every $\mathcal{J} \subset \{1, 2, \ldots, n\}$ by

$$\mathbf{R}^\mathcal{J}(\mathbf{x}) := \sum_{i=1}^n (-1)^{\chi_\mathcal{J}(i)} x_i \mathbf{e}_i, \quad \chi_\mathcal{J}(i) := \begin{cases} 1, & \text{if } i \in \mathcal{J}, \\ 0, & \text{if } i \notin \mathcal{J}. \end{cases}$$

where $\mathbf{e}_i$ is the standard ith unit vector in $\mathbb{R}^n$. Thus, $\mathbf{R}^\mathcal{J}(\mathbf{x})$ is the vector $\mathbf{x}$, except for a minus has been put in front of the coordinate x_i whenever $i \in \mathcal{J}$.

We first define the triangulation $\mathcal{T}^{\mathrm{std}}$ and use it to construct the intermediate triangulation $\mathcal{T}_K$, which in turn is used to define our desired triangulation $\mathcal{T}_K^{\mathbf{F}}$.

The standard triangulation $\mathcal{T}^{\mathrm{std}}$ consists of the simplices

$$\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \mathrm{co}\left\{\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right\},$$

where

$$\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma} := \mathbf{R}^\mathcal{J}\left(\mathbf{z} + \sum_{i=1}^j \mathbf{e}_{\sigma(i)}\right), \tag{4}$$

for all $\mathbf{z} \in \mathbb{N}_0^n = \{0, 1, \ldots\}$, all $\mathcal{J} \subset \{1, 2, \ldots, n\}$, all $\sigma \in S_n$, and $j = 0, 1, \ldots, n$; S_n denotes the set of all permutations of $\{1, 2, \ldots, n\}$ and $\mathbf{e}_{\sigma(i)}$, $j = \sigma(i)$, the standard jth unit vector.

Now fix a $K \in \mathbb{N}_+ = \{1, 2, \ldots\}$ and define the hypercube $\mathcal{H}_K := [-K, K]^n$. Consider the simplices $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} \subset \mathcal{H}_K$ in $\mathcal{T}^{\mathrm{std}}$, that intersect the boundary of $\mathcal{H}_K$. We are only interested in those intersections that are $(n-1)$-simplices, i.e. we take every simplex with vertices $\mathbf{x}_j := \mathbf{R}^\mathcal{J}\left(\mathbf{z} + \sum_{i=1}^j \mathbf{e}_{\sigma(i)}\right)$, $j \in \{0, 1, \ldots, n\}$, where exactly one vertex $\mathbf{x}_{j^*}$ satisfies $\|\mathbf{x}_{j^*}\|_\infty < K$ and the other n of the $n+1$ vertices satisfy

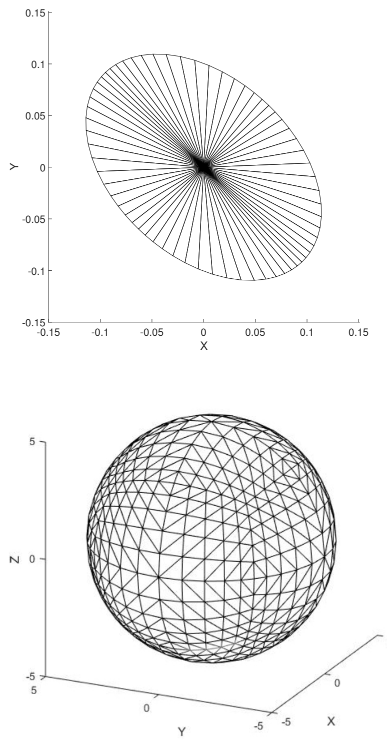

Fig. 1. The triangulation $\mathcal{T}_{\mathbf{8}}^{\mathbf{F}}$ in two dimensions, where the vertices of the triangulation have additionally been mapped with a matrix $\boldsymbol{R} \in \boldsymbol{\mathcal{S}}_{\mathbf{2}}$, $\boldsymbol{R} \succ \mathbf{0}$ (above), and $\mathcal{T}_{\mathbf{5}}^{\mathbf{F}}$ in three dimension (below); note that the origin is a vertex of all the triangles in $\mathcal{T}_{\mathbf{8}}^{\mathbf{F}}$ and all the tetrahedra in $\mathcal{T}_{\mathbf{5}}^{\mathbf{F}}$.

$\|\mathbf{x}_j\|_\infty = K$, i.e. for $j \in \{0, 1, \dots, n\} \setminus \{j^*\}$. Then we replace the vertex $\mathbf{x}_{j^*}$ by $\mathbf{0}$; it is not difficult to see that j^* is necessarily equal to 0. The collection of such vertices triangulates $\mathcal{H}_K$ and this new triangulation of $\mathcal{H}_K$ is our desired triangulation $\mathcal{T}_K$.

It has been shown [3] that it is often advantageous in the CPA method to map the vertices of the triangulation by the mapping $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and

$$\mathbf{F}(\mathbf{x}) := \frac{\|\mathbf{x}\|_2}{\|\mathbf{x}\|_\infty}\mathbf{x}, \quad \text{if } \mathbf{x} \neq \mathbf{0}. \tag{5}$$

Note that $\mathbf{F}$ maps the hypercubes $\{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x}\|_\infty = r\}$ to the spheres $\{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x}\|_2 = r\}$.

Finally, we define the triangulation $\mathcal{T}_K^{\mathbf{F}}$ we will use in the LP problem to parameterize CPA Lyapunov functions. Let $\mathcal{T}_K^{\mathbf{F}}$ to be the triangulation consisting of the simplices

$$\mathfrak{S}_\nu := \operatorname{co}\{\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \dots, \mathbf{x}_n^\nu\}, \quad \mathbf{x}_i^\nu = \mathbf{F}(\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}),$$

where

$$\operatorname{co}\left\{\mathbf{0}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma}, \dots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right\} \in \mathcal{T}_K.$$

Subsequently the vertices of the simplices in the new triangulation $\mathcal{T}_K^{\mathbf{F}}$ can be mapped $\mathbf{x} \mapsto R\mathbf{x}$, where $R \in \mathbb{R}^{n \times n}$ is a nonsingular matrix; in practice usually $R \succ 0$. This is studied in some detail in [21]. Figure 1 depicts two exemplary triangulations of the type $\mathcal{T}_K^{\mathbf{F}}$ for two and three dimension. In our approach for the switched system (1), however, we will not use such a mapping $\mathbf{x} \mapsto R\mathbf{x}$, $R \succ 0$, of the vertices to better adapt the triangulation to the problem, but instead use a coordinate transform for the system to adapt the problem better to the triangulation, see Section III-D below.

B. LP Problem to Parameterize CPA Lyapunov functions

We are now ready to state our LP problem to parameterize a CPA common Lyapunov function for system (1). Note that it is a feasibility problem, i.e. any feasible solution can be used to parameterize a CPA Lyapunov functions.

We use two constants $\varepsilon_1, \varepsilon_2 > 0$ in the LP problem. In theory both can w.l.o.g. be set equal to one, as explained below. In practice different values can be useful.

The variables of the LP problem are $V_{\mathbf{x}} \in \mathbb{R}$ for every vertex of a simplex in $\mathcal{T}_K^{\mathbf{F}}$.

The constraints of the LP problem are:

C1) The first set of constraints are $V_{\mathbf{0}} = 0$ and for every vertex $\mathbf{x}$ of a simplex in $\mathcal{T}_K^{\mathbf{F}}$:

$$V_{\mathbf{x}} \geq \varepsilon_1 \|\mathbf{x}\|_2 \tag{6}$$

C2) The second set of constraints is more involved. For every simplex $\mathfrak{S}_\nu := \operatorname{co}\{\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu \dots, \mathbf{x}_n^\nu\} \in \mathcal{T}_K^{\mathbf{F}}$ we define the matrix $X_\nu = (\mathbf{x}_1^\nu \; \mathbf{x}_2^\nu \; \cdots \mathbf{x}_n^\nu)$, i.e. $\mathbf{x}_k^\nu$ is the kth column of X_ν. Further, we define the vector of variables $\mathbf{v}_\nu = \left(V_{\mathbf{x}_1^\nu} \; V_{\mathbf{x}_2^\nu} \; \cdots \; V_{\mathbf{x}_n^\nu}\right)^T$.
The constraints are: for every simplex $\mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}$, for all $j = 0, 1, \dots, n$ and all $i = 1, 2, \dots, N$:

$$\mathbf{v}_\nu^T X_\nu^{-1} A_i \mathbf{x}_j^\nu \leq -\varepsilon_2 \|\mathbf{x}_j^\nu\|_2. \tag{7}$$

Note that by multiplying the variables $V_{\mathbf{x}}$ of a feasible solution by $(\min\{\varepsilon_1, \varepsilon_2\})^{-1}$ the inequalities (6) and (7) are fulfilled with $\varepsilon_1 = \varepsilon_2 = 1$ on the right-hand-side. Further, there either exists a feasible solution for all pairs of constants $\varepsilon_1, \varepsilon_2 > 0$ or for none; i.e. their numerical values are only an implementation issue.

C. Feasible Solution delivers a CPA Lyapunov functions

Assume the LP problem in Section III-B has a feasible solution. We then define the CPA function $V\colon \mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}} \to \mathbb{R}$ in the following way: For every $\mathbf{x} \in \mathcal{D}_{\mathcal{T}_K^{\mathbf{F}}}$ there exists a simplex $\mathfrak{S}_\nu = \operatorname{co}\{\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu \dots, \mathbf{x}_n^\nu\} \in \mathcal{T}_K^{\mathbf{F}}$ such that $\mathbf{x} \in \mathfrak{S}_\nu$ and there exist a unique $\lambda \in [0,1]^n$, $\sum_{j=1}^n \lambda_j \leq 1$, such that $\mathbf{x} = \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu$. We define $V(\mathbf{x}) = \sum_{j=1}^n \lambda_j V_{\mathbf{x}_j^\nu}$. It is not difficult to see that V is a continuous function that is linear on each simplex $\mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}$, in particular it has the constant gradient $\nabla V_\nu := \mathbf{v}_\nu^T X_\nu^{-1}$ (row vector) on the interior of $\mathfrak{S}_\nu$, see e.g. [19, Rem. 9]. Hence, for any $\mathbf{x} \in \mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}$, $\mathbf{x} = \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu$, we have for any $i = 1, 2, \dots, N$ by C2 and the convexity of the norm that

$$\begin{aligned}
\nabla V_\nu \bullet A_i \mathbf{x} &= \mathbf{v}_\nu^T X_\nu^{-1} A_i \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu = \sum_{j=1}^n \lambda_j \mathbf{v}_\nu^T X_\nu^{-1} A_i \mathbf{x}_j^\nu \\
&\leq -\varepsilon_2 \sum_{j=1}^n \lambda_j \|\mathbf{x}_j^\nu\|_2 \leq -\varepsilon_2 \Big\| \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu \Big\|_2 \\
&= -\varepsilon_2 \|\mathbf{x}\|_2.
\end{aligned}$$

Now, for any $\mathbf{x} \in \mathcal{D}^{\circ}_{\mathcal{T}^{\mathbf{F}}_K}$ we have that for any $i = 1, 2, \ldots, N$ there exists a simplex $\mathfrak{S}_\nu \in \mathcal{T}^{\mathbf{F}}_K$ and an $h > 0$, such that $\mathbf{x} + [0, h]A_i\mathbf{x} \subset \mathfrak{S}_\nu$; note that the ν depends on $\mathbf{x}$ and i. Because V is linear on $\mathfrak{S}_\nu$ we have

$$\limsup_{h \to 0+} \frac{V(\mathbf{x} + hA_i\mathbf{x}) - V(\mathbf{x})}{h} = \nabla V_\nu \bullet A_i\mathbf{x} \leq -\varepsilon_2 \|\mathbf{x}\|_2$$

and since this holds true for all $i = 1, 2, \ldots, N$ we have $D^+V(\mathbf{x}) \leq -\varepsilon_2\|\mathbf{x}\|_2$. Since

$$V(\mathbf{x}) = \sum_{j=1}^{n} \lambda_j V_{\mathbf{x}^\nu_j} \geq \varepsilon_1 \sum_{j=1}^{n} \lambda_j \|\mathbf{x}^\nu_j\|_2 \geq \varepsilon_1 \|\mathbf{x}\|_2$$

by the constraints C1, it is clear that V fulfills the conditions (2) for a Lyapunov function for the switched system (1) for every $\mathbf{x} \in \mathcal{D}^{\circ}_{\mathcal{T}^{\mathbf{F}}_K}$. By extending V to $\mathbb{R}^n$ in the obvious way, i.e. for every $\mathbf{x} \in \mathbb{R}^n$ there exists an $\mathfrak{S}_\nu$ and unique numbers $\lambda_i \geq 0$ such that $\mathbf{x} = \sum_{j=1}^{n} \lambda_j \mathbf{x}^\nu_j$ (a cone defined by the vertices of $\mathfrak{S}_\nu$) and we set $V(\mathbf{x}) = \sum_{j=1}^{n} \lambda_j V_{\mathbf{x}^\nu_j}$ as before, we see that V fulfills, with $\alpha_1(y) = \varepsilon_1 y$, $\alpha_2(y) = \max_{\|\mathbf{x}\|_2=1} V(\mathbf{x}) \cdot y$, and $\alpha_3(y) = \varepsilon_2 y$, the conditions (2) for a Lyapunov function for all $\mathbf{x} \in \mathbb{R}^n$.

D. Augmented LP approach

Note that sublevel sets of Lyapunov functions are forward invariant for the dynamics and their shape influences how many simplices we need in the LP problem. Hence, one can improve the approach by coordinate transforms that lead to Lyapunov functions with level sets that are closer to being hyperspheres and thus requiring fewer simplices and fewer constraints in the LP problem. Intuitively, a long and thin ellipsoid, say $x^2 + c^2y^2 = 1$ in the plane with $c^2 \gg 1$, has most of its structure in simplices with $y \approx 0$, whereas the structure of $x^2 + y^2 = 1$ is evenly distributed on the simplices of our triangulation $\mathcal{T}^{\mathbf{F}}_K$.

For a $P \in \mathcal{S}_n$, $P \succeq 0$, there exists for every $k \in \mathbb{N}$ a unique $Q \in \mathcal{S}_n$, $Q \succeq 0$, such that $Q^k = P$, see e.g. [25, Th. 7.2.6]. We define $P^{\frac{1}{k}} := Q$ and $P^{-\frac{1}{k}} := Q^{-1}$.

We first show that for one system we can find a particularly simple Lyapunov function for a suitably transformed system. Indeed, if $\mathbf{x} \mapsto \mathbf{x}^T P\mathbf{x}$ is a Lyapunov function for the system $\dot{\mathbf{x}} = A\mathbf{x}$, i.e. $A^TP + PA = -Q$ for a $Q \succ 0$, then $\mathbf{y} \mapsto \mathbf{y}^T\mathbf{y}$ is a Lyapunov function for the system $\dot{\mathbf{y}} = P^{\frac{1}{2}}AP^{-\frac{1}{2}}\mathbf{y}$ (coordinate transform $\mathbf{y} = P^{\frac{1}{2}}\mathbf{x}$); just note that

$$P^{\frac{1}{2}}\left(P^{-\frac{1}{2}}A^TP^{\frac{1}{2}} + P^{\frac{1}{2}}AP^{-\frac{1}{2}}\right)P^{\frac{1}{2}} = A^TP + PA = -Q$$

and $P^{-\frac{1}{2}}QP^{-\frac{1}{2}} \succ 0$. Thus, using the coordinate transform $\mathbf{y} = P^{\frac{1}{2}}\mathbf{x}$ delivers a particularly simple Lyapunov function for the transformed system.

Now we assume that we have found a Lyapunov function for the transformed system and show how to find a Lyapunov function for the original system: if $\mathbf{x} \mapsto \mathbf{x}^T\widetilde{P}\mathbf{x}$ is a Lyapunov function for the system $\dot{\mathbf{x}} = R^{\frac{1}{2}}AR^{-\frac{1}{2}}\mathbf{x}$, $R \in \mathcal{S}_n$ and $R \succ 0$, i.e.

$$R^{-\frac{1}{2}}A^TR^{\frac{1}{2}}\widetilde{P} + \widetilde{P}R^{\frac{1}{2}}AR^{-\frac{1}{2}} = -\widetilde{Q}, \quad \widetilde{Q} \succ 0,$$

then $\mathbf{x} \mapsto \mathbf{x}^TP\mathbf{x}$, $P = R^{\frac{1}{2}}\widetilde{P}R^{\frac{1}{2}}$, is a Lyapunov function for the system $\dot{\mathbf{x}} = A\mathbf{x}$ and

$$A^TP + PA = -R^{\frac{1}{2}}\widetilde{Q}R^{\frac{1}{2}}.$$

We will now apply these general ideas for switched systems with more than one linear system. In this case, it is not obvious which transformation will result in a simple Lyapunov function. In our augmented approach we first compute quadratic Lyapunov functions $\mathbf{x} \mapsto \mathbf{x}^TP_i\mathbf{x}$ for the individual subsystems $\dot{\mathbf{x}} = A_i\mathbf{x}$ and then use a weighted sum, w.l.o.g. a convex sum, i.e. $\lambda \in [0, 1]^N$ with $\sum_{i=1}^{N} \lambda_i = 1$, namely

$$R := \sum_{i=1}^{N} \lambda_i P_i \tag{8}$$

and attempt to compute a CLF for the switched system (1) with A_i replaced by $R^{\frac{1}{2}}A_iR^{-\frac{1}{2}}$, $i = 1, 2, \ldots, N$. Once we have computed a Lyapunov function for the transformed system, we can easily compute the corresponding Lyapunov function for the original system. The advantage of this approach will become clear in the example in the next section. However, we will also see that it is not obvious which convex combination of the P_is is optimal.

E. Example

Consider the system $\ddot{x} + 2\dot{x} + f(t)x = 0$ for a measurable function $f\colon [0, \infty) \to \mathbb{R}$ fulfilling $a \leq f(t) \leq b$ for $a, b \in \mathbb{R}$. Solutions are understood in the sense of Carathéodory, see e.g. [46, §10,Supp. II]. Its state-space form is

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -f(t) & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{9}$$

and the asymptotic stability of $(x_1, x_2) = (x, \dot{x}) = (0, 0)$ for the system can be asserted by showing the stability of the origin for system (1) with $N = 2$,

$$A_1 = \begin{pmatrix} 0 & 1 \\ -a & -2 \end{pmatrix}, \text{ and } A_2 = \begin{pmatrix} 0 & 1 \\ -b & -2 \end{pmatrix}. \tag{10}$$

The reason for this is that this arbitrarily switched system can by the Filippov-Ważewski Theorem approximate any solution trajectory of the differential inclusion $\dot{\mathbf{x}} \in \mathrm{co}\{A_1\mathbf{x}, A_2\mathbf{x}\}$ and a solution to (9) is a solution to the differential inclusion, see e.g. [4], [16]. System trajectories of $\dot{\mathbf{x}} = A_1\mathbf{x}$ and $\dot{\mathbf{x}} = A_2\mathbf{x}$ are shown in Figure 2 for different values for the constants a and b. The case with fixed $a = 0.01$ was studied in [37] and it was shown that stability can be affirmed for $b = 11.3$, in comparison to $b = 4.3$ using the circle criterion, a generalization of the Nyquist criterion, see [13]. In [13] it is additionally stated that stability can be guaranteed for $b = 11.6$ using variational analysis. Further, it is shown in [39] that the origin is unstable for $b \geq 12.5$.

We implemented the LMI approach for computing a QCLF as

$$\begin{cases} P - \varepsilon I \succeq 0 \\ A_i^TP + PA_i + \varepsilon I \preceq 0 \quad \text{for } i = 1, 2, \end{cases} \tag{11}$$

with $\varepsilon = 10^{-5}$. Since semidefinite solvers sometimes erroneously report feasibility, we verify (3) for the computed

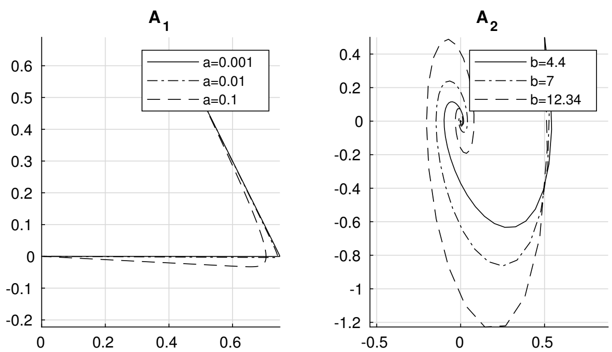

Fig. 2. Trajectories for $\dot{\mathbf{x}} = \boldsymbol{A_1}\mathbf{x}$ for different $\boldsymbol{a}$ (left) and for $\dot{\mathbf{x}} = \boldsymbol{A_2}\mathbf{x}$ for different $\boldsymbol{b}$ (right). The matrices $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ come from (10) and define the subsystems for the arbitrarily switched system (1) used to investigate the stability of the origin for system (9).

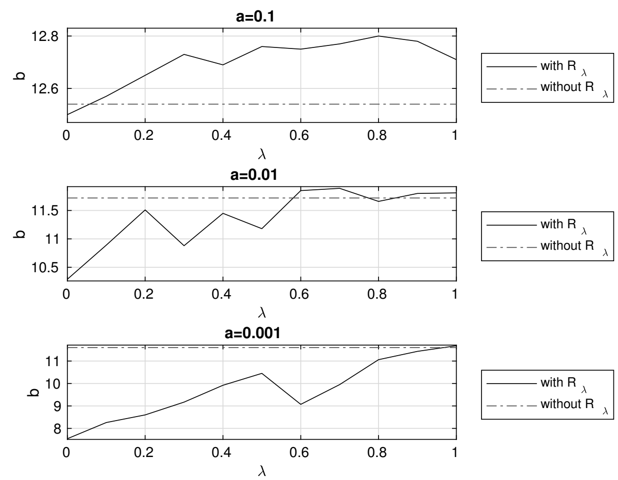

Fig. 3. The optimal $\boldsymbol{b}$ as a function of $\boldsymbol{\lambda}$ with $\mathcal{T}^{\mathbf{F}}_{50}$ for $\boldsymbol{a} = \mathbf{0.1, 0.01,}$ and $\mathbf{0.001}$

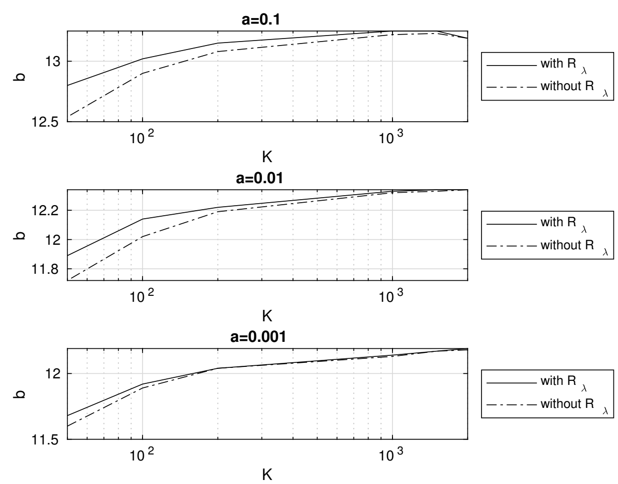

Fig. 4. The optimal $\boldsymbol{b}$ as a function of $\boldsymbol{K}$ in $\mathcal{T}^{\mathbf{F}}_{\boldsymbol{K}}$, both using the LP approach and the augmented LP approach, for $\boldsymbol{a} = \mathbf{0.1, 0.01,}$ and $\mathbf{0.001}$.

solution. Using the LMI approach the best results we obtained with $a = 0.01$ was $b = 4.40$; for higher values of b we either did not get a solution or the reported solution did not fulfill (3). We used the LMI solver sdpt3 [44] implemented in Matlab with YALMIP [29], with a subsequent verification of the inequalities. We additionally used the solvers Mosek [1] and SeDuMi [42] and got the same results.

Using the LP approach from Section III-B with the triangulation $\mathcal{T}^{\mathbf{F}}_{50}$ we obtained with $a = 0.01$ the value $b = 11.72$, a better result than $b = 11.3$ in [37] or $b = 11.6$ reported in [13], and a much better result than $b = 4.4$ obtained with LMI.

Let us compare these results with the augmented LP approach, outlined in Section III-D. First, we define

$$
\begin{aligned}
R_\lambda &= (1-\lambda)\,P_1 + \lambda P_2, \\
\text{where} \quad A_i^T P_i + P_i A_i &= -I, \quad i = 1, 2.
\end{aligned}
$$

Then we attempt to compute a common CPA Lyapunov function for the switched system (1) with A_i from (10) replaced by $R_\lambda^{\frac{1}{2}} A_i R_\lambda^{-\frac{1}{2}}$, $i = 1, 2$. We compared the results for the values $\lambda = 0, 0.1, \ldots, 1$ for the parameter $\lambda \in [0, 1]$, see Figure 3 (middle). The highest value obtained was $b = 11.89$ with $\lambda = 0.7$ (recall that $a = 0.01$).

We also studied the influence of having more triangles in the triangulation $\mathcal{T}^{\mathbf{F}}_K$, i.e. $K = 50, 100, 200, 1000, 1500, 2000$, see Figure 4 (middle). For each K we first used the LP approach from Section III-B (without R_λ) and subsequently the augmented LP approach from Section III-D (with R_λ), where we put the best value obtained with $\lambda = 0, 0.1, \ldots, 1$ in the graph. The best value obtained was $b = 12.34$, either with $K = 2000$ and no coordinate transform, or with $K = 1500$ and the coordinate transform with $\lambda = 0.9$. Note that in particular for lower K, the results are notably better when using the coordinate transform. However, the choice of the optimal values for λ is far from transparent, see Figure 3 (middle).

We repeated all these computations with $a = 0.01$ and $a = 0.1$, see Figure 3 (top, bottom) and Figure 4 (top, bottom). Using the LMI approach we obtained $b = 4.12$ and $b = 5.36$ for $a = 0.001$ and $a = 0.1$ respectively. With $a = 0.1$ we obtained $b = 13.23$ with the LP method and $K = 2000$ and with the augmented LP we obtained $b = 13.25$, either with $\lambda = 0$ and $K = 1500$ or $\lambda = 0.8$ and $K = 1000$. With $a = 0.001$ we obtained $b = 12.18$ with the LP method and $K = 2000$ and with the augmented LP we obtained $b = 12.19$ with $\lambda = 1$ and $K = 2000$. Some of these results are compiled in Table I. Thus, again our new methods compare very favorably to the LMI approach, but the choice of the optimal λ parameter is not transparent. Indeed, for $a = 0.1$ the value of the optimal b is not even a monotonically increasing function of K. The dependence of the optimal b as a function of λ and K remains an open question and will be investigated in the future.

IV. Conclusions

We presented a linear programming (LP) algorithm to compute a common continuous and piecewise affine (CPA) Lyapunov function for the arbitrarily switched system (1). It is an adaptation of the CPA method to compute Lyapunov functions to (1). Further, we presented an adapted linear programming (LP) approach, where a coordinate transform using quadratic Lyapunov functions for the individual subsystems of the switched system (1) is used to improve the LP approach.

TABLE I

THE BEST RESULTS FOR $\boldsymbol{b}$ FOR $\boldsymbol{a}$ FROM DIFFERENT APPROACHES.

a	LMI	LP	Aug LP	Circ. cri.	[13]	[37]
0.1	5.36	13.23	13.25			
0.01	4.40	12.34	12.34	4.3	11.6	11.3
0.001	4.12	12.18	12.19			

We compare our novel methods to different approaches in the literature for an example. In particular, we compare our methods to the usual quadratic common Lyapunov function (QCLF) computed by linear matrix inequalities (LMIs). In all cases our new methods compare favorably.

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