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Computation of complete Lyapunov functions for three-dimensional systems

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Abstract— **Complete Lyapunov functions are of much interest in control theory because of their capability to describe the long-time behaviour of nonlinear dynamical systems. The state-space of a system can be divided in two different regions determined by a complete Lyapunov function: the region of the gradient-like flow, where the Lyapunov function is strictly decreasing along solution trajectories, and the chain-recurrent set whose chain-transitive components are level sets of the Lyapunov function. There has been continuous effort to properly identify both regions and in this paper we discuss the extension of our methods to compute complete Lyapunov functions in the plane to the three-dimensional case, which is directly applicable to higher dimensions, too. When extending the methods to higher dimensions, the number of points for collocation and evaluation grows exponentially. To keep the number of evaluation points under control, we propose a new way to choose them, which does not depend on the dimension.**

I. INTRODUCTION

Throughout this paper, we describe the application of our algorithms to compute complete Lyapunov functions for a system expressed by a general autonomous ordinary differential equation (ODE)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$; in this paper, we choose $n = 3$.

In 1893, Lyapunov [7] obtained results concerning the stability of an equilibrium using only the differential equation itself and without solving it. The centrepiece of his approach was an auxiliary scalar-valued function defined on a subset of the state-space. Such a function is usually referred to as a *Lyapunov function* in his honour. Classically, a (strict) Lyapunov function [8] is decreasing along all solution trajectories in a neighbourhood of an attractor such as an equilibrium or a periodic orbit. The function attains its minimum at the attractor and is strictly decreasing along solutions of the ODE in its neighbourhood and therefore solutions that start close to the attractor are attracted to it. A complete Lyapunov function [4], [5], [10], [11] is a generalization of this idea.

Definition 1.1: We say a point ξ belongs to the *chain-recurrent set*, if an ε-trajectory through it comes back to ξ after any given time, see [4] for more details.

Definition 1.2: A *complete Lyapunov function* is a scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}$ that is defined on the whole phase space. It is non-increasing along all solutions and strictly decreasing outside of the chain-recurrent set.

As a consequence of its definition, a complete Lyapunov function characterizes the absolute behaviour of the dynamical system on the whole phase space and not just in a neighbourhood of one particular attractor. By means of its broader information, the state-space can be divided into two disjoint areas, on which the system behaves in fundamentally different ways: where the system flows through, being strictly non-increasing and the regions in which is constant. A complete Lyapunov function is strictly decreasing where the flow is gradient-like and constant on each chain transitive component of the chain-recurrent set.

For a dynamical system defined on a compact metric space Conley proved the existence of complete Lyapunov functions [4] and Hurley [11] extended this results to separable metric spaces.

There are many methods to analyze the general behaviour of dynamical systems. Among them we can point out techniques like the direct simulation of solutions with many different initial conditions, computation of invariant manifolds which form the boundaries of attractors' basins of attraction [13] (these techniques require an additional analysis of the part with gradient-like form) and the cell mapping approach [9] or set oriented methods [6] that work by dividing the phase space into different cells and study the dynamics between them [14]. However, all these methods require large computational effort.

Our algorithm in this paper follows a method to compute classical Lyapunov functions for an equilibrium using Radial Basis Functions [15]. We aim to approximately solve $V'(\mathbf{x}) = -1$, where

$$V'(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \tag{2}$$

is the derivative along solutions of the ODE, i.e. the orbital derivative. To approximately solve this partial differential equation (PDE), we employ mesh-free collocation techniques with Radial Basis Functions: we choose a finite set of collocation points X and compute an approximation v to V which solves the PDE at all collocation points. This technique is further explained in [1], [2], [15]. The discretized problem is well-posed and possesses a unique solution. However, the computed function v cannot fulfill the PDE at all points of the chain-recurrent set, such as an equilibrium or periodic orbit. The basic idea is to use the area where v fails to fulfill the PDE to localize the chain-recurrent set. Furthermore, our method has the advantage of not being overly computationally demanding.

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II. ALGORITHM

We base our method to compute complete Lyapunov functions on the algorithms described in [1], [3], [2]. In this paper, however, we extend the methodology from two- to three- and higher-dimensional systems.

Like in the methodology introduced in [2], we also normalize the right-hand side of the ODE. This means that the original dynamical system (1) gets substituted by,

$$\dot{\mathbf{x}} = \hat{\mathbf{f}}(\mathbf{x}), \quad \text{where} \quad \hat{\mathbf{f}}(\mathbf{x}) = \frac{\mathbf{f}(\mathbf{x})}{\sqrt{\delta^2 + \|\mathbf{f}(\mathbf{x})\|^2}} \tag{3}$$

with parameter $\delta > 0$ and Euclidean norm $\|\cdot\|$. In [2] there is a long discussion on why this normalization was introduced. To sum it up, the systems (1) and (3) have the same trajectories but the speed of the solutions is nearly uniform for (3), which makes it easier to localize the chain-recurrent sets. In the following we assume that (1) is already in the form (3).

A. MESH-FREE COLLOCATION

Mesh-free collocation methods, e.g. based on Radial Basis Functions, constitute a strong tool for solving generalized interpolation problems [15]. There are many examples of Radial Basis Functions, i.e., Gaussians, multiquadrics and Wendland functions. Solving PDEs with mesh-free collocation enables us to use locally a high resolution of collocation points since scattered points can be added to improve the approximation. These methods deliver smooth functions that approximate the true solution and the methods work in any dimension.

In this paper we use Wendland functions because they are compactly supported [18] and have been intensively studied for computing Lyapunov functions. They have the form $\psi(\mathbf{x}) := \psi_{l,k}(c\|\mathbf{x}\|)$, where $c > 0$, $k \in \mathbb{N}$ is a smoothness parameter and $l = \lfloor \frac{n}{2} \rfloor + k + 1$.

The Wendland functions are positive definite functions that are polynomials on their compact support; the corresponding Reproducing Kernel Hilbert Space is norm-equivalent to the Sobolev space $W_2^{k+(n+1)/2}(\mathbb{R}^n)$. Note that in this context a continuous function $\Psi\colon \mathbb{R}^n \to \mathbb{R}$ is positive definite if for all sets of pairwise distinct centers $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{R}^n$ the quadratic form

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \alpha_k \Psi(\mathbf{x}_j - \mathbf{x}_k)$$

is positive for all $\alpha \in \mathbb{R}^N \setminus \{\mathbf{0}\}$, see [17, Def. 6.1]. The Wendland functions are defined by recursion: for $l \in \mathbb{N}$ and $k \in \mathbb{N}_0$ we define

$$\begin{aligned} \psi_{l,0}(r) &= (1-r)_+^l, \\ \psi_{l,k+1}(r) &= \textstyle\int_r^1 t\,\psi_{l,k}(t)\mathrm{d}t \end{aligned} \tag{4}$$

for $r \in \mathbb{R}_0^+$, where $x_+ = x$ for $x \geq 0$ and $x_+ = 0$ for $x < 0$.

To build our collocation points $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{R}^n$, we use a subset of a hexagonal grid with fineness-parameter $\alpha_{\text{Hexa-basis}} \in \mathbb{R}^+$ which is constructed according to the next equation:

$$\left\{ \alpha_{\text{Hexa-basis}} \sum_{k=1}^{n} i_k \omega_k : i_k \in \mathbb{Z} \right\}, \text{ where} \tag{5}$$

$$\begin{aligned} \omega_1 &= (2\varepsilon_1, 0, 0, \ldots, 0) \\ \omega_2 &= (\varepsilon_1, 3\varepsilon_2, 0, \ldots, 0) \\ &\vdots \quad\quad \vdots \\ \omega_n &= (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, (n+1)\varepsilon_n) \text{ and} \\ \varepsilon_k &= \sqrt{\frac{1}{2k(k+1)}}, \quad k \in \mathbb{N}. \end{aligned}$$

The hexagonal grid is optimal to balance the opposing aims of a dense grid in order to achieve a small error and a large separation distance of collocation points so that the condition number of the collocation matrices is not overly large [12]. In Figures 1, 4, 7, this grid is shown with red dots.

In our construction, we have to remove all equilibria of the system to be studied from the set of the collocation points X, i.e. points $\mathbf{x}$ where $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The approximation v is then given by the function that satisfies the PDE at all collocation points with minimal norm in the corresponding Reproducing Kernel Hilbert space. Practically, we find v by solving a system of N linear equations, where N is the number of collocation points.

To determine the chain-recurrent set, we evaluate for each collocation point whether the approximation was good or poor, i.e. whether the approximation v satisfies the PDE. Since at each collocation point, the PDE is satisfied by construction, we build a special evaluation grid $Y_{\mathbf{x}_j}$ around each collocation point $\mathbf{x}_j$, given by

$$Y_{\mathbf{x}_j} = \left\{ \mathbf{x}_j \pm \frac{r \cdot k \cdot \alpha_{\text{Hexa-basis}} \cdot \mathbf{f}(\mathbf{x}_j)}{m \cdot \|\mathbf{f}(\mathbf{x}_j)\|} : k \in \{1, \ldots, m\} \right\} \tag{6}$$

where $\alpha_{\text{Hexa-basis}}$ is the parameter used to build the hexagonal grid defined above, $r \in (0,1)$ is a given ratio up to which the evaluation points will be placed and controls their overlapping. If we want to avoid overlapping of evaluation points then we need to set $r < 1$. The number $m \in \mathbb{N}$ denotes the number of points in the evaluation grid that will be placed on both sides of the collocation points aligned to the flow. Altogether, will have $2m$ points per collocation point. Note that by equation (6) the evaluation points corresponding to a collocation point are aligned with the direction of the flow which is given by the normal vector $\frac{\mathbf{f}(\mathbf{x}_j)}{\|\mathbf{f}(\mathbf{x}_j)\|}$. This grid is chosen to avoid an exponential growth of evaluation points for it will always be a grid in one dimension regardless of the dimension of the problem. Since we know that a complete Lyapunov function V is non-increasing along the solutions of the ODE, we start by approximating the solution of $V'(\mathbf{x}) = -1$ by v, using the collocation points X. Extending the idea carried out in two dimensions in [1], [2], now for three-dimensional systems, we also define a tolerance parameter $-1 < \gamma \leq 0$, and mark a collocation point $\mathbf{x}_j$ to be

poorly approximated, i.e., an element of our approximation of the chain-recurrent set ($\mathbf{x}_j \in X^0$), if there is at least one point $\mathbf{y} \in Y_{\mathbf{x}_j}$ such that $v'(\mathbf{y}) > \gamma$. The well approximated points, i.e., for which the condition $v'(\mathbf{y}) \leq \gamma$ holds for all $\mathbf{y} \in Y_{\mathbf{x}_j}$ belong to our approximation of the area of the gradient-like flow ($\mathbf{x}_j \in X^-$).

Since we know that the PDE $V'(\mathbf{x}) = -1$ has no solution on the chain-recurrent set, in the next step of our recursive algorithm we use the information obtained in the previous step to improve our estimate of the right-hand side: using the previous approximation v_i, we compute the next approximation v_{i+1} by solving the PDE

$$V'(\mathbf{x}_j) = r_j := \left(\frac{1}{2m} \sum_{\mathbf{y} \in Y_{\mathbf{x}_j}} v_i'(\mathbf{y}) \right)_- , \tag{7}$$

where $x_- = x$ if $x \leq 0$ and $x_- = 0$ otherwise. Note that the method only requires us to know the right-hand side of the PDE at the collocation points. Summarizing, in subsequent iterations, the new right-hand side value is obtained by averaging the values around each collocation point and bounding by 0.

B. Methodology

Our new algorithm to compute complete Lyapunov function in three and higher dimensions can be summarized as follows:

1) Fix the set of collocation points X in $\mathbb{R}^n$ and compute the approximate solution v_0 to $V'(\mathbf{x}) = -1$; set $i = 0$.
2) To approximate the chain-recurrent set by X^0, for each collocation point $\mathbf{x}_j$, compute $v_i'(\mathbf{y})$ for all $\mathbf{y} \in Y_{\mathbf{x}_j}$, see (6). If $v_i'(\mathbf{y}) > \gamma$ for any $\mathbf{y} \in Y_{\mathbf{x}_j}$, then $\mathbf{x}_j \in X^0$, else $\mathbf{x}_j \in X^-$, where $-1 < \gamma \leq 0$ is a chosen critical value
3) Define $r_j = \left(\frac{1}{2m} \sum_{\mathbf{y} \in Y_{\mathbf{x}_j}} v_i'(\mathbf{y}) \right)_-$
4) Compute the approximate solution v_{i+1} of $V'(\mathbf{x}_j) = r_j$ for $j = 1, \ldots, N$
5) Set $i \to i+1$ and repeat steps 2 to 4 until no more points are added to X^0

C. Results

Along the results, we have marked as iteration 0 where we obtain the first Lyapunov function with the approximate solution v_0 of $V'(\mathbf{x}) = -1$ (just step 1). In our results we analyse several benchmark systems. In the first example no iterations are needed as v_0 already delivers excellent results. For the other two examples, iterations improve the results. For all systems, we have replaced $\mathbf{f}$ by $\hat{\mathbf{f}}$, see (3), with $\delta = 10^{-8}$.

1) Attractive to origin: Let us consider the system given by

$$\mathbf{f}(x,y,z) = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}. \tag{8}$$

This simple system has only one equilibrium, the origin, in the chain recurrent set. All the flux of this system converges to the stable equilibrium at the origin. We apply our method using the hexagonal collocation points grid from equation (5), with $\alpha_{\text{Hexa-basis}} = 0.08$ and a fixed ratio for the directional grid of $r = 0.495$. To compute a Lyapunov function for this system, we use Wendland functions as Radial Basis Functions with their lowest possible parameters in three dimensions, $l = 4$ and $k = 2$, as well as $c = 1$. The Lyapunov function was computed on $[-1,1]^3 \subset \mathbb{R}^3$. The directional grid has a total of 20 points for each collocation point ($m = 10$). The critical value used for this problem was $\gamma = -0.5$.

Fig. 1 shows the collocation and evaluation points and Fig. 2 displays the level sets of the Lyapunov function for system (8). It is intuitively showing the *path* to the origin.

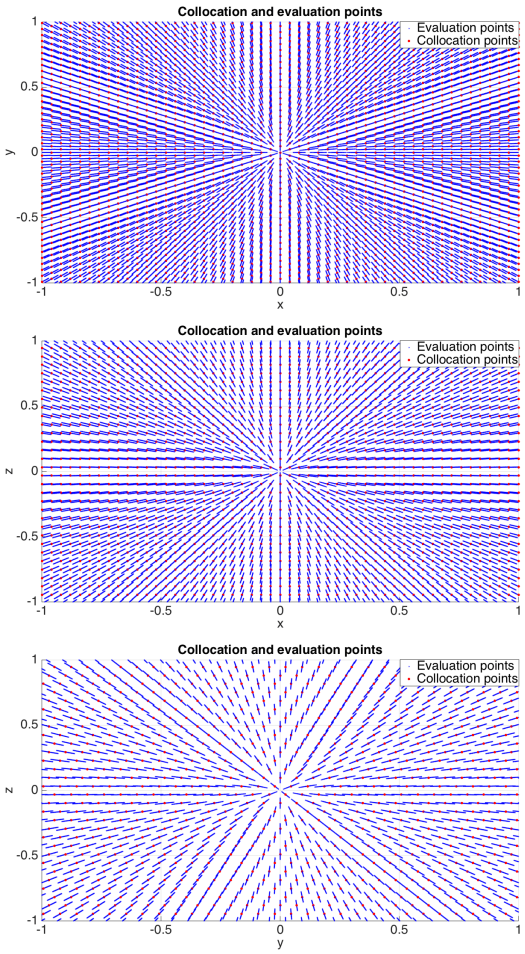

Fig. 1. Collocation and evaluation grids for system (8). First, second and third figures show the XY, XZ and YZ planes, respectively.

The collocation points closest to the origin are failing the orbital derivative conditions and approximate the chain-recurrent set, see Fig. 3.

2) Periodic orbit I: The following system given by

$$\mathbf{f}(x,y,z) = \begin{pmatrix} -x\left(x^2+y^2-1\right) - y \\ -y\left(x^2+y^2-1\right) + x \\ -z \end{pmatrix} \tag{9}$$

has been previously studied in [16]. This is a system whose chain-recurrent set is given by the attractive periodic orbit

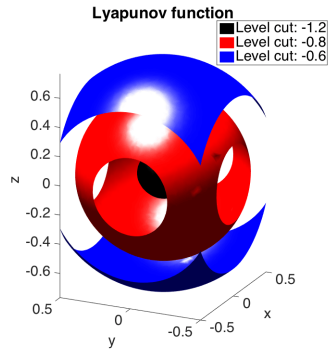

Fig. 2. Level sets of the complete Lyapunov function for system (8).

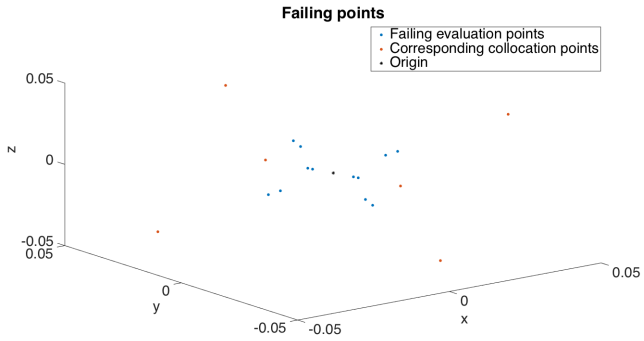

Fig. 3. Chain-recurrent set for system (8). The red points represent the collocation points. The blue points represent the evaluation points to the directional grid. The dark point in the centre of the image shows the origin.

$\Omega = \{(x,y,0) : x^2+y^2 = 1\}$ and an unstable equilibrium at the origin. For our method we use a hexagonal collocation points grid, with $\alpha_{\text{Hexa-basis}} = 0.09$ and a fixed ratio for the directional grid of $r = 0.495$. To compute the Lyapunov function for this system, we use Wendland functions as Radial Basis Functions, again with parameters $l = 4$, $k = 2$, and $c = 1$.

The Lyapunov function was computed on the domain $[-1,1]^2 \times [-1.5,1.5] \subset \mathbb{R}^3$. For the directional grid we used $m = 10$. The critical value used for this problem was $\gamma = -0.25$. In Fig. 4 we depict the directional evaluation grid; in this figure $\alpha_{\text{Hexa-basis}} = 0.108$ to make it easier to see the flux. The flow is circular towards the z-axis.

Fig. 5 shows the chain-recurrent set, which approximates the periodic orbit well already after step 1 (upper figure). After 11 iterations of the method as described in Section II-B, we also cover the unstable equilibrium, see Fig. 5, lower figure.

Fig. 6 shows a level set of the Lyapunov function and our approximation of the chain-recurrent set.

3) Periodic orbit II: In the last example we choose the system given by

$$\mathbf{f}(x,y,z) = \begin{pmatrix} x\left(1-x^2-y^2\right)(x+1/2)-y; \\ y\left(1-x^2-y^2\right)(x+1/2)+x \\ 10z(x+0.55)\left[(x-0.7)^2+0.1\right] \end{pmatrix}. \qquad (10)$$

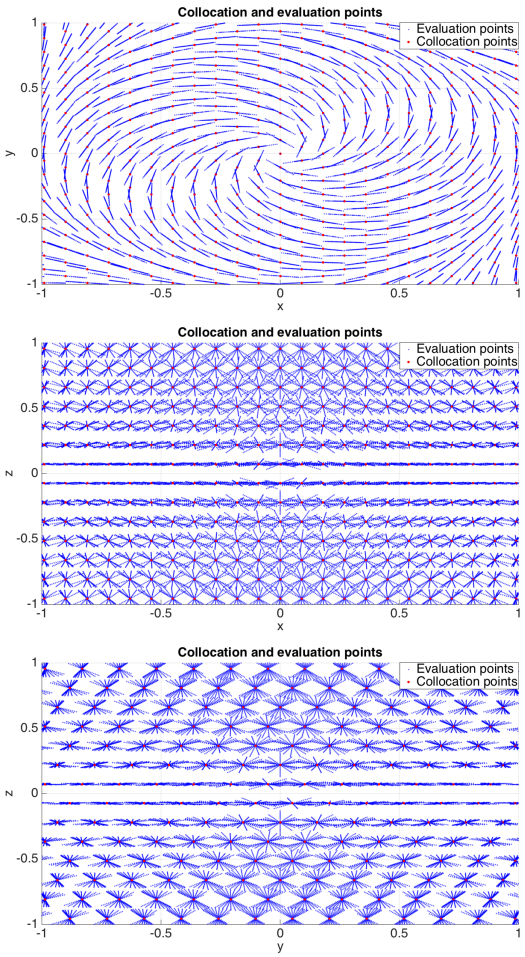

Fig. 4. Collocation and evaluation grids for system (9). First, second and third figures show the XY, XZ and YZ planes, respectively.

The system has an unstable equilibrium at the origin and an exponentially stable periodic orbit $\Omega = \{(x,y,0) : x^2+y^2 = 1\}$, however, the attraction to the periodic orbit is not uniform along the periodic orbit with respect to the Euclidean norm, as can be seen in Fig. 7.

In our computations we used $\alpha_{\text{Hexa-basis}} = 0.108$ in the domain $[-1.5,1.5]^3 \subset \mathbb{R}^3$. The parameters for the Wendland Radial Basis Functions are $l = 4$, $k = 2$, and $c = 1$. The evaluation grid was constructed with $m = 10$. In Fig. 7 we show the collocation and evaluation grid; using $\alpha_{\text{Hexa-basis}} = 0.216$ so that the flow can be seen more clearly.

Fig. 8 and Fig. 9 show some results after 13 iterations for system (10). The method adds points to the approximation of the chain-recurrent set up to when it is possible to see the whole circumference of the periodic orbit. Fig. 10 shows a level set of the Lyapunov function and the approximation of the chain-recurrent set after the 13 iteration.

III. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

Our methodology has proven efficient to compute complete Lyapunov functions and to properly localize the chain-recurrent sets. We had already established that our method

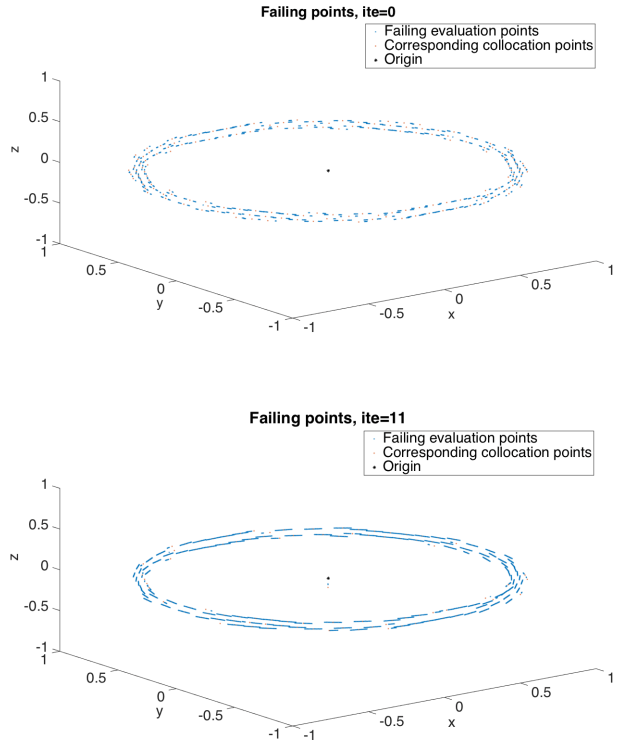

Fig. 5. Approximations of the chain-recurrent set for system (9). The red dots represent the collocation points. The blue dots represent the evaluation points of the directional grid. Finally, the dark point in the centre of the image shows the origin. The first figure shows the results after step 1, covering the periodic orbit, while the second figure shows the results after the 11th iteration, also covering the equilibrium at the origin.

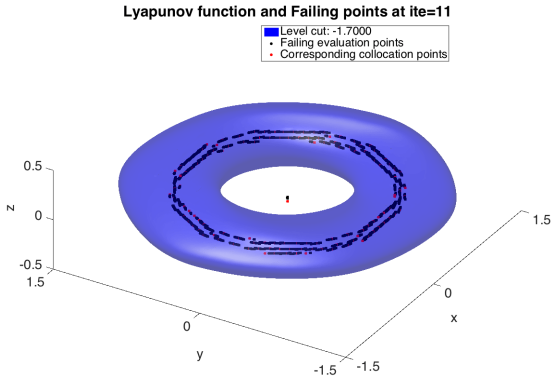

Fig. 6. The approximation of the chain-recurrent set and a level set of the complete Lyapunov function for system (9).

works in two dimensions and have extended it to three dimensions in this paper. For the extension to higher dimensions we have introduced a directional evaluation grid that follows the direction of the flow. It uses a fixed number of evaluation points per collocation point in the one-dimensional direction of the flow and thus does not increase the number of evaluation points depending on the dimension. This enables us to tackle higher-dimensional problems more efficiently.

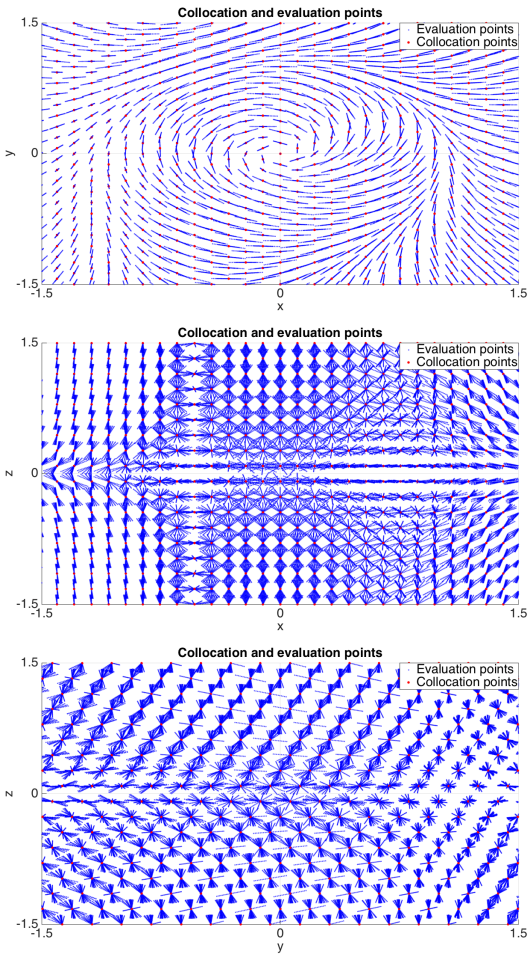

Fig. 7. Collocation and evaluation grids for system (10). First, second and third figures show the XY, XZ and YZ planes, respectively.

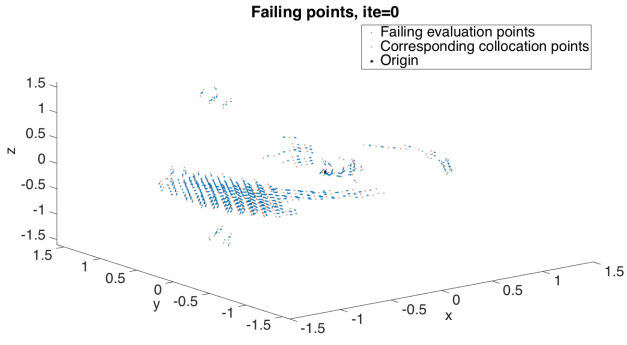

Fig. 8. The approximation of the chain-recurrent set and their corresponding collocation points for system (10) after step 1.

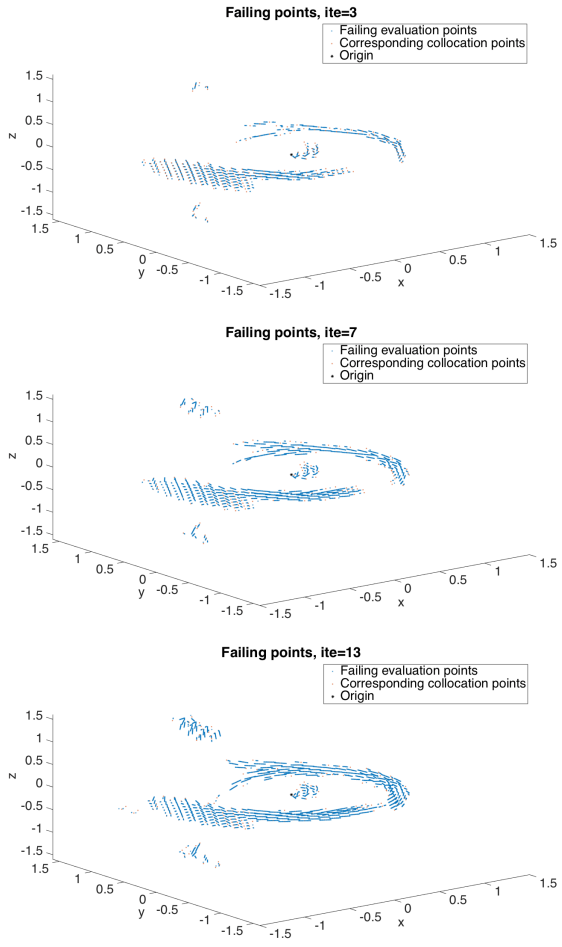

Fig. 9. Third, seventh and thirteenth iterations for system (10) with the approach described in Section II-B.

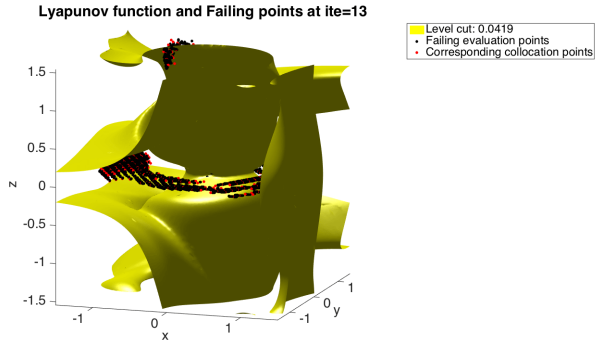

Fig. 10. The approximation of the chain-recurrent set and a level set of the complete Lyapunov function for system (10).

B. Future Work

The current algorithm adds points to the approximation of the chain-recurrent set in each iterative step. In future work it would be desirable to modify the algorithm so that it could also remove points from the approximation of the chain-recurrent set as, e.g. as seen in Fig. 9, it can overestimate the chain-recurrent set.

IV. ACKNOWLEDGMENTS

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