DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B

doi:10.3934/dcdsb.2019080

COMPUTATION OF THE STOCHASTIC BASIN OF ATTRACTION BY RIGOROUS CONSTRUCTION OF A LYAPUNOV FUNCTION

Hjörtur Björnsson and Sigurdur Hafstein

Faculty of Physical Sciences
University of Iceland
107 Reykjavik, Iceland

Peter Giesl and Enrico Scalas

Department of Mathematics
University of Sussex
Falmer BN1 9QH, United Kingdom

Skuli Gudmundsson

Svensk Exportkredit
Klarabergsviadukten 61-63
111 64 Stockholm, Sweden

(Communicated by Xiaoying Han)

Abstract. The γ-basin of attraction of the zero solution of a nonlinear stochastic differential equation can be determined through a pair of a local and a non-local Lyapunov function. In this paper, we construct a non-local Lyapunov function by solving a second-order PDE using meshless collocation. We provide a-posteriori error estimates which guarantee that the constructed function is indeed a non-local Lyapunov function. Combining this method with the computation of a local Lyapunov function for the linearisation around an equilibrium of the stochastic differential equation in question, a problem which is much more manageable than computing a Lyapunov function in a large area containing the equilibrium, we provide a rigorous estimate of the stochastic γ-basin of attraction of the equilibrium.

1. **Introduction.** In deterministic dynamical systems given by autonomous ordinary differential equations (ODE), the basin of attraction of an asymptotically stable equilibrium is the set of all initial conditions, such that the corresponding solutions converge to the equilibrium as time tends to infinity. When considering a stochastic differential equation (SDE), this notion can be replaced by the γ-basin of attraction, i.e. the set of all initial conditions, such that sample paths will converge to the equilibrium as time tends to infinity with probability at least γ. This concept will be defined in Section 2, Definition 2.2.

2010 *Mathematics Subject Classification.* Primary: 37B25, 65P40, 93E03, 93E15, 93D30; Secondary: 60H35, 65C30.

Key words and phrases. Nonlinear stochastic differential equation, Lyapunov function, Radial basis function, Basin of attraction.

The research for this paper was supported by the Icelandic Research Fund (Rannís) in the project 'Lyapunov Methods and Stochastic Stability' (152429-051), which is gratefully acknowledged.

It turns out that the γ-basin of attraction can be determined using Lyapunov functions. In [8], a combination of a local and a non-local Lyapunov function was used to determine a subset of the γ-basin of attraction. A Lyapunov function $V\colon \mathbb{R}^d \to \mathbb{R}$ for a SDE satisfies $LV(\mathbf{x}) \leq 0$, where L is a second-order differential operator, which arises from the SDE. A local Lyapunov function is only defined in a small neighborhood of the equilibrium and can often be determined by linearisation. A non-local Lyapunov function, however, is defined on a superset $\widetilde{\mathcal{U}} \subset \mathbb{R}^d$ of the γ-basin of attraction apart from a small neighborhood, where the negativity condition is not necessarily satisfied. Local Lyapunov functions will be defined in Section 2, Definition 2.3, and non-local ones in Section 2, Definition 2.4.

In this paper, we present a constructive method to compute a non-local Lyapunov function for a general SDE. In particular, we use meshless collocation to solve a PDE boundary value problem of the form $LV(\mathbf{x}) = \tilde{\nu} < 0$ for all $\mathbf{x} \in \widetilde{\mathcal{U}}$ and with fixed boundary values for $V(\mathbf{x})$ at all $\mathbf{x} \in \partial\widetilde{\mathcal{U}}$. After choosing a kernel, in particular a Radial Basis Function, as well as collocation points in $\widetilde{\mathcal{U}}$ and $\partial\widetilde{\mathcal{U}}$, the approximate solution v to the problem is determined by using a certain ansatz and by computing coefficients by solving a linear equation.

To ensure that the approximation v is itself a valid Lyapunov function, we provide rigorous a-posteriori estimates on $Lv(\mathbf{x})$. This is achieved by evaluating $Lv(\mathbf{x})$ at all $\mathbf{x}$ in a test grid and using Taylor-type estimates for the points in between. These make use of the specific ansatz and corresponding estimates. The method is applied to two examples in one and two dimensions, respectively.

The outline of the paper is as follows: In Section 2 we recall the definition of the γ-basin of attraction and its determination using a pair of a local and a non-local Lyapunov function. In Section 3 we discuss meshless collocation for general PDE boundary value problems and in particular for the PDE related to the SDE under study. Moreover, we present a-posteriori error estimates based on first and second derivatives of Lv. Section 4 applies these results to the construction of a non-local Lyapunov function. Finally, we apply the method to two examples in Section 5. The appendix contains explicit formulas for the ansatz using meshless collocation, as well as tables for the estimates.

Note on notations:

If not specified, we use the Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^d$, i.e. $\|\mathbf{x}\| := \|\mathbf{x}\|_2$. We denote the closed ϵ-neighborhood with respect to the $\|\cdot\|_1$ norm of a compact set $K \subset \mathbb{R}^d$ by

$$K_{\epsilon,\|\cdot\|_1} = \{\mathbf{x} \in \mathbb{R}^d \,:\, \operatorname*{dist}_{\|\cdot\|_1}(\mathbf{x}, K) \leq \epsilon\},$$

where $\operatorname*{dist}_{\|\cdot\|_1}(\mathbf{x}, K) = \min_{\mathbf{y}\in K} \|\mathbf{x} - \mathbf{y}\|_1$. We sometimes denote the i-th component of a vector $\mathbf{x} - \mathbf{y}$ by $(\mathbf{x} - \mathbf{y})_i$ to shorten formulas.

2. Stochastic basin of attraction and Lyapunov functions.

In this section we introduce the type of SDE that we study as well as the stochastic basin of attraction of the zero (trivial) solution. We also recall the definition of (stochastic) Lyapunov functions; in particular, we will consider an appropriate combination of a local and a non-local Lyapunov function to determine the stochastic basin of attraction.

We study the stability of the trivial solution of the SDE of Itô type

$$d\mathbf{X}(t) \quad = \quad \mathbf{f}(\mathbf{X}(t))\, dt + \mathbf{g}(\mathbf{X}(t)) d\mathbf{W}(t), \tag{2.1}$$

where $\mathbf{W}(t)$ is a Q-dimensional Wiener process. The functions $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^d$ and $\mathbf{g} : \mathbb{R}^d \to \mathbb{R}^{d\times Q}$ are Lipschitz continuous on a neighbourhood of the origin $\mathcal{O}$, i.e. there exists a $K > 0$ such that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| + \|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\| \leq K\|\mathbf{x} - \mathbf{y}\| \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathcal{O}.$$

Moreover, we assume that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{g}(\mathbf{0}) = 0$, so that $\mathbf{X}(t) = \mathbf{0}$ is a solution of (2.1) for all $t \geq 0$.

Since we are interested in local stability, i.e. γ-basins of attraction within $\mathcal{O}$, we can extend $\mathbf{f}$ and $\mathbf{g}$ to Lipschitz continuous functions on $\mathbb{R}^d$ and consider *strong solutions* to (2.1) on $[0, \infty)$. This simplifies technical matters considerably, cf. [8, §2].

For the SDE (2.1) the associated *generator* is given by

$$LV(\mathbf{x}) := \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) + \frac{1}{2}\sum_{i,j=1}^{d} \left[\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^\top\right]_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}(\mathbf{x}), \tag{2.2}$$

for $V : \mathcal{U} \to \mathbb{R}$ with $\mathcal{U} \subset \mathbb{R}^d$.

Remark 2.1. *If the matrix $\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^\top$ is positive definite for all $\mathbf{x} \in \mathcal{U}$ in a compact set $\mathcal{U} \subset \mathbb{R}^d$, then the second-order linear differential operator L is strictly elliptic in $\mathcal{U}$. In this (non-degenerate) case, results about the existence of classical solutions are available, however, in this paper we will discuss the general case and make no requirement on the positive definiteness of the matrix.*

Let us now define the γ-basin of attraction which describes the set of initial conditions so that sample paths converge to the origin with probability at least γ, see [8, Definition 2.4].

Definition 2.2 (γ-basin of attraction (γ-BOA))**.** *Consider the system (2.1) and let $0 < \gamma \leq 1$. We refer to the set*

$$\left\{\mathbf{x} \in \mathbb{R}^d \,:\, \mathbb{P}\left\{\lim_{t\to\infty} \|\mathbf{X}^{\mathbf{x}}(t)\| = 0\right\} \geq \gamma\right\} \qquad (\gamma\text{-BOA})$$

as the γ-basin of attraction *or short γ-BOA of the origin. Here, $\mathbf{X}^{\mathbf{x}}(t)$ denotes the unique strong solution (stochastic process) of the SDE with initial condition $\mathbf{x}$.*

In the following definition [8, Definition 2.5], we introduce a local Lyapunov function in the set $\mathcal{N}$ (see also [8, Theorem 2.7]). A local Lyapunov function U is a positive definite function such that LU is negative definite in a (small) neighbourhood $\mathcal{N}$ of $\mathbf{0}$. This is most conveniently defined using so-called $\mathcal{K}_\infty$ functions; a function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, $\mu(0) = 0$, and $\lim_{x\to\infty} \mu(x) = \infty$.

Definition 2.3 (Local Lyapunov function)**.** *Consider the system (2.1). A function $U \in C(\mathcal{N}) \cap C^2(\mathcal{N} \setminus \{\mathbf{0}\})$, where $\mathbf{0} \in \mathcal{N} \subset \mathbb{R}^d$ is a domain, is called a (local) Lyapunov function for the system (2.1), if there are functions $\mu_1, \mu_2, \mu_3 \in \mathcal{K}_\infty$, such that U fulfills the properties:*

(i) $\mu_1(\|\mathbf{x}\|) \leq U(\mathbf{x}) \leq \mu_2(\|\mathbf{x}\|)$ *for all* $\mathbf{x} \in \mathcal{N}$
(ii) $LU(\mathbf{x}) \leq -\mu_3(\|\mathbf{x}\|)$ *for all* $\mathbf{x} \in \mathcal{N} \setminus \{\mathbf{0}\}$

Let $U_{\max} > 0$ be such that $U^{-1}([0, U_{\max}])$ is a compact subset of $\mathcal{N}$.

Next we introduce a non-local Lyapunov function in the set $\mathcal{U}$ as in [8, Definition 2.9, (2a)]; note that we have replaced 0 by b and 1 by a. A non-local Lyapunov function satisfies $LV < 0$ in a large set $\mathcal{U}$, not including a small neighborhood $\mathcal{B}$ of the equilibrium.

Definition 2.4 (Non-local Lyapunov function). *Let $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^d$, $\mathcal{B} \subset \mathcal{A}^\circ$, be simply connected compact neighbourhoods of the origin with C^2 boundaries and set $\mathcal{U} := \mathcal{A} \setminus \mathcal{B}^\circ$. A function $V \in C^2(\mathcal{U})$ for the system (2.1) such that*

(1) $b \le V(\mathbf{x}) \le a$ for all $\mathbf{x} \in \mathcal{U}$, $V^{-1}(b) = \partial\mathcal{B}$, $V^{-1}(a) = \partial\mathcal{A}$ with $b < a$, and
(2) $LV(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathcal{U}$,

is called a non-local Lyapunov function *for the system (2.1). We refer to ∂A as the outer boundary of $\mathcal{U}$ and ∂B as the inner boundary of $\mathcal{U}$.*

The following result from [8, Theorem 2.11] shows how a local and a non-local Lyapunov function provide information about the γ-BOA. For an illustration of the various sets, see [8, Figure 1]. The proof uses the non-local Lyapunov function to estimate the probability that solutions starting in $\mathcal{U}$ leave the set through the boundary $\partial\mathcal{B}$, and then the local Lyapunov function estimates the probability that they converge to the origin once they are in $\mathcal{B}$. The combined probability can be bounded by γ.

Theorem 2.5. *Consider the system (2.1) and assume there exists a local Lyapunov function $U : \mathcal{N} \to \mathbb{R}_+$ as in Definition 2.3 with the constant $U_{\max} > 0$ and a non-local Lyapunov function $V : \mathcal{U} \to \mathbb{R}_+$ as in Definition 2.4. Let $0 < \beta < 1$ and $b < \lambda < \alpha < a$ and the set $\mathcal{B}$ from Definition 2.4 be such that*

$$U^{-1}(U_{\max}) \subset V^{-1}([b, \lambda]) \quad \text{and} \quad \partial\mathcal{B} = V^{-1}(b) \subset U^{-1}([0, \beta\, U_{\max}]).$$

Then the set $V^{-1}([b, \alpha]) \cup \mathcal{B}$ is a subset of the γ-BOA of the origin, where

$$\gamma := \frac{(a - \alpha)(1 - \beta)}{a - b - \beta(a - \lambda)}. \tag{2.3}$$

Note that the bound (2.3) has a different formula than in [8, Theorem 2.11], because here $\partial\mathcal{B} = V^{-1}(b)$ and $\partial\mathcal{A} = V^{-1}(a)$ with b and a not necessarily equal to 0 and 1, respectively. Thus our formula is the formula from [8, Theorem 2.11] with γ replaced by $(\gamma - b)/(a - b)$ and α replaced by $(\alpha - b)/(a - b)$.

In this paper, we focus on a general method to compute non-local Lyapunov functions. Local Lyapunov functions can often be found directly in specific examples: for example, if the noise is small and the origin is an asymptotically stable equilibrium of the corresponding deterministic system with no noise, then the deterministic Lyapunov function can serve as a local Lyapunov function. Another way to construct a local Lyapunov function is similar to the construction of local Lyapunov functions for deterministic systems: by linearising the system around the origin and constructing a Lyapunov function for the linearised system, which is a local Lyapunov function for the nonlinear system, see [1].

For the examples in this paper, we are able to construct local Lyapunov functions with one of these two approaches. For a more general discussion on the construction of Lyapunov functions for linear systems see also [9].

3. **Meshless collocation.** In this section we will recall meshless collocation and its use to approximate solutions of boundary value problems for general linear PDEs of the form

$$\begin{cases} LV(\mathbf{x}) &= r(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega, \\ V(\mathbf{x}) &= c(\mathbf{x}) \text{ for } \mathbf{x} \in \partial\Omega, \end{cases} \tag{3.1}$$

where L is a linear differential operator and Ω is a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary. Meshless collocation seeks to find the solution v of an interpolation problem, which minimises the norm in a Reproducing Kernel Hilbert Space (RKHS), in our case a Sobolev space. The interpolation problem will ensure that v satisfies the PDE and the boundary values (3.1) at given collocation points.

If the PDE boundary value problem has a solution V, then v approximates V and we have error estimates of $\|V(\mathbf{x}) - v(\mathbf{x})\|_{L_\infty(\partial\Omega)}$ as well as $\|LV(\mathbf{x}) - Lv(\mathbf{x})\|_{L_\infty(\Omega)}$. The error estimates involve the fill distance of the collocation points, measuring how dense they are in Ω and $\partial\Omega$, respectively. Unfortunately, these estimates also involve unknown quantities, such as the norm of V. Thus, these error estimates ensure that by adding more and more collocation points the error converges to zero, but they do not provide explicit, computable bounds on the error.

We can, however, compute explicit a-posteriori bounds on the errors $\|V(\mathbf{x}) - v(\mathbf{x})\|_{L_\infty(\partial\Omega)}$ as well as $\|LV(\mathbf{x}) - Lv(\mathbf{x})\|_{L_\infty(\Omega)}$ by first computing $|LV(\mathbf{x}) - Lv(\mathbf{x})|$ for a finite, but large set of points $Y \subset \Omega$. Taylor's theorem and estimates on the first and second derivatives by using the explicit form of v provide us with explicit bounds on these errors as shown in Section 3.2.

3.1. **Meshless collocation: PDE boundary value problems.** Meshless collocation, in particular by Radial Basis Functions, is a powerful method to solve linear PDEs [11, 2, 12]. For a general introduction to meshless collocation and RKHS, see [14]. Meshless collocation has been applied to the computation of Lyapunov functions in deterministic systems [4, 7]. For an overview of this and other methods to compute Lyapunov functions, see the review [5].

In this section, we will outline the method, apply it to our particular case, and recall known results, in particular error estimates from [4].

We consider a general linear operator L of order m given by

$$LV(\mathbf{x}) \quad = \sum_{|\boldsymbol{\alpha}|\leq m} c_{\boldsymbol{\alpha}}(\mathbf{x})\partial_{\boldsymbol{\alpha}} V(\mathbf{x}). \tag{3.2}$$

In our case, $m = 2$ and the operator is given by

$$Lv(\mathbf{x}) \quad = \quad \frac{1}{2}\sum_{i,j=1}^{d} m_{ij}(\mathbf{x})\frac{\partial^2}{\partial x_i \partial x_j} v(\mathbf{x}) + \sum_{i=1}^{d} f_i(\mathbf{x})\frac{\partial}{\partial x_i} v(\mathbf{x}), \tag{3.3}$$

where $(m_{ij}(\mathbf{x}))_{i,j=1,\ldots,d} = \mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^\top$, i.e. $m_{ij}(\mathbf{x}) = \sum_{q=1}^{Q} g_{iq}(\mathbf{x})g_{jq}(\mathbf{x})$. We denote the q-th column of $\mathbf{g}$ by g^q.

Hence, our operator is of the form (3.2) with $c_{\mathbf{e}_i}(\mathbf{x}) = f_i(\mathbf{x})$ and $c_{\mathbf{e}_i+\mathbf{e}_j}(\mathbf{x}) = \frac{1}{2}m_{ij}(\mathbf{x})$. A singular point of L is a point $\mathbf{x}$ with $c_{\boldsymbol{\alpha}}(\mathbf{x}) = 0$ for all $|\boldsymbol{\alpha}| \leq 2$, see [4, Definition 3.2].

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\Gamma := \partial\Omega$. Our goal is to (approximately) solve the boundary value problem with a PDE given by:

$$\left\{\begin{array}{lcl} Lv(\mathbf{x}) & = & r(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega, \\ v(\mathbf{x}) & = & c(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma. \end{array}\right. \tag{3.4}$$

Our approximation will be a function in a RKHS, which is a Hilbert space H of functions $\Omega \to \mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle_H$, and a kernel $\Phi\colon \Omega \times \Omega \to \mathbb{R}$ such that

1. $\Phi(\cdot, \mathbf{x}) \in H$ for all $\mathbf{x} \in \Omega$,
2. $g(\mathbf{x}) = \langle g, \Phi(\cdot, \mathbf{x})\rangle_H$ for all $\mathbf{x} \in \Omega$ and $g \in H$.

In our case, we choose the radially symmetric kernel $\Phi(\mathbf{x},\mathbf{y}) = \psi(\|\mathbf{x}-\mathbf{y}\|)$, where $\psi = \psi_{\ell,k}$ is given by a Wendland function [13], see also Table 1. Setting $\ell = \lfloor \frac{d}{2} \rfloor + k + 1$, the parameter $k \in \mathbb{N}$ is a smoothness index and the function $\Phi(\mathbf{x},\mathbf{y})$ is a C^{2k} function in $\mathbf{x}$ for fixed $\mathbf{y}$, and the RKHS with this kernel is norm-equivalent to the Sobolev space W_2^τ with $\tau = k + \frac{d+1}{2}$.

Given sets of pairwise distinct points $X_1 = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \Omega \subset \mathbb{R}^d$, none of which is a singular point of L, and pairwise distinct points $X_2 = \{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_M\} \subset \Gamma = \partial\Omega$, we seek to find the (unique) solution v to the interpolation problem

$$\begin{aligned} Lv(\mathbf{x}_i) &= r(\mathbf{x}_i) \text{ for all } i = 1, \dots, N, \\ v(\boldsymbol{\xi}_i) &= c(\boldsymbol{\xi}_i) \text{ for all } i = 1, \dots, M, \end{aligned}$$

which minimises the norm of the RKHS. It turns out that the solution is given by

$$\begin{aligned} v(\mathbf{x}) &= \sum_{k=1}^{N} \alpha_k (\delta_{\mathbf{x}_k} \circ L)^{\mathbf{y}} \psi(\|\mathbf{x}-\mathbf{y}\|) \\ &\quad + \sum_{k=1}^{M} \alpha_{N+k} (\delta_{\boldsymbol{\xi}_k} \circ L^0)^{\mathbf{y}} \psi(\|\mathbf{x}-\mathbf{y}\|), \end{aligned} \tag{3.5}$$

where $L^0 = \mathrm{id}$, $\delta_{\mathbf{y}} v(\mathbf{x}) = v(\mathbf{y})$, the superscript $\mathbf{y}$ denotes that the operator is applied with respect to the variable $\mathbf{y}$, and the coefficients α_k are computed by solving the linear system $A\alpha = \beta$, where $\beta_k = r(\mathbf{x}_k)$ for $k = 1, \dots, N$ and $\beta_{N+k} = c(\boldsymbol{\xi}_k)$ for $k = 1, \dots, M$. $A = (a_{kl})$ is a symmetric $(N+M) \times (N+M)$ matrix given by $A = \begin{pmatrix} B & C \\ C^\top & D \end{pmatrix}$ with $B \in \mathbb{R}^{N\times N}$, $C \in \mathbb{R}^{N\times M}$, $D \in \mathbb{R}^{M\times M}$, where

$$\begin{aligned} &\text{for } k,l = 1,\dots,N: \\ &\quad b_{kl} = (\delta_{\mathbf{x}_k} \circ L)^{\mathbf{x}} (\delta_{\mathbf{x}_l} \circ L)^{\mathbf{y}} \psi(\|\mathbf{x}-\mathbf{y}\|), \\ &\text{for } k = 1,\dots,N, l = 1,\dots,M: \\ &\quad c_{kl} = (\delta_{\mathbf{x}_k} \circ L)^{\mathbf{x}} (\delta_{\boldsymbol{\xi}_l} \circ L^0)^{\mathbf{y}} \psi(\|\mathbf{x}-\mathbf{y}\|) = (\delta_{\mathbf{x}_k} \circ L)^{\mathbf{x}} \psi(\|\mathbf{x}-\boldsymbol{\xi}_l\|), \\ &\text{for } k,l = 1,\dots,M: \\ &\quad d_{kl} = (\delta_{\boldsymbol{\xi}_k} \circ L^0)^{\mathbf{x}} (\delta_{\boldsymbol{\xi}_l} \circ L^0)^{\mathbf{y}} \psi(\|\mathbf{x}-\mathbf{y}\|) = \psi(\|\boldsymbol{\xi}_k - \boldsymbol{\xi}_l\|). \end{aligned}$$

Explicit formulas for v and Lv are given in the Appendix A.

If the PDE has a solution V, then error estimates imply that the function v is an approximation to V as stated in Theorem 3.1 below. Note that the mesh norms measure how dense the points in X_1 and X_2 are in the domain and boundary, respectively. The following is [4, Corollary 3.12] adapted to our linear operator.

Theorem 3.1. *Let $k > 3/2$, if d is odd, and $k > 2$, if d is even. Let $f_i, m_{ij} \in W_\infty^{k-1+\lfloor \frac{d+1}{2} \rfloor}(\Omega)$ and let the solution V of (3.4) satisfy $V \in W^{k+(d+1)/2}(\Omega)$. Then the approximation v as above, for sufficiently small mesh norms, satisfies*

$$\|LV - Lv\|_{L_\infty(\Omega)} \le C h_{X_1,\Omega}^{k-3/2} \|V\|_{W_2^{k+(d+1)/2}(\Omega)}, \tag{3.6}$$

$$\|V - v\|_{L_\infty(\partial\Omega)} \le C h_{X_2,\partial\Omega}^{k+1/2} \|V\|_{W_2^{k+(d+1)/2}(\Omega)}, \tag{3.7}$$

where $h_{X_1,\Omega} = \sup_{\mathbf{x}\in\Omega} \min_{\mathbf{x}_j \in X_1} \|\mathbf{x}-\mathbf{x}_j\|$ and the constant $h_{X_2,\partial\Omega}$ is the mesh norm for the boundary part, for the precise definition see [4].

3.2. **A-posteriori error estimates.** Note that, unless L is non-degenerate, we have no results on the existence of classical solutions and thus we cannot use Theorem 3.1. Even in that case, the error estimates in Theorem 3.1 contain quantities that are not known explicitly, such as $\|V\|_{W_2^{k+(d+1)/2}(\Omega)}$.

Hence, in this section we derive estimates that only contain explicitly computable constants. They do not require us to prove the existence of a solution, but are a verification that the computed function satisfies an inequality at all points. The main idea is to evaluate the function at many points on a test grid and then use a Taylor-type argument in between. As we have an explicit formula for the approximation, we can derive explicit bounds on the derivatives. As these are multiplied by the mesh norm of the test grid, which can be made arbitrarily small, we can make the estimate as accurate as necessary.

Let us first present the Taylor-type estimates for a general function u, which later will be either the approximation v or Lv. These theorems, as well as a more detailed discussion of a suitable choice of the test grid are taken from [10], see also [6].

As test grids, we will use the following:

Definition 3.2. *Define the following grids in $\mathbb{R}^d$ with $h > 0$:*

- $S_h = h\mathbb{Z}^d$
- $C_h = S_h \cup \left(\frac{h}{2}\mathbf{1} + S_h\right)$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^d$

The following theorem is based on the mean-value theorem and uses the specific structure of the grid points S_h.

Theorem 3.3 (First derivative)**.** *Let $u \in C^1(\mathbb{R}^d, \mathbb{R})$ and let $K \subset \mathbb{R}^d$ be compact. Fix $h > 0$ and let $Y := C_h \cap K_{h\,d/4, \|\cdot\|_1}$.*

Define

$$e_h = \frac{d}{4} \max_{\mathbf{z} \in K_{h\,d/4,\|\cdot\|_1}} \max_{l \in \{1,\ldots,d\}} \left| \frac{\partial u}{\partial x_l}(\mathbf{z}) \right| h.$$

Then we have for all $\mathbf{x} \in K$ that

$$\min_{\mathbf{y} \in Y} u(\mathbf{y}) - e_h \le u(\mathbf{x}) \le \max_{\mathbf{y} \in Y} u(\mathbf{y}) + e_h.$$

Proof. Let $\mathbf{x} \in K$. Then there is a $\mathbf{y} \in C_h$ with $\|\mathbf{x} - \mathbf{y}\|_1 \le \frac{d}{4}h$, see [10, Theorem 5.5], and thus $\mathbf{y} \in Y$. The mean value theorem shows that there is a $\theta \in [0, 1]$ such that

$$\begin{aligned} |u(\mathbf{x}) - u(\mathbf{y})| &= |\nabla u(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})| \\ &\le \|\nabla u(\theta\mathbf{x} + (1-\theta)\mathbf{y})\|_\infty \|\mathbf{x} - \mathbf{y}\|_1 \\ &\le \max_{l \in \{1,\ldots,d\}} \left| \frac{\partial u}{\partial x_l}(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \right| \frac{d}{4}h\,. \end{aligned}$$

Note that $\theta\mathbf{x} + (1-\theta)\mathbf{y} \in K_{h\,d/4,\|\cdot\|_1}$, since

$$\|\theta\mathbf{x} + (1-\theta)\mathbf{y} - \mathbf{x}\|_1 = (1-\theta)\|\mathbf{y} - \mathbf{x}\|_1 \le \frac{d}{4}h.$$

This shows the statement. □

The next theorem relies on a triangulation of the phase space with vertices in S_h, C_h, respectively. Using Taylor's theorem in each simplex, we can derive the estimates below. Note that, as discussed in [10], depending on odd or even dimension, we use either S_h or C_h to obtain an estimate with as few points as possible.

Theorem 3.4 (Second derivative)**.** *Let $u \in C^2(\mathbb{R}^d, \mathbb{R})$ and let $K \subset \mathbb{R}^d$ be compact. Fix $h > 0$. If $d > 1$ define*

$$e_h = \frac{d^2}{4} \max_{\mathbf{z} \in K_{d\,h, \|\cdot\|_1}} \max_{l,p \in \{1,\ldots,d\}} \left| \frac{\partial^2 u}{\partial x_p \partial x_l}(\mathbf{z}) \right| h^2.$$

- *If d is even, then let $Y := S_h \cap K_{d\,h, \|\cdot\|_1}$.*
- *If $d \geq 3$ is odd, then let $Y := C_h \cap K_{(d-1)\,h, \|\cdot\|_1}$.*

In the case $d = 1$ let $Y := C_h \cap K_{h/2, \|\cdot\|_1}$ and define

$$e_h = \frac{1}{4} \max_{\mathbf{z} \in K_{h/2, \|\cdot\|_1}} |u''(\mathbf{z})|\, h^2.$$

In all cases we then have for all $\mathbf{x} \in K$

$$\min_{\mathbf{y} \in Y} u(\mathbf{y}) - e_h \quad \leq \quad u(\mathbf{x}) \ \leq\ \max_{\mathbf{y} \in Y} u(\mathbf{y}) + e_h.$$

Proof. We consider the case where d is even. Let $\mathbf{x} \in K$. Then there is a simplex S with vertices $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_d\} \subset S_h$, such that $\mathbf{x} = \sum_{i=0}^d \lambda_i \mathbf{x}_i \in S$, where $\sum_{i=0}^d \lambda_i = 1$ and $0 \leq \lambda_i \leq 1$. Since $\max_{\mathbf{y},\mathbf{z} \in S} \|\mathbf{y} - \mathbf{z}\|_1 = dh$, we have $S \subset K_{d\,h, \|\cdot\|_1}$ and thus $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_d\} \subset Y$.

Now we use the following result from [10, Proposition 5.2]: Denote by $h^* := \max_{j=0,\ldots,d} \|\mathbf{x}_0 - \mathbf{x}_j\|_1$ the maximal distance from any vertex to the fixed vertex $\mathbf{x}_0$. For $w \in C^2(\mathbb{R}^d, \mathbb{R})$ we have for all $0 \leq \lambda_i \leq 1$ with $\sum_{i=0}^d \lambda_i = 1$ that

$$\left| w\left(\sum_{i=0}^d \lambda_i \mathbf{x}_i \right) - \sum_{i=0}^d \lambda_i w(\mathbf{x}_i) \right| \quad \leq \quad \max_{\mathbf{z} \in S} \max_{l,p \in \{1,\ldots,d\}} \left| \frac{\partial^2 w(\mathbf{z})}{\partial x_p \partial x_l} \right| (h^*)^2. \tag{3.8}$$

In our case, we can choose the vertex $\mathbf{x}_0$ such that $h^* = \frac{d}{2} h$. As $\mathbf{x} \in S$ there are $0 \leq \lambda_i \leq 1$ with $\sum_{i=0}^d \lambda_i = 1$ such that $\mathbf{x} = \sum_{i=0}^d \lambda_i \mathbf{x}_i$. Hence, by (3.8)

$$\left| u(\mathbf{x}) - \sum_{i=0}^d \lambda_i u(\mathbf{x}_i) \right| \quad \leq \quad \max_{\mathbf{z} \in K_{d\,h, \|\cdot\|_1}} \max_{l,p \in \{1,\ldots,d\}} \left| \frac{\partial^2 u(\mathbf{z})}{\partial x_p \partial x_l} \right| \frac{d^2}{4} h^2$$

and then

$$u(\mathbf{x}) \leq \max_{\mathbf{y} \in Y} u(\mathbf{y}) \underbrace{\sum_{i=0}^d \lambda_i}_{=1} + \max_{\mathbf{z} \in K_{d\,h, \|\cdot\|_1}} \max_{l,p \in \{1,\ldots,d\}} \left| \frac{\partial^2 u(\mathbf{z})}{\partial x_p \partial x_l} \right| \frac{d^2}{4} h^2$$

and similarly for the other inequality.

The result for odd dimensions follows in a similar way, noting that we choose a simplex S with vertices $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_d\} \subset C_h$. Since $\max_{\mathbf{y},\mathbf{z} \in S} \|\mathbf{y} - \mathbf{z}\|_1 = (d-1)h$ for $d \geq 2$ and $\frac{h}{2}$ for $d = 1$, and for a simplex with vertices in C_h we can choose the vertex $\mathbf{x}_0$ such that $h^* := \max_{j=0,\ldots,d} \|\mathbf{x}_0 - \mathbf{x}_j\|_1 = \frac{d}{2} h$, see [10, Theorem 5.8], the result follows. $\square$

The following theorem provides us with explicit bounds on the first and second derivatives of both v and Lv, as required in Theorems 3.3 and 3.4 for $u = v$ and $u = Lv$, respectively. Note that they involve quantities depending on $\mathbf{f}$ and $\mathbf{g}$ as well as their first and second derivatives, and the (computed) coefficients α_i, $i = 1, \ldots, N + M$. Moreover, the bounds $\psi_{i,k}$ as defined below are calculated for specific Wendland functions ψ_0 in the appendix. Note that the requirement on ψ_i is satisfied for Wendland functions with smoothness index $k \geq 6$.

Theorem 3.5. *Let $v \in C^4(\mathbb{R}^d,\mathbb{R})$ be given by (3.5) with kernel $\psi(r) =: \psi_0(r) \in C^6$. Let $C \subset \mathbb{R}^d$ be a compact set. Denote*

- $\psi_i(r) = \frac{1}{r}\frac{d}{dr}\psi_{i-1}(r)$ *for $r > 0$ and $i = 1,\ldots,6$ and assume that $\psi_i(r)$ can be continuously extended to $r = 0$,*
- $\psi_{i,k} = \sup_{r\in[0,\infty)} |\psi_i(r)|r^k < \infty$ *for $i,k \in \mathbb{N}_0$,*
- $F = \max_{\mathbf{x}\in C} \|\mathbf{f}(\mathbf{x})\|$,
  $F_1 = \max_{\mathbf{x}\in C} \max_{l\in\{1,\ldots,d\}} \left\|\frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_l}\right\|$, *and*
  $F_2 = \max_{\mathbf{x}\in C} \max_{l,p\in\{1,\ldots,d\}} \left\|\frac{\partial^2 \mathbf{f}(\mathbf{x})}{\partial x_p \partial x_l}\right\|$.
- $$\begin{aligned} G &= \frac{1}{2}\sum_{q=1}^{Q} \max_{\mathbf{x}\in C} \|g^q(\mathbf{x})\|^2, \\ G_1 &= \sum_{q=1}^{Q} \max_{\mathbf{x}\in C} \max_{l\in\{1,\ldots,d\}} \|g^q(\mathbf{x})\| \left\|\frac{\partial g^q(\mathbf{x})}{\partial x_l}\right\|, \text{ and} \\ G_2 &= \sum_{q=1}^{Q} \max_{\substack{\mathbf{x}\in C\\ l,p\in\{1,\ldots,d\}}} \left[\left\|\frac{\partial^2 g^q(\mathbf{x})}{\partial x_p \partial x_l}\right\| \|g^q(\mathbf{x})\| + \left\|\frac{\partial g^q(\mathbf{x})}{\partial x_p}\right\| \left\|\frac{\partial g^q(\mathbf{x})}{\partial x_l}\right\|\right], \end{aligned}$$
  where $g^q(\mathbf{x}) \in \mathbb{R}^d$ denotes the vector $(g_{iq}(\mathbf{x}))_{i=1,\ldots,d}$ for all $q = 1,\ldots,Q$.
- $\boldsymbol{\alpha}_1 = \sum_{k=1}^N |\alpha_k|$ *and* $\boldsymbol{\alpha}_2 = \sum_{k=N+1}^{N+M} |\alpha_k|$.

Then we have the following bounds for all $\mathbf{x} \in C$ and all $l,p \in \{1,\ldots,d\}$:

$$\left|\frac{\partial v}{\partial x_l}(\mathbf{x})\right| \leq \boldsymbol{\alpha}_1\{G[\psi_{3,3} + 3\psi_{2,1}] + F[\psi_{2,2} + \psi_{1,0}]\} + \boldsymbol{\alpha}_2\psi_{1,1},$$

$$\begin{aligned} \left|\frac{\partial^2 v}{\partial x_p \partial x_l}(\mathbf{x})\right| &\leq \boldsymbol{\alpha}_1\{G[\psi_{4,4} + 6\psi_{3,2} + 3\psi_{2,0}] + F[\psi_{3,3} + 3\psi_{2,1}]\} \\ &\quad + \boldsymbol{\alpha}_2[\psi_{2,2} + \psi_{1,0}], \end{aligned}$$

$$\begin{aligned} \left|\frac{\partial Lv}{\partial x_l}(\mathbf{x})\right| \leq\ & \boldsymbol{\alpha}_1\Big\{G^2[\psi_{5,5} + 10\psi_{4,3} + 15\psi_{3,1}] \\ & + (2F + G_1)G[\psi_{4,4} + 6\psi_{3,2} + 3\psi_{2,0}] \\ & + (F_1 G + F G_1 + F^2)[\psi_{3,3} + 3\psi_{2,1}] \\ & + F F_1[\psi_{2,2} + \psi_{1,0}]\Big\} \\ & + \boldsymbol{\alpha}_2\Big\{G[\psi_{3,3} + 3\psi_{2,1}] + (F + G_1)[\psi_{2,2} + \psi_{1,0}] + F_1\psi_{1,1}\Big\}, \text{ and} \end{aligned}$$

$$\begin{aligned} \left|\frac{\partial^2 Lv}{\partial x_p \partial x_l}(\mathbf{x})\right| \leq\ & \boldsymbol{\alpha}_1\Big\{G^2[\psi_{6,6} + 15\psi_{5,4} + 45\psi_{4,2} + 15\psi_{3,0}] \\ & + 2(F + G_1)G[\psi_{5,5} + 10\psi_{4,3} + 15\psi_{3,1}] \\ & + (F^2 + GG_2 + 2F_1 G + 2FG_1)[\psi_{4,4} + 6\psi_{3,2} + 3\psi_{2,0}] \\ & + (2FF_1 + F_2 G + FG_2)[\psi_{3,3} + 3\psi_{2,1}] \\ & + FF_2[\psi_{2,2} + \psi_{1,0}]\Big\} \end{aligned}$$

$$
\begin{aligned}
+\boldsymbol{\alpha}_2 \Big\{ & G[\psi_{4,4} + 6\psi_{3,2} + 3\psi_{2,0}] \\
& +(F + 2G_1)[\psi_{3,3} + 3\psi_{2,1}] + (2F_1 + G_2)[\psi_{2,2} + \psi_{1,0}] \Big\}.
\end{aligned}
$$

Proof. The proof follows directly by differentiation and estimating terms of similar type, where we use the explicit formulas derived in the appendix. Using formula (A.1) for v we get

$$
\begin{aligned}
& \frac{\partial v}{\partial x_j}(\mathbf{x}) \\
= & \sum_{k=1}^{N} \alpha_k \Big\{ -\psi_2(\|\mathbf{x}-\mathbf{x}_k\|)(\mathbf{x}-\mathbf{x}_k)_l \langle \mathbf{x}-\mathbf{x}_k, \mathbf{f}(\mathbf{x}_k)\rangle \\
& \quad -\psi_1(\|\mathbf{x}-\mathbf{x}_k\|) f_l(\mathbf{x}_k) \\
& \quad +\frac{1}{2}\sum_{i,j=1}^{d} m_{ij}(\mathbf{x}_k)\Big[\psi_3(\|\mathbf{x}-\mathbf{x}_k\|)(\mathbf{x}-\mathbf{x}_k)_l(\mathbf{x}-\mathbf{x}_k)_i(\mathbf{x}-\mathbf{x}_k)_j \\
& \quad +\psi_2(\|\mathbf{x}-\mathbf{x}_k\|)[\delta_{il}(\mathbf{x}-\mathbf{x}_k)_j + (\mathbf{x}-\mathbf{x}_k)_i\delta_{lj} + (\mathbf{x}-\mathbf{x}_k)_l\delta_{ij}]\Big]\Big\} \\
& + \sum_{k=1}^{M} \alpha_{N+k}\psi_1(\|\mathbf{x}-\boldsymbol{\xi}_k\|)(\mathbf{x}-\boldsymbol{\xi}_k)_l \\
\leq & \sum_{k=1}^{N} |\alpha_k| \Big\{ |\psi_2(\|\mathbf{x}-\mathbf{x}_k\|)| \; \|\mathbf{x}-\mathbf{x}_k\|^2 F + |\psi_1(\|\mathbf{x}-\mathbf{x}_k\|)| F \\
& \quad +\frac{1}{2}\sum_{i,j=1}^{d}\sum_{q=1}^{Q} g_{iq}(\mathbf{x}_k) g_{jq}(\mathbf{x}_k)\Big[|\psi_3(\|\mathbf{x}-\mathbf{x}_k\|)| \|\mathbf{x}-\mathbf{x}_k\|^3 \\
& \quad +3|\psi_2(\|\mathbf{x}-\mathbf{x}_k\|)| \|\mathbf{x}-\mathbf{x}_k\|\Big]\Big\} \\
& + \sum_{k=1}^{M} |\alpha_{N+k}| \; |\psi_1(\|\mathbf{x}-\boldsymbol{\xi}_k\|)| \; \|\mathbf{x}-\boldsymbol{\xi}_k\|.
\end{aligned}
$$

This shows the first estimate, using the definitions of $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, and $\psi_{l,k}$. The other estimates are proved in a similar way. □

4. **Non-local Lyapunov function.** In this section we will present a method to use meshless collocation, as discussed in the previous section, to compute a non-local Lyapunov function and combine it with a given, local Lyapunov function.

We seek to find a non-local Lyapunov function v satisfying $Lv(\mathbf{x}) < 0$, see Definition 2.4. This is done by finding an approximate solution of the PDE $LV(\mathbf{x}) = \tilde{\nu}$ with $\tilde{\nu} < 0$ in $\widetilde{\mathcal{U}}$ by meshless collocation and using the a-posteriori estimates for Lv to show that v satisfies $Lv(\mathbf{x}) \leq \nu < 0$. Note, however, that the boundary of $\widetilde{\mathcal{U}}$ is only approximately given by the level sets with level 0 and 1 of v, apart from the case $d = 1$. Hence, we compute the minimum of v at the outer boundary of $\widetilde{\mathcal{U}}$ and the maximum of v at the inner boundary of $\widetilde{\mathcal{U}}$, using the a-posteriori estimates for

v. Then we can define $\mathcal{U} = \mathcal{A} \setminus \mathcal{B}^\circ$ via $\mathcal{A}$ and $\mathcal{B}$ through the level sets of v with levels a and b, respectively, and thus show that v satisfies the conditions in Definition 2.4. Theorem 2.5 applied to v then gives us a rigorous result for the stochastic basin of attraction of the equilibrium at the origin.

Let v be the approximate solution of the following boundary-value problem:

$$LV(\mathbf{x}) = \tilde{\nu} \quad \text{for all} \quad \mathbf{x} \in \widetilde{\mathcal{U}}^\circ, \tag{4.1}$$

$$V(\mathbf{x}) = \begin{cases} 0 & \text{for all} \quad \mathbf{x} \in \partial\widetilde{\mathcal{B}}, \\ 1 & \text{for all} \quad \mathbf{x} \in \partial\widetilde{\mathcal{A}}, \end{cases} \tag{4.2}$$

where L is given by (2.2), $\tilde{\nu} < 0$ and $\widetilde{\mathcal{U}} = \widetilde{\mathcal{A}} \setminus \widetilde{\mathcal{B}}^\circ$, where $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{A}}^\circ$ and $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ are both simply connected compact neighborhoods of the origin with C^2 boundaries.

We use Theorem 3.3 or Theorem 3.4 with the set $K = \partial\widetilde{\mathcal{B}}$ and fixed $h > 0$ for the function v. We set

$$m := \max_{\mathbf{x} \in Y} v(\mathbf{x}) + e_h,$$

where e_h and Y are defined in Theorem 3.3 or Theorem 3.4, respectively.

We use Theorem 3.3 or Theorem 3.4 with the set $K = \partial\widetilde{\mathcal{A}}$ and fixed $h > 0$ for the function v. We set

$$M := \min_{\mathbf{x} \in Y} v(\mathbf{x}) - e_h,$$

where e_h and Y are defined in Theorem 3.3 or Theorem 3.4, respectively.

Lemma 4.1. *In the situation described above, assume that $v \in C^2$ and $m < M$, and choose $m < b < a < M$. Define $\mathcal{A} = v^{-1}((-\infty, a])$, $\mathcal{B} = v^{-1}((-\infty, b])$ and $\mathcal{U} = \mathcal{A} \setminus \mathcal{B}^\circ$. Assume that $\mathcal{A}$ and $\mathcal{B}$ are simply connected compact neighborhoods of the origin, and assume that $\mathbb{R}^d \setminus \mathcal{B}$ is connected. Then $\mathcal{A}$ and $\mathcal{B}$ have C^2 boundaries, $\mathcal{B} \subset \mathcal{A}^\circ$, and $\mathcal{U} \subset \widetilde{\mathcal{U}}$.*

Proof. The sets $\mathcal{A}$ and $\mathcal{B}$ have C^2 boundaries since $v \in C^2$. $\mathcal{B} \subset \mathcal{A}^\circ$ follows from $b < a$.

We first show now that $\mathcal{A} \subset \widetilde{\mathcal{A}}$. Assuming the opposite, there is a point $\mathbf{x}^* \in \mathcal{A} \setminus \widetilde{\mathcal{A}}$ and, since $\mathcal{A}$ is a connected neighborhood of the origin, there is a continuous path from $\mathbf{x}^*$ to the origin within $\mathcal{A}$, which has to intersect with $\partial\widetilde{\mathcal{A}}$ as the origin is in $\widetilde{\mathcal{A}}$. Hence, there is a point $\mathbf{x} \in \mathcal{A} \cap \partial\widetilde{\mathcal{A}}$. This means that $v(\mathbf{x}) \le a$ and, because of Theorem 3.3 or 3.4 and the arguments above, that $v(\mathbf{x}) \ge \min_{\mathbf{y} \in Y} v(\mathbf{y}) - e_h = M > a$, which is a contradiction.

Next we show that $\widetilde{\mathcal{B}} \subset \mathcal{B}$. Since both $\widetilde{\mathcal{B}}$ and $\mathcal{B}$ are compact, there is a point $\widetilde{\mathbf{x}} \in \mathbb{R}^d$ with $\widetilde{\mathbf{x}} \notin \widetilde{\mathcal{B}}$ and $\widetilde{\mathbf{x}} \notin \mathcal{B}$. Now assume the opposite to the statement $\widetilde{\mathcal{B}} \subset \mathcal{B}$, namely that there is a point $\mathbf{x}^* \in \widetilde{\mathcal{B}} \setminus \mathcal{B}$ and, since $\mathbb{R}^d \setminus \mathcal{B}$ is a connected neighborhood of $\widetilde{\mathbf{x}}$, there is a continuous path from $\mathbf{x}^*$ to $\widetilde{\mathbf{x}}$ within $\mathbb{R}^d \setminus \mathcal{B}$, which has to intersect with $\partial\widetilde{\mathcal{B}}$ as $\widetilde{\mathbf{x}}$ is in $\mathbb{R}^d \setminus \widetilde{\mathcal{B}}$. Hence, there is a point $\mathbf{x} \in (\mathbb{R}^d \setminus \mathcal{B}) \cap \partial\widetilde{\mathcal{B}}$. This means that $v(\mathbf{x}) > b$ and, because of Theorem 3.3 or 3.4 and the arguments above, that $v(\mathbf{x}) \le \max_{\mathbf{y} \in Y} v(\mathbf{y}) + e_h = m < b$, which is a contradiction. □

Now we use Theorem 3.3 or 3.4, respectively, to establish that v is a non-local Lyapunov function. To estimate $C_{\mathcal{U}}$ we use Theorem 3.5 with $C = \mathcal{U}_{h\,d/4, \|\cdot\|_1}$. Together with a local Lyapunov function, we can then use Theorem 2.5 to determine a γ-basin of attraction.

Theorem 4.2 (First derivative)**.** *Let $v \in C^3$ be a function given by meshless collocation as described above.*

Let $\mathcal{A} = v^{-1}((-\infty, a])$ and $\mathcal{B} = v^{-1}((-\infty, b])$ and assume that $\mathcal{B} \subset \mathcal{A}^\circ$ and that $\mathcal{A}$ and $\mathcal{B}$ are simply connected compact neighbourhoods of the origin with C^2 boundaries. Set $\mathcal{U} := \mathcal{A} \setminus \mathcal{B}^\circ$.

Fix $h > 0$ and define $Y_{\mathcal{U}} := C_h \cap \mathcal{U}_{h\,d/4\|\cdot\|_1}$,

$$C_{\mathcal{U}} := \max_{\mathbf{z} \in \mathcal{U}_{h\,d/4,\|\cdot\|_1}} \max_{l \in \{1,\ldots,d\}} \left| \frac{\partial Lv}{\partial x_l}(\mathbf{z}) \right|$$

and

$$\nu \quad := \quad \max_{\mathbf{y} \in Y_{\mathcal{U}}} Lv(\mathbf{y}) + C_{\mathcal{U}} \frac{d}{4} h.$$

If $\nu < 0$, then v is a non-local Lyapunov function.

Proof. For all $\mathbf{x} \in \mathcal{U}$ we have by Theorem 3.3 for $u = Lv$

$$Lv(\mathbf{x}) \quad \leq \quad \max_{\mathbf{y} \in Y_{\mathcal{U}}} Lv(\mathbf{y}) + C_{\mathcal{U}} \frac{d}{4} h = \nu < 0.$$

Hence, v satisfies the assumptions of Definition 2.4. □

Theorem 4.3 (Second derivative)**.** *Let $v \in C^4$ be a function given by meshless collocation as described above. Let $\mathcal{A} = v^{-1}((-\infty, a])$ and $\mathcal{B} = v^{-1}((-\infty, b])$ and assume that $\mathcal{B} \subset \mathcal{A}^\circ$ and that $\mathcal{A}$ and $\mathcal{B}$ are simply connected compact compact neighbourhoods of the origin with C^2 boundaries. Set $\mathcal{U} := \mathcal{A} \setminus \mathcal{B}^\circ$. Fix $h > 0$.*

- *If $d = 1$, then let*
$$Y_{\mathcal{U}} := C_h \cap \mathcal{U}_{h/2,\|\cdot\|_1} \quad \textit{and} \quad C_{\mathcal{U}} := \max_{\mathbf{z} \in \mathcal{U}_{h/2,\|\cdot\|_1}} |(Lv)''(\mathbf{z})|.$$
- *If d is even, then let*
$$Y_{\mathcal{U}} := S_h \cap \mathcal{U}_{d\,h,\|\cdot\|_1} \quad \textit{and} \quad C_{\mathcal{U}} := \max_{\mathbf{z} \in \mathcal{U}_{d\,h,\|\cdot\|_1}} \max_{p,l \in \{1,\ldots,d\}} \left| \frac{\partial^2 Lv}{\partial x_p \partial x_l}(\mathbf{z}) \right|.$$
- *If $d \geq 3$ is odd, then let $Y_{\mathcal{U}} := C_h \cap \mathcal{U}_{(d-1)\,h,\|\cdot\|_1}$ and*
$$C_{\mathcal{U}} := \max_{\mathbf{z} \in \mathcal{U}_{(d-1)\,h,\|\cdot\|_1}} \max_{p,l \in \{1,\ldots,d\}} \left| \frac{\partial^2 Lv}{\partial x_p \partial x_l}(\mathbf{z}) \right|.$$

Let

$$\nu \quad := \quad \max_{\mathbf{y} \in Y_{\mathcal{U}}} Lv(\mathbf{y}) + C_{\mathcal{U}} \frac{d^2}{4} h^2$$

If $\nu < 0$, then v is a non-local Lyapunov function.

Proof. For all $\mathbf{x} \in \mathcal{U}$ we have with Theorem 3.4 for $u = Lv$

$$Lv(\mathbf{x}) \quad \leq \quad \max_{\mathbf{y} \in Y_{\mathcal{U}}} Lv(\mathbf{y}) + C_{\mathcal{U}} \frac{d^2}{4} h^2 = \nu < 0.$$

Hence, v satisfies the assumptions of Definition 2.4. □

Remark 4.4. *Note that due to Lemma 4.1 we have $\mathcal{U} \subset \widetilde{\mathcal{U}}$ and thus we can replace $\mathcal{U}$ in the previous two theorems by $\widetilde{\mathcal{U}}$. However, we can use Theorems 4.2 and 4.3 directly with suitable a and b, without employing Lemma 4.1 as well.*

5. Examples.

5.1. One-dimensional example.

We consider the example from [8]:

$$dx = \sin x\, dt + \frac{3x}{1+x^2}\, dW, \tag{5.1}$$

where W is a one-dimensional Wiener-process. As $\sin x$ and $3x/(1+x^2)$ are Lipschitz, this equation has a unique strong solution. As local Lyapunov function we take $U(x) = |x|^{1/2}$ as in [8]. Then

$$LU(x) = -\frac{1}{2}|x|^{1/2}\left(3^2\frac{\frac{1}{2}}{2(1+x^2)^2} - \frac{\sin(x)}{x}\right)$$

and $LU(x) < 0$ for all $x \in [-2^{-1/2}, 2^{-1/2}] \setminus \{0\} =: \mathcal{B} \setminus \{0\}$. Therefore we can choose $\{\pm 2^{-1/2}\} = U^{-1}(U_{\max})$ with $U_{\max} = 2^{-1/4}$.

For the non-local Lyapunov function we just consider $x \geq 0$, since the SDE is symmetric. We use the Wendland function $\phi_{7,6}$ with coefficient $c = 2$. We set $\rho_1 = 10^{-2}$ and $\rho_2 = 8$ and determine an approximate solution to the equation

$$LV(x) = -10^{-3} \quad \text{on } (\rho_1, \rho_2)$$

such that $V(\rho_1) = 0$ and $V(\rho_2) = 1$. We have chosen 700 collocation points evenly spaced in the interval $[1.1 \cdot 10^{-2}, 7.99]$.

The approximating function v and Lv are displayed in Figure 1. We obtain the values $\boldsymbol{\alpha}_1 = 653.0140$ and $\boldsymbol{\alpha}_2 = 0.9440$. Since in the 1-dimensional case the boundary values for the approximation are $v(\rho_1) = 0$ and $v(\rho_2) = 1$, we choose $a = 1$ and $b = 0$ and hence $\mathcal{U} = [\rho_1, \rho_2]$. We first use Theorem 3.5 on any compact set C with $F = F_1 = F_2 = 1$, $G = 9/8$, $G_1 = 1.9566$, and $G_2 = 9$ to obtain $\max_{z\in\mathbb{R}} |(Lv)''(z)| = 1.6846 \cdot 10^{12} =: C_{\mathcal{U}}$; for the values $\psi_{k,l}$ see Table 3.

We now use Theorem 4.3 and choose $h = 2.1307 \cdot 10^{-8}$, which corresponds to $7.5 \cdot 10^8$ evenly spaced points $Y_{\mathcal{U}} = C_h \cap [\rho_1 - h/2, \rho_2 + h/2] = \frac{h}{2}\mathbb{Z} \cap [\rho_1 - h/2, \rho_2 + h/2]$ on the interval. We obtain a maximum value of $\max_{y \in Y_{\mathcal{U}}} Lv(y) = -0.281 \cdot 10^{-3}$ and thus

$$\nu = \max_{y \in Y_{\mathcal{U}}} Lv(y) + C_{\mathcal{U}}\frac{h^2}{4} = -0.281 \cdot 10^{-3} + 0.19119 \cdot 10^{-3} < 0.$$

By Theorem 4.3, v is a non-local Lyapunov function.

Now we need to determine constants $0 < \beta < 1$ and $0 < \lambda < \alpha < 1$, see Theorem 2.5, such that

$$U^{-1}(U_{\max}) \subset v^{-1}([0, \lambda]) \quad \text{and} \quad \partial\mathcal{B} = v^{-1}(0) \subset U^{-1}([0, \beta U_{\max}]).$$

Following calculations from [8] we compute a lower estimate $[-r_{1-\beta}, r_{1-\beta}]$ for the $(1-\beta)$-BOA of the equilibrium, by solving $U(r_{1-\beta}) = \beta U_{\max}$. Thus $r_{1-\beta} = \beta^2 2^{-1/2}$. Theorem 2.5 requires

$$\rho_1 = v^{-1}(0) \subset U^{-1}([0, \beta U_{\max}]),$$

which is equivalent to $\rho_1 = 10^{-2} < r_{1-\beta} = \beta^2 2^{-1/2}$, i.e. $\beta > 0.1189$. We need to find λ such that

$$U^{-1}(U_{\max}) \subset v^{-1}([0, \lambda])$$

which is equivalent to $V(2^{-1/2}) \leq \lambda$. We now fix $\beta = 0.1247$, $\lambda = 0.0421$ and we are free to choose $\alpha > \lambda$.

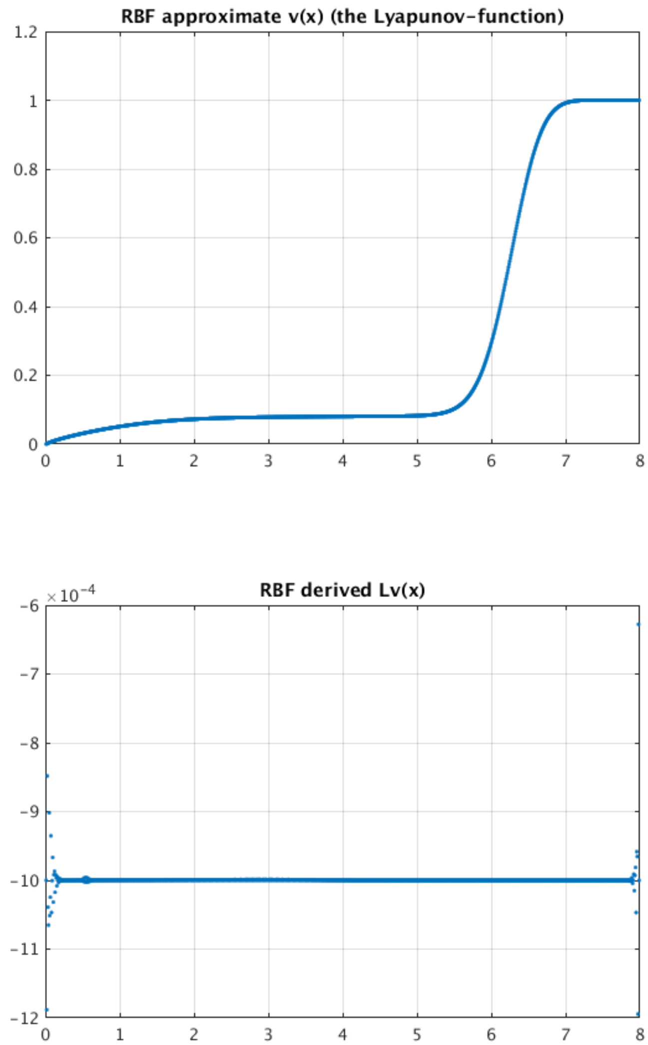


FIGURE 1. Above: the computed non-local Lyapunov function v for system (5.1). Below: the function Lv, approximating -10^{-3}.

Corresponding to our choice of α, we have that the set $v^{-1}([0,\alpha]) \cup \mathcal{B}$ is a subset of the γ-BOA by Theorem 2.5 (note that $b = 0$ and $a = 1$) with

$$\gamma = \frac{(1-\alpha)(1-\beta)}{1-\beta(1-\lambda)}.$$

For $\alpha = 0.044$ we have $V^{-1}([0,\alpha]) \cup \mathcal{B} \approx [-0.803, 0.803]$ and $\gamma \approx 0.95$.

For $\alpha = 0.09$ we have $V^{-1}([0,\alpha]) \cup \mathcal{B} \approx [-5.33, 5.33]$ and $\gamma \approx 0.90$.

Let us compare these results first to the local Lyapunov function: here we obtain $[-0.00177, 0.00177]$ and $[-0.00707, 0.00707]$ as lower estimates of the 0.95- and 0.90-BOAs. By comparing those values with the estimates obtained above we see a very substantial increase.

Our results are comparable to the results in [8] in that we obtained similarly sized γ-BOA, however, our method includes a rigorous verification (numerical proof) that v is indeed a non-local Lyapunov function. This verification is missing in [8] and one can only hope that the computed non-local function is a Lyapunov function for the system.

Lastly, we set up a simple Monte-Carlo simulation using the First-order stochastic Runge-Kutta method to generate 1000 approximate realisations of sample paths, starting at the point $x = 5.33$. We then check when they leave the interval $[10^{-4}, 8]$ and at which end. The result is that 98% of simulations leave through the inner boundary and 2% through the outer, which is close to what we expected since the point 5.33 is inside the 0.9-BOA. Note that this a larger value than predicted by our method. On the one hand, our estimate is indeed just a lower bound and exiting $[10^{-4}, 8]$ through the lower boundary is not the same as the sample trajectories converging to the origin as time tends to infinity. It confirms, however, the validity of our estimate.

5.2. **Two-dimensional example.** We consider the first example from [3, Section 4], namely

$$d\mathbf{x} \quad = \quad (M + \rho(\mathbf{x})I)\mathbf{x}dt + \mathbf{g}(\mathbf{x})dW, \tag{5.2}$$

where W is a one-dimensional Wiener-process, I is the 2×2 identity matrix, and with

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \rho(\mathbf{x}) = \|\mathbf{x}\| - 1, \text{ and } \mathbf{g}(\mathbf{x}) = \theta\|\mathbf{x}\| \left(\|\mathbf{x}\| - \frac{1}{2}\right)\left(\|\mathbf{x}\| - \frac{3}{2}\right)\mathbf{x}.$$

To assert the existence of unique strong solutions we use these formulas for $\mathbf{f}(\mathbf{x}) = (M + \rho(\mathbf{x})I)\mathbf{x}$ and $\mathbf{g}(\mathbf{x})$ inside of a ball, centered at the origin and with radius 4 and outside of this ball we extend $\mathbf{f}$ and $\mathbf{g}$ as Lipschitz functions. For this SDE the generator is

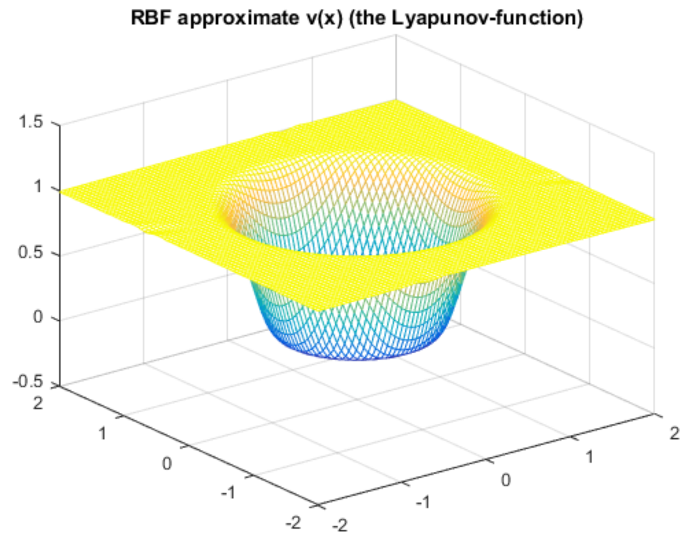
$$L := \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}(\mathbf{x})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{2} f_i(\mathbf{x})\frac{\partial}{\partial x_i}, \text{ where } \ a(\mathbf{x}) := \mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^\top.$$

By solving the continuous time Lyapunov equation $J^\top P + PJ = -2I$ for the deterministic linearised system

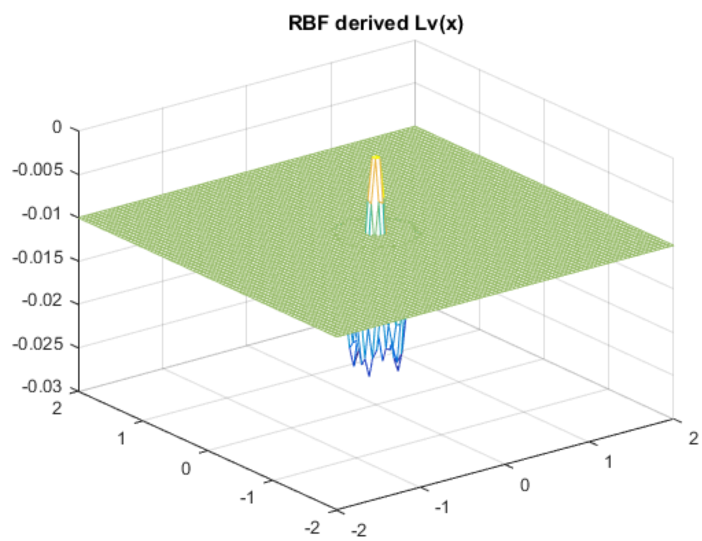
$$\mathbf{x}' = J\mathbf{x} \ \text{ with } \ J = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} = D\mathbf{f}(\mathbf{0}),$$

we get the Lyapunov function $U(\mathbf{x}) = \|\mathbf{x}\|^2$. For our system this delivers with $\mathbf{x} = (x, y)$:

$$\begin{aligned} & U(\mathbf{x}) \\ = \quad & \frac{1}{2}\theta^2\|\mathbf{x}\|^2\left(\|\mathbf{x}\| - \frac{1}{2}\right)^2\left(\|\mathbf{x}\| - \frac{3}{2}\right)^2\left(x^2 \cdot 2 + xy \cdot 0 + yx \cdot 0 + y^2 \cdot 2\right) \\ & +[(\|\mathbf{x}\| - 1)x + y] \cdot 2x + [-x + (\|\mathbf{x}\| - 1)y]2y \\ = \quad & \theta^2\|\mathbf{x}\|^4\left(\|\mathbf{x}\| - \frac{1}{2}\right)^2\left(\|\mathbf{x}\| - \frac{3}{2}\right)^2 + (\|\mathbf{x}\| - 1)(2x^2 + 2y^2) \\ = \quad & -\|\mathbf{x}\|^2\left(2 - 2\|\mathbf{x}\| - \theta^2\|\mathbf{x}\|^2\left(\|\mathbf{x}\| - \frac{1}{2}\right)^2\left(\|\mathbf{x}\| - \frac{3}{2}\right)^2\right). \end{aligned}$$



(A) The function v for system (5.2).



(B) The function Lv for system (5.2), approximating -10^{-2}.

FIGURE 2. Non-local Lyapunov function for system (5.2) with $\theta = 1$. The non-local Lyapunov functions looks very similar to the one computed in [3].

Set

$$h_\theta(r) = 2 - 2r - \theta^2 r^2 \left(r - \frac{1}{2}\right)^2 \left(r - \frac{3}{2}\right)^2.$$

Then $LU(\mathbf{x}) = -\|\mathbf{x}\|^2 h_\theta(\|\mathbf{x}\|)$ and routine calculations show that on the interval $[0, 1/2]$ the function $r \mapsto r^2(r - \frac{1}{2})^2(r - \frac{3}{2})^2$ takes its largest value at $r^* = (4 -$

$\sqrt{7})/6 \approx 0.22571$ and that $h_\theta(r^*) > 1.55 - 6.3 \cdot 10^{-3}\theta^2$, so for any $0 \le \theta \le 15.56$ the function $U(\mathbf{x}) = \|\mathbf{x}\|^2$ is a Lyapunov function for the system on $B_{1/2}(0)$. It is not difficult to verify that if $0 \le \theta \le 1$, then U is a (local) Lyapunov function on $B_{0.9}(0)$.

Now we calculate the constants for $K = \{\mathbf{x} \in \mathbb{R}^2 : R_1 \le \|\mathbf{x}\| \le R_2\}$ with $R_2 = 2$. We have, see appendix, $F = R_2\sqrt{1+(R_2-1)^2} = 2\sqrt{2}$, $F_1 = \sqrt{12}$, $F_2 = 2$, $G = \frac{9}{2}\theta^2$, $G_1 = 33\theta^2$, and $G_2 = 197.5\theta^2$.

We used the Wendland function $\phi_{8,6}$ with $c = 1$, for the system (5.2) with $\theta = 1$. We choose $\rho_1 = 0.4$ and $\rho_2 = 1.9$ and use a 80×80 grid of collocation points on $[-2,2]\times[-2,2]$ to calculate a non-local Lyapunov function, approximating $LV(\mathbf{x}) = -10^{-2}$, see Figure 2. With $\boldsymbol{\alpha}_1 = 401.4572$ and $\boldsymbol{\alpha}_2 = 5.8372$ we obtain the value $C_{\mathcal{U}} = 4.3220 \cdot 10^{12}$. By evaluating LV on a relatively coarse 1000×1000 grid of points on $[-2,2] \times [-2,2]$, we estimated the maximum value of LV not to exceed -0.005. Hence, we require a checking grid with $h = 3.4013 \cdot 10^{-8}$ and thus we need to evaluate LV at $(1.1760 \cdot 10^8)^2 \approx 10^{16}$ points. Our current software and computer setup is not adequate to complete those calculations in a reasonable time frame, but we note that the verification workload is perfectly parallel which can be used to speed up the calculations. The necessary estimates for these computations are included in the appendix for future reference.

Now similarly to Example 5.1, we have to determine constants $0 < \beta < 1$ and $0 < \lambda < \alpha < 1$ (see Theorem 2.5), such that

$$U^{-1}(U_{\max}) \subset v^{-1}([0,\lambda]) \quad \text{and} \quad \partial\mathcal{B} = v^{-1}(0) \subset U^{-1}([0,\beta U_{\max}]),$$

where $U(\mathbf{x}) = \|\mathbf{x}\|^2$ is the local Lyapunov function on $B_{1/2}(0)$. We calculate a lower estimate for the $(1-\beta)$-BOA, $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \le r_{1-\beta}\}$, of the equilibrium by choosing $\overline{B_{1/2}(0)} = U^{-1}([0,U_{\max}])$, i.e. $U_{\max} = 1/4$, and solving $\overline{B_{r_{1-\beta}}(0)} = U^{-1}([0,\beta U_{\max}])$. Thus $r_{1-\beta} = \frac{\sqrt{\beta}}{2}$. Theorem 2.5 requires

$$v^{-1}(0) \subset U^{-1}([0,\beta U_{\max}])$$

which is equivalent to $0.4 \le r_{1-\beta} = \sqrt{\beta}2$, i.e. $\beta > 0.64$. Now we need to find λ such that

$$U^{-1}(U_{\max}) \subset v^{-1}([0,\lambda]).$$

We now fix $\beta = 0.65$, $\lambda = 0.005$ and we are free to choose $\alpha > \lambda$. Corresponding to our choice of α we have that the set $v^{-1}((0,\alpha))$ is a subset of the γ-BOA by Theorem 2.5 (with $b = 0$ and $a = 0$) with

$$\gamma = \frac{(1-\alpha)(1-\beta)}{1-\beta(1-\lambda)}.$$

For $\alpha = 0.01$ we have $v^{-1}([0,\alpha]) \cup \mathcal{B} \approx B_{0.6454}(0)$ and $\gamma \approx 0.9809$

For $\alpha = 0.09$ we have $v^{-1}([0,\alpha]) \cup \mathcal{B} \approx B_{0.839}(0)$ and $\gamma \approx 0.90$.

Let us compare these results to the local Lyapunov function: Here we obtain $B_{0.0707}(0)$ and $B_{0.1581}(0)$ as lower estimates of the 0.98 and 0.90-BOAs. By comparing the estimates above we see a substantial increase.

Our results are comparable to the results in [3] in that we obtained similarly sized γ-BOA. However, our method includes a framework for rigorous verification (numerical proof) that v is indeed a non-local Lyapunov function, although this verification could not be performed at this point due to its huge computational demand.

Acknowledgment. This research and, in particular, Björnsson and Gudmundsson were supported by The Icelandic Research Fund with the project *Lyapunov Methods and Stochastic Stability* (grant nr. 152429-051).

Appendix A. **Explicit formulas for meshless collocation.** We calculate $v(\mathbf{x})$, $Lv(\mathbf{x})$, and the collocation matrix A for the specific operator L given in (3.3). We denote recursively $\psi_{i+1}(r) = \frac{1}{r}\frac{\partial}{\partial r}\psi_i(r)$ for $i = 0, 1, \ldots, 5$ and $\psi_0 = \psi$, where ψ is a certain Wendland functions that can be found below in Tables 1 and 2. Recall that $\|\cdot\| = \|\cdot\|_2$.

We have, see (3.5), that

$$\begin{aligned} v(\mathbf{x}) \quad = \quad & \sum_{k=1}^{N} \alpha_k \Big[-\psi_1(\|\mathbf{x}-\mathbf{x}_k\|)\langle \mathbf{x}-\mathbf{x}_k, f(\mathbf{x}_k)\rangle \\ & \qquad + \frac{1}{2}\sum_{i,j=1}^{d} m_{ij}(\mathbf{x}_k)[\psi_2(\|\mathbf{x}-\mathbf{x}_k\|)(\mathbf{x}-\mathbf{x}_k)_i(\mathbf{x}-\mathbf{x}_k)_j \\ & \qquad\qquad + \delta_{ij}\psi_1(\|\mathbf{x}-\mathbf{x}_k\|)]\Big] \\ & + \sum_{k=1}^{M} \alpha_{N+k}\psi_0(\|\mathbf{x}-\boldsymbol{\xi}_k\|). \end{aligned} \tag{A.1}$$

The formula for $Lv(x)$ is, abbreviating $\boldsymbol{\beta} = \mathbf{x} - \mathbf{x}_k$,

$$\begin{aligned} & Lv(\mathbf{x}) \\ = \quad & \sum_{k=1}^{N} \alpha_k \Big\{ -\psi_2(\|\boldsymbol{\beta}\|)\langle \boldsymbol{\beta}, f(\mathbf{x})\rangle\langle \boldsymbol{\beta}, f(\mathbf{x}_k)\rangle - \psi_1(\|\boldsymbol{\beta}\|)\langle f(\mathbf{x}), f(\mathbf{x}_k)\rangle \\ & + \frac{1}{2}\sum_{i,j=1}^{d} m_{ij}(\mathbf{x}_k)\Big[\psi_3(\|\boldsymbol{\beta}\|)\langle \boldsymbol{\beta}, f(\mathbf{x})\rangle\beta_i\beta_j + \psi_2(\|\boldsymbol{\beta}\|)f_j(\mathbf{x})\beta_i \\ & \qquad + \psi_2(\|\boldsymbol{\beta}\|)f_i(\mathbf{x})\beta_j + \delta_{ij}\psi_2(\|\boldsymbol{\beta}\|)\langle \boldsymbol{\beta}, f(\mathbf{x})\rangle\Big] \\ & + \frac{1}{2}\sum_{i,j=1}^{d} m_{ij}(\mathbf{x})\Big[-\psi_3(\|\boldsymbol{\beta}\|)\langle \boldsymbol{\beta}, f(\mathbf{x}_k)\rangle\beta_i\beta_j - \psi_2(\|\boldsymbol{\beta}\|)f_j(\mathbf{x}_k)\beta_i \\ & \qquad - \psi_2(\|\boldsymbol{\beta}\|)f_i(\mathbf{x}_k)\beta_j - \delta_{ij}\psi_2(\|\boldsymbol{\beta}\|)\langle \boldsymbol{\beta}, f(\mathbf{x}_k)\rangle\Big] \\ & + \frac{1}{4}\sum_{r,s=1}^{d}\sum_{i,j=1}^{d} m_{rs}(\mathbf{x})m_{ij}(\mathbf{x}_k)\Big[\psi_4(\|\boldsymbol{\beta}\|)\beta_i\beta_j\beta_r\beta_s \\ & \qquad + \psi_3(\|\boldsymbol{\beta}\|)[\delta_{ij}\beta_r\beta_s + \delta_{ir}\beta_j\beta_s \\ & \qquad\qquad + \delta_{is}\beta_j\beta_r + \delta_{jr}\beta_i\beta_s + \delta_{js}\beta_i\beta_r + \delta_{rs}\beta_i\beta_j] \\ & \qquad + \psi_2(\|\boldsymbol{\beta}\|)[\delta_{ij}\delta_{rs} + \delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}]\Big]\Big\} \\ & + \sum_{k=1}^{M} \alpha_{N+k}\Big\{ -\psi_1(\|\boldsymbol{\xi}_k - \mathbf{x}\|)\langle \boldsymbol{\xi}_k - \mathbf{x}, f(\mathbf{x})\rangle \end{aligned}$$

$$+\frac{1}{2}\sum_{i,j=1}^{d} m_{ij}(\mathbf{x})[\psi_2(\|\boldsymbol{\xi}_k-\mathbf{x}\|)(\boldsymbol{\xi}_k-\mathbf{x})_i(\boldsymbol{\xi}_k-\mathbf{x})_j$$
$$\left.+\delta_{ij}\psi_1(\|\boldsymbol{\xi}_k-\mathbf{x}\|)]\right\}.$$

The formulas for the matrix elements are

$$\begin{aligned}
d_{kl} &= \psi_0(\|\boldsymbol{\xi}_k-\boldsymbol{\xi}_l\|),\\
c_{kl} &= -\psi_1(\|\boldsymbol{\xi}_l-\mathbf{x}_k\|)\langle\boldsymbol{\xi}_l-\mathbf{x}_k, f(\mathbf{x}_k)\rangle\\
&\quad+\frac{1}{2}\sum_{i,j=1}^{d} m_{ij}(\mathbf{x}_k)[\psi_2(\|\boldsymbol{\xi}_l-\mathbf{x}_k\|)(\boldsymbol{\xi}_l-\mathbf{x}_k)_i(\boldsymbol{\xi}_l-\mathbf{x}_k)_j\\
&\quad+\delta_{ij}\psi_1(\|\boldsymbol{\xi}_l-\mathbf{x}_k\|)],
\end{aligned}$$

and, abbreviating $\boldsymbol{\beta}=\mathbf{x}_k-\mathbf{x}_l$,

$$\begin{aligned}
b_{kl} &= -\psi_2(\|\boldsymbol{\beta}\|)\langle\boldsymbol{\beta}, f(\mathbf{x}_k)\rangle\langle\boldsymbol{\beta}, f(\mathbf{x}_l)\rangle-\psi_1(\|\boldsymbol{\beta}\|)\langle f(\mathbf{x}_k), f(\mathbf{x}_l)\rangle\\
&\quad+\frac{1}{2}\sum_{i,j=1}^{d} m_{ij}(\mathbf{x}_l)\Big[\psi_3(\|\boldsymbol{\beta}\|)\langle\boldsymbol{\beta}, f(\mathbf{x}_k)\rangle\beta_i\beta_j+\psi_2(\|\boldsymbol{\beta}\|)f_j(\mathbf{x}_k)\beta_i\\
&\qquad+\psi_2(\|\boldsymbol{\beta}\|)f_i(\mathbf{x}_k)\beta_j+\delta_{ij}\psi_2(\|\boldsymbol{\beta}\|)\langle\boldsymbol{\beta}, f(\mathbf{x}_k)\rangle\Big]\\
&\quad+\frac{1}{2}\sum_{i,j=1}^{d} m_{ij}(\mathbf{x}_k)\Big[-\psi_3(\|\boldsymbol{\beta}\|)\langle\boldsymbol{\beta}, f(\mathbf{x}_l)\rangle\beta_i\beta_j-\psi_2(\|\boldsymbol{\beta}\|)f_j(\mathbf{x}_l)\beta_i\\
&\qquad-\psi_2(\|\boldsymbol{\beta}\|)f_i(\mathbf{x}_l)\beta_j-\delta_{ij}\psi_2(\|\boldsymbol{\beta}\|)\langle\boldsymbol{\beta}, f(\mathbf{x}_l)\rangle\Big]\\
&\quad+\frac{1}{4}\sum_{r,s=1}^{d}\sum_{i,j=1}^{d} m_{rs}(\mathbf{x}_k)m_{ij}(\mathbf{x}_l)\Big[\psi_4(\|\boldsymbol{\beta}\|)\beta_i\beta_j\beta_r\beta_s\\
&\qquad+\psi_3(\|\boldsymbol{\beta}\|)[\delta_{ij}\beta_r\beta_s+\delta_{ir}\beta_j\beta_s+\delta_{is}\beta_j\beta_r\\
&\qquad\quad+\delta_{jr}\beta_i\beta_s+\delta_{js}\beta_i\beta_r+\delta_{rs}\beta_i\beta_j]\\
&\qquad+\psi_2(\|\boldsymbol{\beta}\|)[\delta_{ij}\delta_{rs}+\delta_{ir}\delta_{js}+\delta_{is}\delta_{jr}]\Big].
\end{aligned}$$

Appendix B. **Two-dimensional example.** In this section we give the details of the estimates for F_i and G_i of the 2-dimensional example from Section 5.2.

With $\mathbf{f}(x_1,x_2)=\begin{pmatrix}(\|\mathbf{x}\|-1)x_1+x_2\\ -x_1+(\|\mathbf{x}\|-1)x_2\end{pmatrix}$ we obtain

$$\begin{aligned}
F &= \Big((\|\mathbf{x}\|-1)^2x_1^2+x_2^2+2x_1x_2(\|\mathbf{x}\|-1)\\
&\qquad+x_1^2-2x_1x_2(\|\mathbf{x}\|-1)+x_2^2(\|\mathbf{x}\|-1)^2\Big)^{1/2}\\
&= \|\mathbf{x}\|\sqrt{(\|\mathbf{x}\|-1)^2+1},\\
\frac{\partial\mathbf{f}}{\partial x_1} &= \begin{pmatrix}\frac{x_1^2}{\|\mathbf{x}\|}+\|\mathbf{x}\|-1\\ -1+\frac{x_1x_2}{\|\mathbf{x}\|}\end{pmatrix}=\begin{pmatrix}\frac{2x_1^2+x_2^2}{\|\mathbf{x}\|}-1\\ -1+\frac{x_1x_2}{\|\mathbf{x}\|}\end{pmatrix},
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathbf{f}}{\partial x_2} &= \begin{pmatrix} \frac{x_1 x_2}{\|\mathbf{x}\|} + 1 \\ \frac{x_2^2}{\|\mathbf{x}\|} + \|\mathbf{x}\| - 1 \end{pmatrix} = \begin{pmatrix} \frac{x_1 x_2}{\|\mathbf{x}\|} + 1 \\ \frac{x_1^2 + 2x_2^2}{\|\mathbf{x}\|} - 1 \end{pmatrix}, \\
\left\| \frac{\partial \mathbf{f}}{\partial x_2} \right\|^2 &= \frac{x_1^2 x_2^2 + x_1^4 + 4x_1^2 x_2^2 + 4x_2^4}{\|\mathbf{x}\|^2} + \frac{2x_1 x_2 - 2x_1^2 - 4x_2^2}{\|\mathbf{x}\|} + 2 \\
&\leq x_1^2 + 4x_2^2 - \frac{x_1^2 + 3x_2^2}{\|\mathbf{x}\|} + 2 \\
&= x_1^2 \left(1 - \frac{1}{\|\mathbf{x}\|}\right) + x_2^2 \left(4 - \frac{3}{\|\mathbf{x}\|}\right) + 2 \\
&\leq x_1^2 \left(1 - \frac{1}{R_2}\right) + x_2^2 \left(4 - \frac{3}{R_2}\right) + 2 \\
&\leq R_2^2 \max\left(0, \left(1 - \frac{1}{R_2}\right), \left(4 - \frac{3}{R_2}\right)\right) + 2, \\
\frac{\partial^2 \mathbf{f}}{\partial x_1^2} &= \begin{pmatrix} \frac{x_1(2x_1^2 + 3x_2^2)}{\|\mathbf{x}\|^3} \\ \frac{x_2^3}{\|\mathbf{x}\|^3} \end{pmatrix}, \\
\frac{\partial^2 \mathbf{f}}{\partial x_1 \partial x_2} &= \begin{pmatrix} \frac{x_2^3}{\|\mathbf{x}\|^3} \\ \frac{x_1^3}{\|\mathbf{x}\|^3} \end{pmatrix}, \\
\frac{\partial^2 \mathbf{f}}{\partial x_2^2} &= \begin{pmatrix} \frac{x_1^3}{\|\mathbf{x}\|^3} \\ \frac{x_2(3x_1^2 + 2x_2^2)}{\|\mathbf{x}\|^3} \end{pmatrix}, \quad \text{and} \\
\left\| \frac{\partial^2 \mathbf{f}}{\partial x_2^2} \right\| &= \frac{\sqrt{x_1^6 + 9x_1^4 x_2^2 + 12x_1^2 x_2^4 + 4x_2^6}}{\|\mathbf{x}\|^3} \\
&\leq \frac{\sqrt{4x_1^6 + 12x_1^4 x_2^2 + 12x_1^2 x_2^4 + 4x_2^6}}{\|\mathbf{x}\|^3} \\
&= \frac{\sqrt{4(x_1^2 + x_2^2)^3}}{\|\mathbf{x}\|^3} = 2.
\end{aligned}
$$

We now calculate the estimates for $\mathbf{g}(\mathbf{x}) = \theta r(r-0.5)(r-1.5)\mathbf{x}$, denoting $\|\mathbf{x}\| = r$. For $r \in [0, 2]$ we have

$$
\|\mathbf{g}(\mathbf{x})\| \leq \theta 4 \cdot \frac{3}{2} \cdot \frac{1}{2} = 3\theta.
$$

Furthermore,

$$
\frac{\partial \mathbf{g}}{\partial x_i} = \theta \left\{ \frac{x_i}{\|\mathbf{x}\|} \left[3r^2 - 4r + \frac{3}{4} \right] \mathbf{x} + r(r-0.5)(r-1.5)\mathbf{e}_i \right\}
$$

Hence,

$$
\begin{aligned}
\left\| \frac{\partial \mathbf{g}}{\partial x_i} \right\| &\leq \theta \left(\left| 3r^2 - 4r + \frac{3}{4} \right| r + |r(r-0.5)(r-1.5)| \right) \\
&\leq 11\theta
\end{aligned}
$$

for $r \in [0, 2]$.

Finally,

$$\begin{aligned}
\frac{\partial^2 \mathbf{g}}{\partial x_1^2} &= \theta\left[3r^2-4r+\frac{3}{4}\right]\left(\frac{x_2^2}{\|\mathbf{x}\|^3}\mathbf{x}+2\frac{x_1}{\|\mathbf{x}\|}\mathbf{e}_1\right)+\theta\frac{x_1^2}{\|\mathbf{x}\|^2}(6r-4)\mathbf{x}\\
&= \frac{\theta}{r^3}\left[\left(3r^2-4r+\frac{3}{4}\right)x_2^2+(6r^2-4r)x_1^2\right]\mathbf{x}\\
&\quad +2\theta\left(3r^2-4r+\frac{3}{4}\right)\frac{x_1}{\|\mathbf{x}\|}\mathbf{e}_1,\\
\left\|\frac{\partial^2 \mathbf{g}}{\partial x_1^2}\right\| &\le \theta\max\left(\left|3r^2-4r+\frac{3}{4}\right|,\left|6r^2-4r\right|\right)\\
&\quad +2\theta\left|3r^2-4r+\frac{3}{4}\right|\\
&\le 25.5\theta \text{ for } r\in[0,2],\\
\frac{\partial^2 \mathbf{g}}{\partial x_1\partial x_2} &= \theta\left[3r^2-4r+\frac{3}{4}\right]\left(\frac{-x_1x_2}{\|\mathbf{x}\|^3}\mathbf{x}+\frac{x_2}{\|\mathbf{x}\|}\mathbf{e}_1+\frac{x_1}{\|\mathbf{x}\|}\mathbf{e}_2\right)\\
&\quad +\theta\frac{x_1x_2}{\|\mathbf{x}\|^2}(6r-4)\mathbf{x}\\
&= \frac{\theta}{r^3}\left[-3r^2+4r-\frac{3}{4}+6r^2-4r\right]x_1x_2\mathbf{x}\\
&\quad +\frac{\theta}{r}\left(3r^2-4r+\frac{3}{4}\right)(x_2\mathbf{e}_1+x_1\mathbf{e}_2), \quad \text{and}\\
\left\|\frac{\partial^2 \mathbf{g}}{\partial x_1\partial x_2}\right\| &\le \theta\left|3r^2-\frac{3}{4}\right|+\theta\left|3r^2-4r+\frac{3}{4}\right|\\
&\le 16\theta.
\end{aligned}$$

Appendix C. **Wendland functions.** In this appendix we give the explicit formulas of the Wendland functions $\phi_{8,6}$ and $\phi_{7,6}$ as well as the corresponding auxiliary functions ψ_i, $i=1,\dots,6$. Furthermore, we give the relevant estimates for $\psi_{k,i}=\sup_{r\in[0,\infty)}|\psi_k(r)|r^i$.

In particular, in Table 1 and Table 2, we give the formulas for the Wendland function $\psi_0(r)=\phi_{8,6}(cr)$ and $\psi_0(r)=\phi_{7,6}(cr)$, respectively, as well as ψ_i, $i=1,\dots,6$. In Table 3 we give the formulas for the expressions $\psi_{k,i}=\sup_{r\in[0,\infty)}\psi_k(r)r^i$, required for the estimates for the same Wendland functions $\phi_{8,6}$ and $\phi_{7,6}$. Note that $x_+ := \max\{x,0\}$.

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	$\phi_{8,6}$
$\psi_0(r)$	$[46,189(cr)^6 + 73,206(cr)^5 + 54,915(cr)^4 + 24,500(cr)^3$ $+6,755(cr)^2 + 1,078cr + 77]\,(1-cr)_+^{14}$
$\psi_1(r)$	$-380\,c^2\,[2,431(cr)^5 + 2,931(cr)^4 + 1,638(cr)^3 + 518(cr)^2$ $+91cr + 7]\,(1-cr)_+^{13}$
$\psi_2(r)$	$12,920\,c^4\,[1,287(cr)^4 + 1,108(cr)^3 + 426(cr)^2$ $+84cr + 7]\,(1-cr)_+^{12}$
$\psi_3(r)$	$-620,160\,c^6\,[429(cr)^3 + 239(cr)^2 + 55cr + 5]\,(1-cr)_+^{11}$
$\psi_4(r)$	$112,869,120\,c^8\,[33(cr)^2 + 10cr + 1]\,(1-cr)_+^{10}$
$\psi_5(r)$	$-4,966,241,280\,c^{10}\,[9cr + 1]\,(1-cr)_+^{9}$
$\psi_6(r)$	$446,961,715,200\,c^{12}\,(1-cr)_+^{8}$

TABLE 1. The table shows the Wendland function $\psi_0(r) := \phi_{8,6}(cr)$ as well as the related functions ψ_1 to ψ_6, defined recursively by $\psi_{k+1}(r) := \frac{\partial_r \psi_k(r)}{r}$ for $k = 0, 1, \ldots, 5$.

	$\phi_{7,6}$
$\psi_0(r)$	$[4,096(cr)^6 + 7,059(cr)^5 + 5,751(cr)^4 + 2,782(cr)^3 + 830(cr)^2$ $+143cr + 11]\,(1-cr)_+^{13}$
$\psi_1(r)$	$-38\,c^2\,[2,048(cr)^5 + 2,697(cr)^4 + 1,644(cr)^3 + 566(cr)^2$ $+108cr + 9]\,(1-cr)_+^{12}$
$\psi_2(r)$	$10,336\,c^4\,[128(cr)^4 + 121(cr)^3 + 51(cr)^2 + 11cr + 1]\,(1-cr)_+^{11}$
$\psi_3(r)$	$-62,016\,c^6\,[320(cr)^3 + 197(cr)^2 + 50cr + 5]\,(1-cr)_+^{10}$
$\psi_4(r)$	$3,224,832\,c^8\,[80(cr)^2 + 27cr + 3]\,(1-cr)_+^{9}$
$\psi_5(r)$	$-354,731,520\,c^{10}\,[8cr + 1]\,(1-cr)_+^{8}$
$\psi_6(r)$	$25,540,669,440\,c^{12}\,(1-cr)_+^{7}$

TABLE 2. The table shows the Wendland function $\psi_0(r) := \phi_{7,6}(cr)$ as well as the related functions ψ_1 to ψ_6, defined recursively by $\psi_{k+1}(r) := \frac{\partial_r \psi_k(r)}{r}$ for $k = 0, 1, \ldots, 5$.

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$\psi_{k,i}$	$\phi_{8,6}$	$\phi_{7,6}$
$\psi_{6,6}$	$3.148511062 \cdot 10^7 \cdot c^6$	$3.240130299 \cdot 10^6 \cdot c^6$
$\psi_{5,5}$	$2.363249538 \cdot 10^6 \cdot c^5$	$2.588617377 \cdot 10^5 \cdot c^5$
$\psi_{5,4}$	$6.409097287 \cdot 10^6 \cdot c^6$	$6.534280933 \cdot 10^5 \cdot c^6$
$\psi_{4,4}$	$1.947997580 \cdot 10^5 \cdot c^4$	$2.262550039 \cdot 10^4 \cdot c^4$
$\psi_{4,3}$	$6.000016519 \cdot 10^5 \cdot c^5$	$6.515237949 \cdot 10^4 \cdot c^5$
$\psi_{4,2}$	$2.215560450 \cdot 10^6 \cdot c^6$	$2.237953342 \cdot 10^5 \cdot c^6$
$\psi_{3,3}$	$1.807542870 \cdot 10^4 \cdot c^3$	$2.219149087 \cdot 10^3 \cdot c^3$
$\psi_{3,2}$	$6.618581621 \cdot 10^4 \cdot c^4$	$7.625999381 \cdot 10^3 \cdot c^4$
$\psi_{3,1}$	$3.172360616 \cdot 10^5 \cdot c^5$	$3.414789975 \cdot 10^4 \cdot c^5$
$\psi_{3,0}$	$3.1008 \cdot 10^6 \cdot c^6$	$3.1008 \cdot 10^5 \cdot c^6$
$\psi_{2,2}$	$1.970990855 \cdot 10^3 \cdot c^2$	$2.550970282 \cdot 10^2 \cdot c^2$
$\psi_{2,1}$	$9.418422390 \cdot 10^3 \cdot c^3$	$1.147899628 \cdot 10^3 \cdot c^3$
$\psi_{2,0}$	$9.044 \cdot 10^4 \cdot c^4$	$1.0336 \cdot 10^4 \cdot c^4$
$\psi_{1,1}$	$2.767275907 \cdot 10^2 \cdot c$	$3.766803387 \cdot 10^1 \cdot c$
$\psi_{1,0}$	$2.66 \cdot 10^3 \cdot c^2$	$3.42 \cdot 10^2 \cdot c^2$

TABLE 3. The table shows values for $\psi_{k,i} := \sup_{r\in[0,\infty)} |\psi_i(r)|r^k$ for the Wendland functions $\psi_0(r) := \phi_{8,6}(cr)$ and $\psi_0(r) := \phi_{7,6}(cr)$.

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Received April 2018; revised October 2018.

E-mail address: hjb6@hi.is
E-mail address: P.A.Giesl@sussex.ac.uk
E-mail address: skgu@sek.se
E-mail address: shafstein@hi.is
E-mail address: E.Scalas@sussex.ac.uk