# Computational Approach for Complete Lyapunov Functions

**Carlos Argáez, Peter Giesl and Sigurdur Freyr Hafstein**

**Abstract** Ordinary differential equations arise in a variety of applications, including climate modeling, electronics, predator-prey modeling, etc., and they can exhibit highly complicated dynamical behaviour. Complete Lyapunov functions capture this behaviour by dividing the phase space into two disjoint sets: the chain-recurrent part and the transient part. If a complete Lyapunov function is known for a dynamical system the qualitative behaviour of the system's solutions is transparent to a large degree. The computation of a complete Lyapunov function for a given system is, however, a very hard task. We present significant improvements of an algorithm recently suggested by the authors to compute complete Lyapunov functions. Previously this methodology was incapable to fully detect chain-recurrent sets in dynamical systems with high differences in speed. In the new approach we replace the system under consideration with another one having the same solution trajectories but such that they are traversed at a more uniform speed. The qualitative properties of the new system such as attractors and repellers are the same as for the original one. This approach gives a better approximation to the chain-recurrent set of the system under study.

**Keywords** Complete Lyapunov Function · Dynamical Systems
Lyapunov theory · Meshless collocation · Radial Basis Functions

C. Argáez (✉) · S. F. Hafstein
Science Institute, University of Iceland, Dunhagi 5, 107,
Reykjavík, Iceland
e-mail: carlos@hi.is

S. F. Hafstein
e-mail: shafstein@hi.is

P. Giesl
Department of Mathematics, University of Sussex,
Falmer BN1 9QH, UK
e-mail: P.A.Giesl@sussex.ac.uk

 
J. Awrejcewicz (ed.), *Dynamical Systems in Theoretical Perspective*,
Springer Proceedings in Mathematics & Statistics 248,
https://doi.org/10.1007/978-3-319-96598-7_1

## 1 Introduction

Let us consider a general autonomous ordinary differential equation (ODE) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$. A (classical) Lyapunov function [1] is a scalar-valued function defined in a neighborhood of an invariant set. It is built to show the stability of such a set and can be used to analyse its basin of attraction. Hence, it is linked to one attractor, e.g. an equilibrium or a periodic orbit. In particular, a (strict) Lyapunov function attains its minimum on the attractor and is strictly decreasing along solutions of the ODE.

This idea is generalized to a complete Lyapunov function [2–5], which completely characterizes the behaviour of the dynamical system in the whole phase space.

> A complete Lyapunov function is a scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}$ which is defined not only on a neighbourhood of one attractor but in the whole phase space under the condition of being non-increasing along solutions of the ODE.

The phase space can be divided into the area where the complete Lyapunov function strictly decreases along solution trajectories and the area where it is constant along solution trajectories. If the complete Lyapunov function is sufficiently smooth, these properties can be expressed by the orbital derivative $V'(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$, i.e. the derivative along solutions of the ODE. The first area, where $V'(\mathbf{x}) < 0$, characterizes the region where solutions pass through and the larger this area is, the more information is obtained from the complete Lyapunov function. The second area, where $V'(\mathbf{x}) = 0$, includes the chain-recurrent set; the complete Lyapunov function is constant on each transitive component of the chain-recurrent set. In short, the first one determines where solutions pass through while the second accounts for determining the long-time behaviour.

Dynamical systems model real-world systems and describe their often complicated behaviour, e.g. the double [6] and triple pendulum with periodic forcing [7] and dry friction [8], leading to time-periodic and non-smooth systems, or the dynamics of the wobblestone [9]. There are many methods to analyse the qualitative behaviour of a given dynamical systems: one of them directly simulates solutions with many different initial conditions. This becomes very expensive and unable to provide general information on the behaviour of a given system, unless estimates are available, e.g. when shadowing solutions. More sophisticated methods include invariant manifolds and their computation, which form boundaries of basins of attraction for the attractors [10]. The cell mapping approach [11] or set oriented methods [12] divide the phase space into cells and compute the dynamics between them, see e.g. [13]. These ideas have been used for a computational approach to construct complete Lyapunov functions [14], where the authors consider the discrete system given by the time-$T$ map, divide the phase space into cells and compute the dynamics between them through an induced multivalued map. This is done with the computer package GAIO [15]. Then, using graphs algorithms, an approximate complete Lyapunov function is computed [16]. However, even for low dimensions, a high number of cells is required to compute the Lyapunov function under this approach. We will use a different methodology, significantly improving the method described in [17].

Our new approach follows from a method to compute classical Lyapunov functions for a given equilibrium by approximating the solution to $V'(\mathbf{x}) = -1$, i.e. the orbital derivative. We approximate the solution of this partial differential equation (PDE) by means of mesh-free collocation with Radial Basis Functions: over a finite set of collocation points $X$, we compute an approximation $v$ to $V$ that solves the PDE in all collocation points.

At points of the chain-recurrent set, such as an equilibrium or periodic orbit, the PDE does not have a solution; the numerical method, however, always has one. The idea is to use the area $F$, where the approximation is poor, to approximate the chain-recurrent set. Following the fact that a complete Lyapunov function should be constant in the chain-recurrent set, in the next step, we solve the PDE $V'(\mathbf{x}) = 0$ for $\mathbf{x} \in F$ and $V'(\mathbf{x}) = -1$ elsewhere.

For the numerical method we thus split the collocation points $X$ into a set $X^0 = X \cap F$, where the approximation is poor, and $X^- = X \setminus X^0$, where it works correctly. Then we solve the PDE $V'(\mathbf{x}) = 0$ for all $\mathbf{x} \in X^0$ and $V'(\mathbf{x}) = -1$ for all $\mathbf{x} \in X^-$.

As a result, the approximated function $v$ gives us information about the solution to the ODE under consideration. On the one hand, the set $X^0$ where $v'(\mathbf{x}) \approx 0$ approximates the chain-recurrent set, including equilibria, periodic orbits and homoclinic orbits, and on the other hand, the set $X^-$ in which $v'(\mathbf{x}) \approx -1$ approximates the part where the flow is gradient-like. Information about the stability and attraction properties is obtained through the level sets of the function $v$: minima of $v$ correspond to attractors while maxima represent repellers. For more details of the method see [17].

In this paper we significantly improve the method from [17], described above. Firstly, the method in [17] was not able to accurately identify the chain-recurrent set in more complicated examples, in particular examples where the speed $\|\mathbf{f}(\mathbf{x})\|$ with which solutions of the ODE are passed through varies considerably. Hence, in this paper we replace the original system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with the system

$$\dot{\mathbf{x}} = \hat{\mathbf{f}}(\mathbf{x}), \quad \text{where} \quad \hat{\mathbf{f}}(\mathbf{x}) = \frac{\mathbf{f}(\mathbf{x})}{\sqrt{\delta^2 + \|\mathbf{f}(\mathbf{x})\|^2}} \tag{1}$$

with parameter $\delta > 0$.

The new system has the same solution trajectories as the original system, but these are traversed at a more uniform speed, namely $\|\hat{\mathbf{f}}(\mathbf{x})\| = \frac{\|\mathbf{f}(\mathbf{x})\|}{\sqrt{\delta^2+\|\mathbf{f}(\mathbf{x})\|^2}} \approx 1$. The smaller $\delta$ is, the closer the speed is to 1.

This modification improves the ability of the method to find the chain-recurrent set significantly, as we will show in the paper.

Secondly, the function $V$ satisfying $V'(\mathbf{x}) = 0$ for $\mathbf{x} \in F$ and $V'(\mathbf{x}) = -1$ elsewhere is not smooth due to the jump in the orbital derivative, while the error estimates in mesh-free collocation require the solution of the PDE to be smooth. To overcome this problem, we propose to replace the discontinuous right-hand side function by a smooth function.

Let us give an overview of the paper: In Sect. 2 we present the method with the modified system (1) and show the improvements over the previous method from [17] in three examples. Section 3 studies the dependence on the parameter $\delta$. Section 4 discusses replacing the discontinuous right-hand side by a smooth function and applies the improved method to the same three examples before ending with conclusions in Sect. 5.

## 2 Normalized Speed

As discussed above, we fix a parameter $\delta > 0$ and consider the modified system (1) with normalized speed. We fix a finite set of collocation points $X$, none of which is an equilibrium point for the system. For our examples we used a subset of the hexagonal grid

$$\alpha_{\text{Hexa-basis}} \left\{ k \begin{pmatrix} 1 \\ 0 \end{pmatrix} + l/2 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \, : \, k, l \in \mathbb{Z} \right\}$$

with parameter $\alpha_{\text{Hexa-basis}} > 0$. We approximate the solution of the PDE $V'(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \hat{\mathbf{f}}(\mathbf{x}) = -1$ using mesh-free collocation with the kernel $\Phi(\mathbf{x}) := \psi_{l,k}(c\|\mathbf{x}\|)$ given by the Wendland function $\psi_{l,k}$ and parameter $c > 0$, for details see [17, 18]. We denote the approximation by $v$.

To identify the collocation points where the approximation is poor, indicating the chain-recurrent set, we evaluate $v'(\mathbf{x})$ near each collocation point – note that in the collocation point the orbital derivative is $-1$ by construction. In particular, in $\mathbb{R}^2$, for a given collocation point $\mathbf{x}_j$, we build a set of points $Y_{\mathbf{x}_j}$ placed in two spheres with center $\mathbf{x}_j$, namely:

$$Y_{\mathbf{x}_j} = \{\mathbf{x}_j + r\alpha_{\text{Hexa-basis}}(\cos(\theta), \sin(\theta)) \, : \, \theta \in \{0, 2\pi/32, 4\pi/32, 6\pi/32, \ldots, 2\pi\}\} \tag{2}$$

$$\cup \{\mathbf{x}_j + \frac{r}{2}\alpha_{\text{Hexa-basis}}(\cos(\theta), \sin(\theta)) \, : \, \theta \in \{0, 2\pi/32, 4\pi/32, 6\pi/32, \ldots, 2\pi\}\} \tag{3}$$

where $r > 0$ is a parameter and $\alpha_{\text{Hexa-basis}}$ is the parameter used to build the hexagonal grid defined above. We define a tolerance parameter $\gamma > -1$ and mark a collocation point $\mathbf{x}_j$ as being in the chain-recurrent set ($\mathbf{x}_j \in X^0$) if there is at least one point $\mathbf{y} \in Y_{\mathbf{x}_j}$ such that $v'(\mathbf{y}) > \gamma$.

We will now present the method applied to three systems with different properties; these are the same systems as in [17] so that we can compare the two methods.

### 2.1 Attractive and Repelling Periodic Orbits

The dynamical system given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{f}(x, y) = \begin{cases} -x(x^2 + y^2 - 1/4)(x^2 + y^2 - 1) - y \\ -y(x^2 + y^2 - 1/4)(x^2 + y^2 - 1) + x \end{cases} \tag{4}$$

has two periodic orbits and an equilibrium. The equilibrium at the origin is asymptotically stable, and so is the periodic orbit with radius 1, while the periodic orbit with radius $1/2$ is unstable.

We used a hexagonal grid with $\alpha_{\text{Hexa-basis}} = 0.02$ in the set $[-1.5, 1.5]^2 \subset \mathbb{R}^2$ which gives a total of 29,440 collocation points, the Wendland function with parameters $(l, k, c) = (5, 3, 1)$, the critical value $\gamma = -0.5$, and $\delta^2 = 10^{-8}$. Furthermore, for the evaluation grid we set $r = 0.5$. We have compared the new method (normalized, right-hand side) with the non-normalized method of [17] (left-hand side), see Fig. 1.

In the lower right figure in Fig. 1, we can see that the equilibrium at the origin is found with less error than in the lower left figure where there are more points around $(0, 0)$. The chain-recurrent set actually looks very well-defined in both cases because of the relatively simple dynamics.

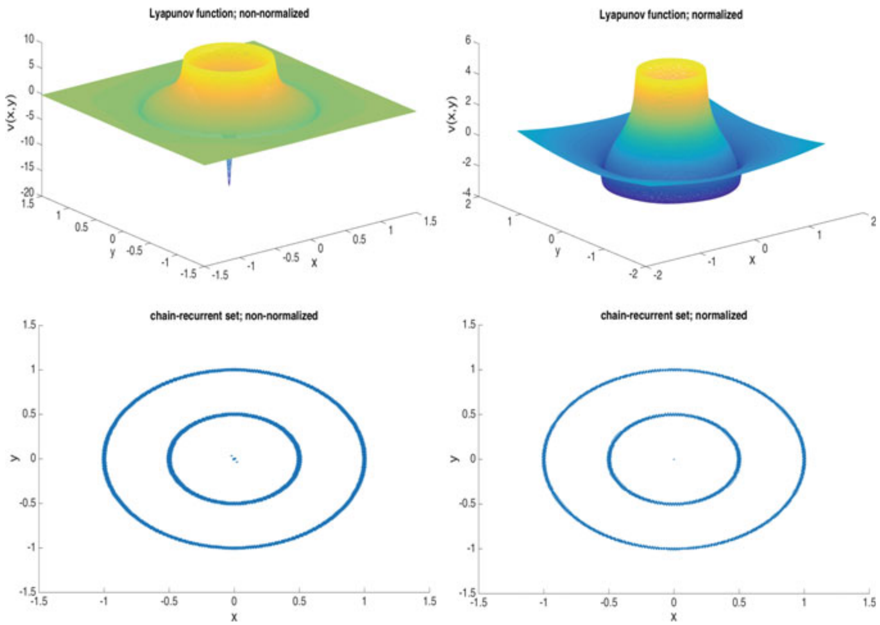


**Fig. 1** Lyapunov functions for system (4) under both non-normalized (upper left) and for the normalized approach (upper right). Chain-recurrent set for both systems non-normalized (lower left) and normalized (lower right)

## 2.2 Van der Pol Oscillator

System (5) is the two-dimensional form of the Van der Pol oscillator. The system has an asymptotically stable periodic orbit and an unstable equilibrium at the origin.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{f}(x, y) = \begin{cases} y \\ (1 - x^2)y - x \end{cases} \tag{5}$$

We have a hexagonal grid with $\alpha_{\text{Hexa-basis}} = 0.1$ in the set $[-4.0, 4.0]^2 \subset \mathbb{R}^2$ which gives a total of 7708 collocation points, the Wendland function with parameters $(l, k, c) = (4, 2, 1)$, the critical value $\gamma = -0.5$, and $\delta^2 = 10^{-8}$. As before we set $r = 0.5$ in the evaluation grid. We have compared the new method (normalized) with the non-normalized method of [17], see Fig. 2.

The improvement of the proposed method can be seen clearly in the lower figures in Fig. 2: the chain-recurrent set is much better detected in the normalized system.

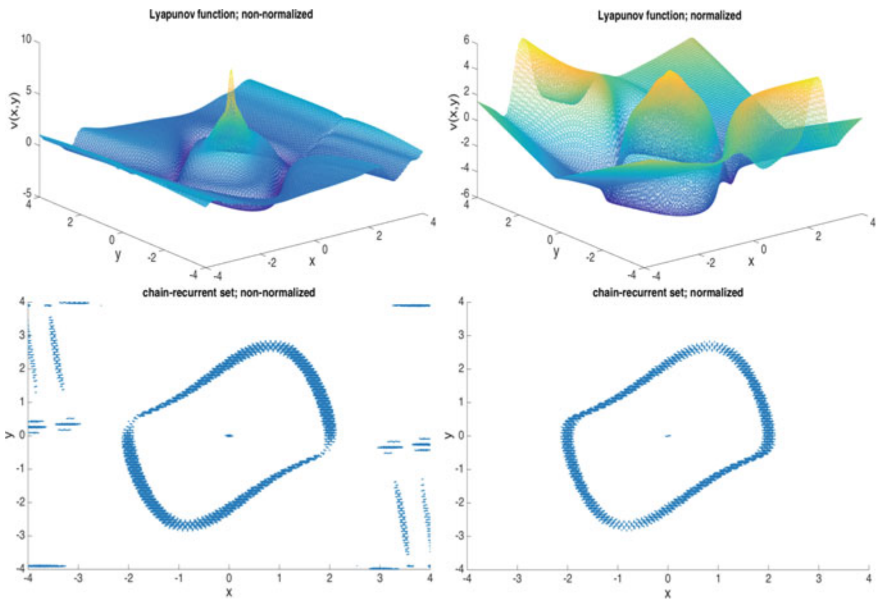


**Fig. 2** Lyapunov functions for system (5) under both non-normalized (upper left) and for the normalized approach (upper right). Chain-recurrent set for both systems non-normalized (lower left) and normalized (lower right)

## 2.3 Homoclinic Orbit

The system (6) has an asymptotically stable homoclinic orbit and an unstable equilibrium at the origin.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{f}(x, y) = \begin{cases} x(1 - x^2 - y^2) - y((x-1)^2 + (x^2 + y^2 - 1)^2) \\ y(1 - x^2 - y^2) + x((x-1)^2 + (x^2 + y^2 - 1)^2) \end{cases} \tag{6}$$

We used a hexagonal grid with $\alpha_{\text{Hexa-basis}} = 0.02$ in the set $[-1.5, 1.5]^2 \subset \mathbb{R}^2$ which gives a total of 29,440 collocation points, the Wendland function with parameters $(l, k, c) = (4, 2, 1)$, the critical value $\gamma = -0.75$, and $\delta^2 = 10^{-8}$. Again we used $r = 0.5$ in the evaluation grid. The new method (normalized) is compared with the non-normalized method of [17] in Fig. 3.

In this case, we can see a clear enhancement on the detection of the chain-recurrent set. In Fig. 3 (lower left) the failing set over-estimates the chain-recurrent set, while in Fig. 3 (lower right) the normalized method detects the chain-recurrent set much better.

Summarizing, the new method is able to better detect chain-recurrent sets.

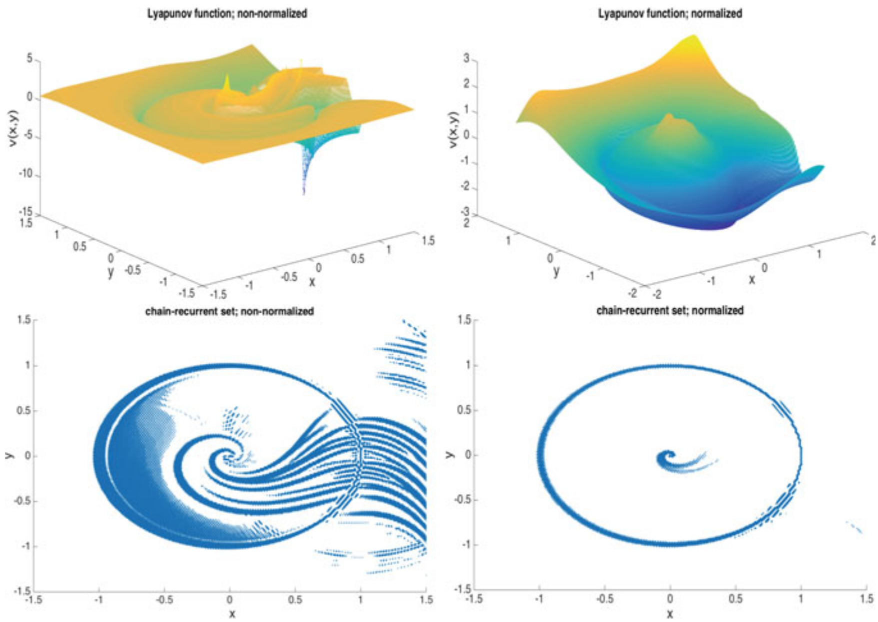


**Fig. 3** Lyapunov functions for system (6) under both non-normalized (upper left) and for the normalized approach (upper right). Chain-recurrent set for both systems non-normalized (lower left) and normalized (lower right)

# 3 Behaviour of the Lyapunov Functions Depending on the Values of $\delta$

Using the system defined in Sect. 2.1 by Eq. (4), we show the dependence of the behaviour of the Lyapunov function for a normalized system with different parameters $\delta$. We have chosen to show examples for $\delta^2 = 10^{-10}$ and $\delta^2 = 1$. Figure 4 shows how the Lyapunov function changes with different values of $\delta^2$: for small $\delta^2$ (black) the function has a derivative close to 0 around the equilibrium point, while for large $\delta^2$ (red) the function has a steep slope. Since Eq. (1) leads to the PDE

$$\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = -\sqrt{\delta^2 + \|\mathbf{f}(\mathbf{x})\|^2},$$

near the equilibrium the right-hand side is $\approx -\delta$. Hence, the gradient of $V$ must become large because $\mathbf{f}(\mathbf{x})$ is small close to the equilibrium.

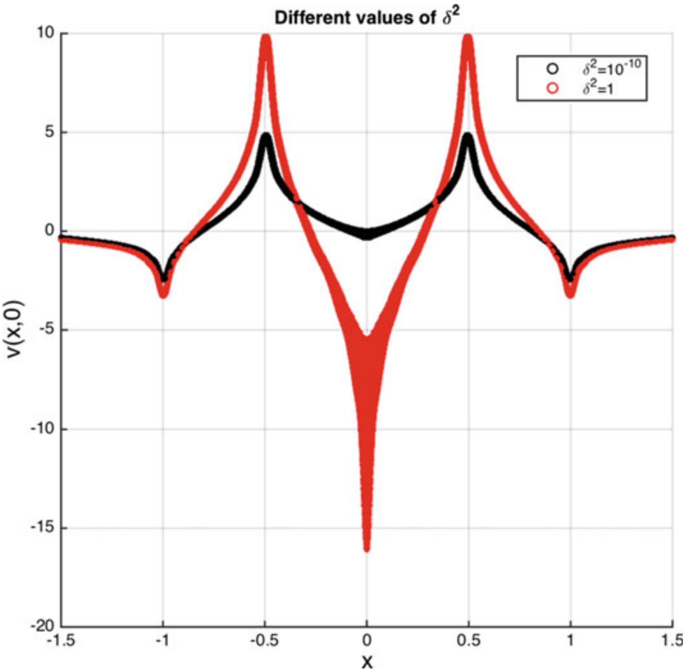


**Fig. 4** Lyapunov function for system (4) around the equilibrium point. With $\delta^2 = 1$ the gradient of $V$ is much larger close to the equilibrium at zero than with $\delta^2 = 10^{-10}$

## 4 Smooth Function

Our second main objective is in the next step to find a PDE which has a smooth solution and, subsequently, approximate its solution numerically.

The method from [17] starts with the PDE $V'(\mathbf{x}) = -1$, which does not have a solution on chain-recurrent sets; for an equilibrium $\mathbf{x}_0$, e.g. we clearly have $V'(\mathbf{x}_0) = 0$. By using mesh-free collocation to approximate a solution of $V'(\mathbf{x}) = -1$ we obtain an approximation $v$ which satisfies $v'(\mathbf{x}) \approx -1$ in areas which are not chain-recurrent and results in a poor approximation in the chain-recurrent set. Let us denote the area where the approximation is poor by $F$.

In the method described in [17] we then study the PDE

$$V'(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in F, \\ -1 & \text{if } \mathbf{x} \notin F. \end{cases}$$

As the right-hand side is discontinuous, the solution $V$ will not be a smooth function.

We assume that $F$ is a compact set and improve the method by considering the following PDE with smooth right-hand side

$$V'(\mathbf{x}) = r(\mathbf{x}) := \begin{cases} 0 & \text{if } \mathbf{x} \in F, \\ -\exp\left(-\frac{1}{\xi \cdot \mathfrak{d}^2(\mathbf{x})}\right) & \text{if } \mathbf{x} \notin F, \end{cases} \tag{7}$$

where $\mathfrak{d}(\mathbf{x}) = \min_{\mathbf{y} \in F} \|\mathbf{x} - \mathbf{y}\|$ is the distance between the point $\mathbf{x}$ and the set $F$ and $\xi > 0$ is a parameter.

To implement the method numerically, we construct the approximation to the complete Lyapunov function with our new approach. We first normalize our system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ by replacing it with the system (1). Note that we only need to evaluate the right-hand side $r(\mathbf{x})$ at the collocation points. Recall that we identify a collocation point $\mathbf{x}_j$ to be in an area of poor approximation $F$, as described above, if there exists at least one $\mathbf{y} \in Y_{\mathbf{x}_j}$ with $v'(\mathbf{y}) > \gamma$. Then we split the set of collocation points $X$ into the subset $X^0$ consisting of points in an area of poor approximation and the remaining points $X^- = X \setminus X^0$.

For all collocation points $\mathbf{x}_j \in X$ we then approximate the distance of $\mathbf{x}$ to the set $F$, represented by $X^0$, by

$$\mathfrak{d}(\mathbf{x}_j) \approx \min_{\mathbf{y} \in X^0} \|\mathbf{x}_j - \mathbf{y}\|;$$

note that $\mathfrak{d}(\mathbf{x}_j) = 0$ for all $\mathbf{x}_j \in X^0$.

Now, the right-hand side $r(\mathbf{x})$ of the Eq. (7) at a collocation point $\mathbf{x}_j \in X$ is set to be $r(\mathbf{x}_j) = 0$ if $\mathbf{x}_j \in X^0$, and $r(\mathbf{x}_j) = \exp\left(-\frac{1}{\xi \cdot \mathfrak{d}^2(\mathbf{x}_j)}\right)$ if $\mathbf{x}_j \in X^-$.

For our test systems (4), (5) and (6) we have already shown the normalized Lyapunov functions in Figs. 1, 2 and 3, respectively, so now we show the solution of

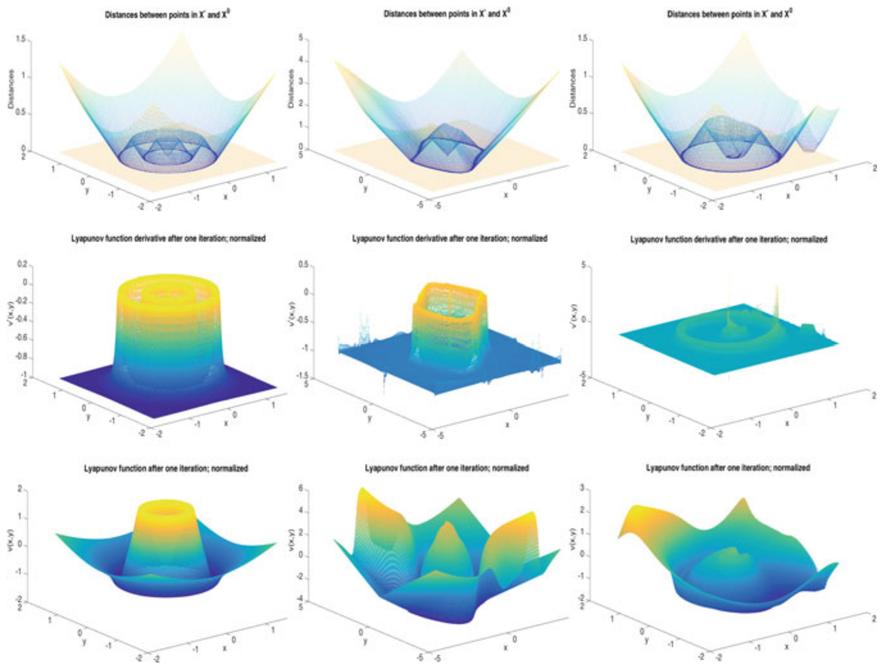


**Fig. 5** First row: values of $\mathfrak{d}$ as a function of the collocation points for systems (4) in column 1, (5) in column 2 and (6) in column 3, respectively. Second and third row: Lyapunov functions (third row) and their derivatives (second row) for systems (4) in column 1, (5) in column 2 and (6) in column 3 respectively, with the modified, smooth right-hand side

(7) as described above in Fig. 5. In this case, for all computations in Fig. 5, the normalization factor used is $\delta = 10^{-8}$ with $\xi = 300$. The second row shows that the orbital derivatives of the approximated functions are smooth functions.

## 5 Conclusions

In this paper we have significantly improved a method to construct complete Lyapunov functions and determine the chain-recurrent set. The two main improvements were firstly to consider a system with normalized speed, which enabled us to detect the chain-recurrent set more accurately. Secondly, we have replaced the discontinuous right-hand side of the PDE under consideration by a smooth function so that the PDE has a smooth solution, which is well approximated by the proposed method.

**Acknowledgements** The first author in this paper is supported by the Icelandic Research Fund (Rannís) grant number 163074-052, Complete Lyapunov functions: Efficient numerical computation. Special thanks to Dr. Jean-Claude Berthet for all his good comments and advice on C++.

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