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COMPUTATION OF LYAPUNOV FUNCTIONS FOR SYSTEMS WITH MULTIPLE LOCAL ATTRACTORS

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Abstract. We present a novel method to compute Lyapunov functions for continuous-time systems with multiple local attractors. In the proposed method one first computes an outer approximation of the local attractors using a graph-theoretic approach. Then a candidate Lyapunov function is computed using a Massera-like construction adapted to multiple local attractors. In the final step this candidate Lyapunov function is interpolated over the simplices of a simplicial complex and, by checking certain inequalities at the vertices of the complex, we can identify the region in which the Lyapunov function is decreasing along system trajectories. The resulting Lyapunov function gives information on the qualitative behavior of the dynamics, including lower bounds on the basins of attraction of the individual local attractors. We develop the theory in detail and present numerical examples demonstrating the applicability of our method.

1. **Introduction.** The decomposition of the flow of a dynamical system into a chain-recurrent part and a part where the flow is gradient-like is characterized by a so-called complete Lyapunov function for the system [6, 13, 21]. This decomposition is sometimes referred to as the *Fundamental Theorem of Dynamical Systems* [19]. A complete Lyapunov function V for a dynamical system is a continuous function that is decreasing along trajectories of the system in the gradient-like part of the

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flow and each chain transitive component of the chain-recurrent part is contained in a preimage $V^{-1}(c)$ for some constant $c \geq 0$.

In this paper we consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is r-times differentiable and $n, r \in \mathbb{N}$; i.e., $\mathbf{f}$ is C^r on $\mathbb{R}^n$. We denote the solution of system (1.1) with initial value $\boldsymbol{\xi}$ at time zero by $t \mapsto \boldsymbol{\phi}(t, \boldsymbol{\xi})$ and we assume that it is defined for all $t \geq 0$. Note that $\boldsymbol{\phi}(0, \boldsymbol{\xi}) = \boldsymbol{\xi}$.

In this paper, we are interested in studying local attractors and their respective basins of attraction; these local attractors could, e.g., be components of the global attractor. If some information about one or several local attractors is available, one approach to constructing local Lyapunov functions for local attractors is to simply deal with each attractor independently. At least for local attractors that are topologically equivalent to points, previous work on computing Lyapunov functions for exponentially or asymptotically stable equilibrium points on a compact domain [8, 10, 11, 12, 20, 22, 23] could be modified in a straightforward manner to replace the equilibrium point with a local attractor, and then apply the chosen method on a compact domain within the basin of attraction of the attractor.

A more complete approach involves both estimating all local attractors and repellers for (1.1), and computing something close to a complete Lyapunov function. Such an approach was proposed in [2, 15], where a discretization of both the phase space and the system dynamics are used to generate a transition graph for how the system evolves which, in turn, can be used to approximate a complete Lyapunov function for the system. Furthermore, these algorithms were implemented and tested for simple systems in [2]. An apparent drawback of this approach for continuous-time systems is that choosing a good time step for discretization to simultaneously obtain a good approximation of all local attractors and repellers as well as computing the Lyapunov function can be quite difficult. In general it seems that the time step needs to be "large" in order to approximate the local attractors and repellers, but "small" in order to give a sufficiently good approximation to the Lyapunov function. In Section 5 we present a technique similar to the one presented in [15] in order to approximate only the local attractors of (1.1) in a given domain.

With an estimate of the local attractors available, the next task is to construct a Lyapunov function on some subset of the basin of attraction for each local attractor. We do this using a Continuous and Piecewise Affine (CPA) approximation to a Lyapunov function construction in a converse Lyapunov theorem. Such an approach has been used in [5, 12] for (1.1) where the origin is an asymptotically stable equilibrium point. Where [12] uses a particular Lyapunov function construction due to Yoshizawa [27], [5] uses a Lyapunov function construction due to Massera [18]. Here, we will demonstrate a converse Lyapunov theorem for multiple local attractors using a construction similar to [5, 18]. The Lyapunov function construction at each point $\mathbf{x} \in \mathbb{R}^n$ of this converse theorem is dependent on solutions of (1.1) with the initial condition $\mathbf{x} \in \mathbb{R}^n$. Therefore, at each vertex of a simplicial complex we numerically solve an initial-value problem and then interpolate the numerically computed values at each vertex to obtain a CPA function. Using recently derived results on CPA functions [10, 11, 12] we then check a system of linear inequalities to verify that the CPA function is indeed a Lyapunov function.

The paper is organized as follows. In Section 2 we present the essential sufficient Lyapunov-based conditions required to prove convergence to a neighborhood of a

local attractor from some (large) subset of its basin of attraction. In Section 3 we provide a converse theorem to the sufficient conditions in Section 2. In Section 4 we summarize some results on approximating Lyapunov functions by Continuous and Piecewise Affine (CPA) functions and in Section 5 we describe a method for approximating the local attractors for (1.1). Finally, in Section 6 we apply our constructive techniques to three particular nonlinear systems to demonstrate the utility of this approach.

2. Lyapunov functions for compact invariant sets.

In this section we provide necessary notation and define a Lyapunov function for a general compact invariant set Ω, not only an equilibrium. Moreover, in contrast to a classical Lyapunov function, we do not assume that the function is decreasing along solution trajectories everywhere outside the compact invariant set, but allow for it to be nonnegative on a larger set, a neighborhood $\mathcal{F}$ of Ω. We generalize this to a Lyapunov function for several compact invariant sets, with the goal of later constructing such a Lyapunov function.

For a set $\mathcal{D} \subset \mathbb{R}^n$, we denote the interior of $\mathcal{D}$ by $\mathcal{D}^\circ$, the closure of $\mathcal{D}$ by $\overline{\mathcal{D}}$, the boundary of $\mathcal{D}$ by $\partial\mathcal{D}$, and the complement of $\mathcal{D}$ by $\mathcal{D}^C$. For a vector $\mathbf{x} \in \mathbb{R}^n$, we denote the 2-norm by $|\mathbf{x}|$, the 1-norm by $|\mathbf{x}|_1$, the maximum-norm by $|\mathbf{x}|_\infty$, and for a matrix $A \in \mathbb{R}^{n\times n}$ we denote its spectral norm by $\|A\| := \max_{|\mathbf{x}|=1} |A\mathbf{x}|$. For an ordered tuple $(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k)$ of vectors in $\mathbb{R}^n$ we define their convex combination as

$$\operatorname{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k) := \left\{ \sum_{i=0}^{k} \lambda_i \mathbf{x}_i \ : \ 0 \le \lambda_i \le 1 \text{ for } i = 0, 1, \dots, k \text{ and } \sum_{i=0}^{k} \lambda_i = 1 \right\}.$$

If the vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k$ are affinely independent, i.e. the vectors $\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \dots, \mathbf{x}_k - \mathbf{x}_0$ are linearly independent, then the set $\operatorname{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k)$ is called a k-simplex. We denote the positive real numbers by $\mathbb{R}_{>0}$ and the nonnegative real numbers by $\mathbb{R}_{\ge 0}$. Given $\varepsilon \in \mathbb{R}_{>0}$ we define $\mathcal{B}_\varepsilon := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < \varepsilon\}$. We denote the distance from a point $\mathbf{x} \in \mathbb{R}^n$ to a set $\Omega \subset \mathbb{R}^n$ by $\operatorname{dist}(\mathbf{x}, \Omega) := \inf_{\mathbf{y}\in\Omega} |\mathbf{x} - \mathbf{y}|$ and we denote the diameter of Ω by $\operatorname{diam}(\Omega) := \sup_{\mathbf{x},\mathbf{y}\in\Omega} |x - y|$. For sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$ we write the Minkowski sum $\mathcal{A}+\mathcal{B} = \{\mathbf{x}+\mathbf{y} \mid \mathbf{x} \in \mathcal{A}, \mathbf{y} \in \mathcal{B}\}$. We denote the empty set by $\emptyset$ and the power set of a set $\mathcal{A}$ by $P(\mathcal{A})$. Finally, for a mapping $f : \mathcal{A} \to \mathcal{A}$ we define f^ℓ as the ℓ-fold composition of f, i.e. $f^1 = f$ and $f^{(\ell+1)} = f \circ f^\ell$.

For $\mathcal{M} \subset \mathbb{R}^n$, the orbital (upper right) Dini derivative of a locally Lipschitz function $V : \mathcal{M} \to \mathbb{R}_{\ge 0}$ along the solution trajectories of (1.1) is defined at every $\mathbf{x} \in \mathcal{M}^\circ$ by

$$D^+V(\mathbf{x}, \mathbf{f}(\mathbf{x})) := \limsup_{h\to 0+} \frac{V(\mathbf{x} + h\mathbf{f}(\mathbf{x})) - V(\mathbf{x})}{h}. \tag{2.1}$$

Since V is locally Lipschitz, it is differentiable almost everywhere and, where V is differentiable $D^+V(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$. If V is continuously differentiable, then the orbital Dini derivative coincides with the usual orbital derivative everywhere.

For a fixed compact set $\Omega \subset \mathbb{R}^n$ we define the set $\mathfrak{N}(\Omega)$ of certain neighborhoods of Ω that we will repeatedly use in this paper.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a compact set. Denote by $\mathfrak{N}(\Omega)$ the set of all subsets $\mathcal{D} \subset \mathbb{R}^n$ that fulfill:

(i) $\mathcal{D}$ is compact.
(ii) The interior $\mathcal{D}^\circ$ of $\mathcal{D}$ is a connected open neighborhood of Ω.

(iii) $\mathcal{D} = \overline{\mathcal{D}^\circ}$.

In general, it is well-known that level sets of a Lyapunov function are forward invariant. To facilitate the development of a numerical procedure for computation of Lyapunov functions for systems with multiple local attractors, we will define a region of interest, $\mathcal{D} \subset \mathbb{R}^n$, that contains the local attractor. In order for this to make sense, we explicitly require the existence of level sets with closed level hypersurfaces lying in the interior of $\mathcal{D}$ to guarantee that trajectories do not leave the region of interest. Furthermore, particularly for the development of numerical procedures, we will use an outer approximation $\mathcal{F} \subset \mathbb{R}^n$ of a local attractor Ω, given by $\Omega \subset \mathcal{F}$, and hence we require that closed level hypersurfaces also lie in the complement of $\mathcal{F}$. This motivates the following level set definition.

Definition 2.2. Let Ω be a compact set, $\mathcal{F}, \mathcal{D} \in \mathfrak{N}(\Omega)$, and $\mathcal{F} \subset \mathcal{D}^\circ$. Let $V : \mathcal{D} \setminus \mathcal{F}^\circ \to \mathbb{R}_{\geq 0}$ be a continuous function and let $m \in \mathbb{R}_{>0}$ be a constant. Define the set

$$\mathcal{O}_m := \mathcal{F} \cup \{\mathbf{x} \in \mathcal{D} \setminus \mathcal{F}^\circ : V(\mathbf{x}) < m\} \subset \mathcal{D}.$$

Denote by $\mathcal{O}_{m,\Omega}$ the connected component of $\mathcal{O}_m$ satisfying $\Omega \subset \mathcal{O}_{m,\Omega} \subset \mathcal{O}_m$. If $\mathcal{F} \subset \mathcal{O}_{m,\Omega}^\circ \subset \overline{\mathcal{O}_{m,\Omega}} \subset \mathcal{D}^\circ$ we define the level set $\mathcal{L}_{V,m} := \mathcal{O}_{m,\Omega}$. If no such $\mathcal{O}_{m,\Omega}$ exists we write $\mathcal{L}_{V,m} := \emptyset$. We further define

$$\mathcal{L}_V^{\inf} := \bigcap_{\substack{m \in \mathbb{R}_{>0} \\ \mathcal{L}_{V,m} \neq \emptyset}} \mathcal{L}_{V,m} \qquad \text{and} \qquad \mathcal{L}_V^{\sup} := \bigcup_{m \in \mathbb{R}_{>0}} \mathcal{L}_{V,m}\,.$$

Observe that it is possible that $\mathcal{L}_V^{\inf}$ and $\mathcal{L}_V^{\sup}$ can be empty if $\mathcal{O}_{m,\Omega}$ fails to exist for all $m \in \mathbb{R}_{>0}$. However, when nonempty, $\mathcal{L}_V^{\inf}$ is a closed set, see Theorem 2.5 (e), $\mathcal{L}_V^{\sup}$ is an open set, and $\mathcal{L}_V^{\inf} \subset \mathcal{L}_V^{\sup}$.

We define a Lyapunov function for Ω on $\mathcal{D} \setminus \mathcal{F}^\circ$ that is strictly positive and is such that there is a reasonable region, outside of $\mathcal{F}$, where the Lyapunov function is strictly decreasing. Here, $\mathcal{F}$ ideally is a tight outer approximation of the local attractor Ω, and $\mathcal{D}$ is tight inner approximation of its basin of attraction $B(\Omega)$.

Definition 2.3. Let $\mathcal{F}, \mathcal{D} \in \mathfrak{N}(\Omega)$ satisfy $\mathcal{F} \subset \mathcal{D}^\circ$. Let $\mathcal{G} \subset \mathbb{R}^n$ satisfy $\mathcal{G} \supseteq \mathcal{D} \setminus \mathcal{F}^\circ$. A Lipschitz function $V : \mathcal{G} \to \mathbb{R}_{\geq 0}$ is called a **Lyapunov function for Ω on $\mathcal{D} \setminus \mathcal{F}^\circ$ for** (1.1) if there exists a constant $\alpha \in \mathbb{R}_{>0}$ such that

(i) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{D} \setminus \mathcal{F}^\circ$,
(ii) $D^+V(\mathbf{x}, \mathbf{f}(\mathbf{x})) \leq -\alpha$ for all $\mathbf{x} \in \mathcal{D}^\circ \setminus \mathcal{F}$, and
(iii) $\mathcal{L}_V^{\inf} \neq \emptyset$.

Note that the definition of a Lyapunov function for (1.1) for a local attractor Ω on a domain $\mathcal{D}$ is similar to the above with the Lyapunov function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ being positive definite with respect to Ω and with a negative definite Dini derivative in the direction of the vector field; i.e., $V(\mathbf{x}) = 0$ for all $\mathbf{x} \in \Omega$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{D} \setminus \Omega$, and $D^+V(\mathbf{x}, \mathbf{f}(\mathbf{x})) < 0$ for all $\mathbf{x} \in \mathcal{D}^\circ \setminus \Omega$. Note that this then implies the existence of an $m > 0$ such that $\mathcal{L}_{V,m} \neq \emptyset$ (with $\mathcal{F} = \Omega$) and hence $\mathcal{L}_V^{\inf}$ and $\mathcal{L}_V^{\sup}$ are nonempty. In particular, since $V(\mathbf{x}) = 0$ for $\mathbf{x} \in \Omega$, $V(\mathbf{x}) > 0$ for $\mathbf{x} \in \mathcal{D}$, and V is continuous, for some $m > 0$ sufficiently small, $\mathcal{L}_{V,m}$ is nonempty. By contrast, in Definition 2.3 since we merely require $V(\mathbf{x}) > 0$ for $\mathbf{x} \in \mathcal{D} \setminus \mathcal{F}^\circ$ and not that $V(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathcal{F}$, we must explicitly assume that $\mathcal{L}_V^{\inf} \neq \emptyset$.

A Lyapunov function provides information about the basin of attraction through its sublevel sets. For a Lyapunov function on $\mathcal{D} \setminus \mathcal{F}^\circ$ as in Definition 2.3 similar statements hold as we show in Theorem 2.5 below.

Definition 2.4. Consider the system (1.1). Let Ω be a compact, invariant subset of $\mathbb{R}^n$. We call Ω a **local attractor** if

(i) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\operatorname{dist}(\mathbf{x}, \Omega) < \delta \quad \Rightarrow \quad \operatorname{dist}(\boldsymbol{\phi}(t, \mathbf{x}), \Omega) < \varepsilon \ \text{ for all } t \geq 0,$$

(ii) the set

$$B(\Omega) := \{\mathbf{x} \in \mathbb{R}^n \ : \ \limsup_{t \to \infty} \operatorname{dist}(\boldsymbol{\phi}(t, \mathbf{x}), \Omega) = 0\}$$

is an open neighborhood of Ω.

$B(\Omega)$ is called the **basin of attraction** of Ω.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a local attractor with basin of attraction $B(\Omega)$. Let $\mathcal{F}, \mathcal{D} \in \mathfrak{N}(\Omega)$, $\mathcal{F} \subset \mathcal{D}^\circ$, and let V be a Lyapunov function for Ω on $\mathcal{D} \setminus \mathcal{F}^\circ$. Let $m \in \mathbb{R}_{>0}$ be a constant such that $\mathcal{L}_{V,m} \neq \emptyset$. Then:*

(a) $\mathcal{L}_{V,m}$, $\mathcal{L}_V^{\inf}$, and $\mathcal{L}_V^{\sup}$ are forward invariant sets.
(b) There is a constant $T > 0$ such that $\boldsymbol{\xi} \in \mathcal{L}_V^{\sup}$ implies $\boldsymbol{\phi}(T, \boldsymbol{\xi}) \in \mathcal{L}_V^{\inf}$.
(c) For every $\boldsymbol{\xi} \in \mathcal{L}_V^{\sup}$ there is a sequence $(t_k)_{k \in \mathbb{N}}$, $t_k \to +\infty$, such that $\boldsymbol{\phi}(t_k, \boldsymbol{\xi}) \in \mathcal{F}$ for all k.
(d) If $\mathcal{F} \subset B(\Omega)$ then $\mathcal{L}_V^{\sup} \subset B(\Omega)$.
(e) Set $a := \max_{\mathbf{x} \in \partial \mathcal{F}} V(\mathbf{x})$. Then $\mathcal{L}_V^{\inf}$ is the connected component of $\mathcal{F} \cup \{\mathbf{x} \in \mathcal{D} \setminus \mathcal{F}^\circ \ : \ V(\mathbf{x}) \leq a\}$ that contains Ω. Especially, $\mathcal{L}_V^{\inf}$ is a closed set.
(f) With $b := \sup\{c \in \mathbb{R} \ : \ \mathcal{L}_{V,c} \neq \emptyset\}$ the set $\mathcal{L}_V^{\sup}$ is the connected component of $\mathcal{F} \cup \{\mathbf{x} \in \mathcal{D} \setminus \mathcal{F}^\circ \ : \ V(\mathbf{x}) < b\}$ that contains Ω.

Proof. The proof of (a)-(d) is largely based on [17, Theorem 1.16].

Proposition (a). That $\mathcal{L}_{V,m}$ is forward invariant follows immediately from the proof of [17, Th. 1.16 (i)] and the forward invariance of $\mathcal{L}_V^{\inf}$ and $\mathcal{L}_V^{\sup}$, the intersection and the union of forward invariant sets, are well known consequences.

Propositions (b) and (c). Consider for all $\boldsymbol{\xi} \in \mathcal{L}_V^{\sup} \setminus \mathcal{F}$ the mapping $\psi_{\boldsymbol{\xi}}(t) := V(\boldsymbol{\phi}(t, \boldsymbol{\xi})) + \alpha t$, where α is as in Definition 2.3. As shown in the proof of [17, Th. 1.16 (ii)], $\psi_{\boldsymbol{\xi}}$ is monotonically decreasing for any such $\boldsymbol{\xi}$ on an interval $[0, T_{\boldsymbol{\xi}}[$, where either $\boldsymbol{\phi}(T_{\boldsymbol{\xi}}, \boldsymbol{\xi}) \in \mathcal{F}$ or $T_{\boldsymbol{\xi}} = +\infty$. Note that since $\mathcal{D} \setminus \mathcal{F}^\circ$ is compact and V is continuous $V_{\min} := \min_{\mathcal{D} \setminus \mathcal{F}^\circ} V(\mathbf{x})$ and $V_{\max} := \max_{\mathcal{D} \setminus \mathcal{F}^\circ} V(\mathbf{x})$ are properly defined. We show that $T_{\boldsymbol{\xi}} \leq (V_{\max} - V_{\min})/\alpha$ for all $\boldsymbol{\xi} \in \mathcal{L}_V^{\sup} \setminus \mathcal{F}$.

Assume on the contrary there is a $\boldsymbol{\xi} \in \mathcal{L}_V^{\sup} \setminus \mathcal{F}$ such that $T_{\boldsymbol{\xi}} > (V_{\max} - V_{\min})/\alpha$. Then

$$\begin{aligned} V_{\max} \geq V(\boldsymbol{\xi}) = \psi_{\boldsymbol{\xi}}(0) \geq \psi_{\boldsymbol{\xi}}(T_{\boldsymbol{\xi}}) = V(\boldsymbol{\phi}(T_{\boldsymbol{\xi}}, \boldsymbol{\xi})) + \alpha T_{\boldsymbol{\xi}} \\ > V_{\min} + \alpha \cdot \frac{V_{\max} - V_{\min}}{\alpha} = V_{\max}, \end{aligned}$$

a contradiction.

Proposition (d). Follows immediately from (c).

Proposition (e). Let $\mathcal{F}^*$ be the connected component of $\mathcal{F} \cup \{\mathbf{x} \in \mathcal{D} \setminus \mathcal{F}^\circ \ : \ V(\mathbf{x}) \leq a\}$ that contains Ω. For every c such that $a < c \leq m$ we have $\mathcal{F}^* \subset \mathcal{L}_{V,c}$ and thus $\mathcal{F}^* \subset \mathcal{L}_V^{\inf}$, since $\mathcal{L}_{V,c} = \emptyset$ for $c \leq a$.

To show $\mathcal{L}_V^{\inf} \subset \mathcal{F}^*$, assume on the contrary that there is an $\mathbf{x} \in \mathcal{L}_V^{\inf}$ but $\mathbf{x} \notin \mathcal{F}^*$. Set $c := V(\mathbf{x})$. Then $a < c \leq m$ and hence $\mathbf{x} \notin \mathcal{L}_{V,(a+c)/2}$. Since $\mathcal{L}_V^{\inf} \subset \mathcal{L}_{V,(a+c)/2}$, this is a contradiction.

Proposition (f). Note that if for an $M \in \mathbb{R}_{>0}$ we have $\mathcal{L}_{V,M} \neq \emptyset$, then $\mathcal{L}_{V,c} \subset \mathcal{L}_{V,M}$ for all $c \leq M$. However, if $\mathcal{L}_{V,M} = \emptyset$ and $M > m$, then $\mathcal{L}_{V,c} = \emptyset$ for all $c \geq M$. Thus, we can define $b := \sup\{c \in \mathbb{R} : \mathcal{L}_{V,c} \neq \emptyset\}$ and let $\mathcal{S}$ be the connected component of $\mathcal{F} \cup \{\mathbf{x} \in \mathcal{D} \setminus \mathcal{F}^\circ : V(\mathbf{x}) < b\}$ that contains Ω.

We first show $\mathcal{L}_V^{\sup} \subset \mathcal{S}$. Indeed, $\mathcal{L}_{V,c} \subset \mathcal{S}$ for all $c \in \mathbb{R}_{>0}$, because $\mathcal{L}_{V,c}$ is either empty or a subset of $\mathcal{S}$.

Now we establish $\mathcal{S} \subset \mathcal{L}_V^{\sup}$. Let $\mathbf{x} \in \mathcal{S} \setminus \mathcal{F}$. Then there is a continuous map $\gamma\colon [0,1] \to \mathcal{S}$ such that $\gamma(0) \in \Omega$ and $\gamma(1) = \mathbf{x}$ as well as $V(\gamma(\theta)) < b$ for all $\theta \in [0,1]$. Since V and γ are continuous and $[0,1]$ is compact, there exists

$$\max_{\theta\in[0,1]} V(\gamma(\theta)) =: c < b.$$

Hence, $\mathbf{x} \in \mathcal{L}_{V,(b+c)/2} \neq \emptyset$, showing $\mathcal{S} \subset \mathcal{L}_V^{\sup}$. □

3. **Converse theorem.** In this section we will show the existence of a Lyapunov-like function, that is a Lyapunov function as defined in the previous section for several different local attractors. To achieve this, assume that Ω_i, $i = 1, 2, \ldots, N$ are local attractors of the system (1.1). Note that the system may have other local attractors, even infinitely many. We fix sets $\mathcal{F}_i, \mathcal{D}_i \in \mathfrak{N}(\Omega_i)$, $\mathcal{F}_i \subset \mathcal{D}_i$, where $\mathcal{F}_i$ should be thought of as a small, and $\mathcal{D}_i$ a large, neighborhood of Ω_i. Preferably $\mathcal{D}_i$ is a close inner approximation to the basin of attraction $B(\Omega_i)$ and $\mathcal{F}_i$ is a close outer approximation of the local attractor Ω_i.

We now show the existence and some properties of a Lyapunov-like function that decreases along solutions in large parts of the phase space and provides information about the basins of attraction of the Ω_i's through sublevel sets; see Theorem 2.5.

To achieve this, we first prove a lemma to show the existence of a time $T > 0$ such that all solutions starting in $\mathcal{D}_i$ reach and stay in $\mathcal{F}_i$ for all times $t \geq T$.

Lemma 3.1. *Consider the system (1.1) and assume it possesses a local attractor Ω. Fix $\mathcal{F}, \mathcal{D} \in \mathfrak{N}(\Omega)$ such that $\mathcal{F} \subset \mathcal{D} \subset B(\Omega)$. Then there is a $T > 0$ such that $\boldsymbol{\phi}(t, \mathcal{D}) \subset \mathcal{F}$ for all $t \geq T$.*

Proof. By the stability of Ω there exists a forward invariant neighborhood $\mathcal{P} \subset \mathcal{F}$ of Ω. We show that there is a finite time $T > 0$ such that for every $\mathbf{x} \in \mathcal{D}$ we have $\boldsymbol{\phi}(t, \mathbf{x}) \in \mathcal{P}$ for all $t \geq T$.

Assume, on the contrary, there is no such finite time $T > 0$. Then there is a sequence of times $(t_k)_{k\in\mathbb{N}}$, $t_k \to \infty$, and a sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ in $\mathcal{D}$, such that $\boldsymbol{\phi}(t_k, \mathbf{x}_k) \notin \mathcal{P}$ for all $k \in \mathbb{N}$. Since $\mathcal{D}$ is compact, there is a convergent subsequence $(\mathbf{x}_{k_j})_{j\in\mathbb{N}}$ of $(\mathbf{x}_k)_{k\in\mathbb{N}}$ with limit $\mathbf{x} \in \mathcal{D}$. Let $\varepsilon > 0$ be so small that $\Omega + \mathcal{B}_\varepsilon \subset \mathcal{P}$. Since $\mathcal{D} \subset B(\Omega)$, there is a $\tau > 0$ such that $\operatorname{dist}(\boldsymbol{\phi}(\tau, \mathbf{x}), \Omega) < \varepsilon/2$. By continuity of $\mathbf{x} \mapsto \boldsymbol{\phi}(t, \mathbf{x})$ there is a $\delta > 0$ such that $|\boldsymbol{\phi}(\tau, \mathbf{x}) - \boldsymbol{\phi}(\tau, \mathbf{y})| < \varepsilon/2$ holds for all $|\mathbf{x} - \mathbf{y}| < \delta$. There is a $j \in \mathbb{N}$ large enough such that both $|\mathbf{x}_{k_j} - \mathbf{x}| < \delta$ and $t_{k_j} \geq \tau$ holds. For this j we have

$$\operatorname{dist}(\boldsymbol{\phi}(\tau, \mathbf{x}_{k_j}), \Omega) \quad \leq \quad |\boldsymbol{\phi}(\tau, \mathbf{x}_{k_j}) - \boldsymbol{\phi}(\tau, \mathbf{x})| + \operatorname{dist}(\boldsymbol{\phi}(\tau, \mathbf{x}), \Omega) \quad < \quad \varepsilon$$

which implies $\boldsymbol{\phi}(\tau, \mathbf{x}_{k_j}) \in \mathcal{P}$, so that $\boldsymbol{\phi}(t, \mathbf{x}) \in \mathcal{F}$ for all $t \geq \tau$, in particular for $t = t_{k_j}$ which is a contradiction. □

An obvious corollary is:

Corollary 1. *Consider the system (1.1) and assume Ω_i, $i = 1, 2, \dots, N$, are finitely many local attractors. For each Ω_i fix neighborhoods $\mathcal{F}_i, \mathcal{D}_i \in \mathfrak{N}(\Omega_i)$, $\mathcal{F}_i \subset \mathcal{D}_i \subset B(\Omega_i)$. Then there is a finite time $T > 0$ such that $\boldsymbol{\phi}(t, \mathcal{D}_i) \subset \mathcal{F}_i$ for all $t \geq T$ and all $i = 1, 2, \dots, N$.*

Theorem 3.2. *Consider the system (1.1) and finitely many local attractors Ω_i, $i = 1, 2, \dots, N$. For each local attractor, fix $\mathcal{F}_i, \mathcal{D}_i \in \mathfrak{N}(\Omega_i)$ such that $\mathcal{F}_i \subset \mathcal{D}_i \subset B(\Omega_i)$. Let $\gamma \colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function such that $\gamma(\mathbf{x}) = 0$ if $\mathbf{x} \in \bigcup_{i=1}^N \mathcal{F}_i$ and $\gamma(\mathbf{x}) > 0$ otherwise.*[1]

Let $T > 0$ be as in Corollary 1, so that, for all $\mathbf{x} \in \mathcal{D}_i$, $\boldsymbol{\phi}(t, \mathbf{x}) \in \mathcal{F}_i$ for all $t \geq T$ and all $i = 1, 2, \dots, N$. Define the function $V \colon \mathbb{R}^n \to \mathbb{R}$ as

$$V(\mathbf{x}) := \int_0^T \gamma(\boldsymbol{\phi}(\tau, \mathbf{x}))\, d\tau. \tag{3.1}$$

Then V is a locally Lipschitz function such that

$$D^+V(\mathbf{x}, \mathbf{f}(\mathbf{x})) = -\gamma(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathcal{D}_i,\ i = 1, 2, \dots, N.$$

Further, if γ is a C^p function and $\mathbf{f}$ is a C^r function, then V is a C^q function with $q := \min\{r, p\}$.

Proof. Let $\mathcal{C} \subset \mathbb{R}^n$ be compact and let $R > 0$ be so large that $\boldsymbol{\phi}(t, \mathbf{x}) \in \mathcal{B}_R$ for all $\mathbf{x} \in \mathcal{C}$ and all $t \in [0, T]$. Let G be a Lipschitz constant for γ on $\overline{\mathcal{B}_R}$. Further, let L be a Lipschitz constant for $\mathbf{f}$ on $\overline{\mathcal{B}_R}$. Then for every $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ we have the standard estimate $|\boldsymbol{\phi}(t, \mathbf{x}) - \boldsymbol{\phi}(t, \mathbf{y})| \leq |\mathbf{x} - \mathbf{y}| e^{Lt}$ and it follows that

$$\begin{aligned}
|V(\mathbf{x}) - V(\mathbf{y})| &= \left| \int_0^T [\gamma(\boldsymbol{\phi}(\tau, \mathbf{x})) - \gamma(\boldsymbol{\phi}(\tau, \mathbf{y}))]\, d\tau \right| \\
&\leq G \int_0^T |\boldsymbol{\phi}(\tau, \mathbf{x}) - \boldsymbol{\phi}(\tau, \mathbf{y})| d\tau \\
&\leq G \int_0^T |\mathbf{x} - \mathbf{y}| e^{L\tau} d\tau \\
&= \frac{G}{L}(e^{LT} - 1)|\mathbf{x} - \mathbf{y}|.
\end{aligned}$$

Hence, V is locally Lipschitz.

Further,

$$\begin{aligned}
V(\boldsymbol{\phi}(h, \mathbf{x})) - V(\mathbf{x}) &= \int_h^{T+h} \gamma(\boldsymbol{\phi}(\tau, \mathbf{x})) d\tau - \int_0^T \gamma(\boldsymbol{\phi}(\tau, \mathbf{x})) d\tau \\
&= \int_T^{T+h} \gamma(\boldsymbol{\phi}(\tau, \mathbf{x})) d\tau - \int_0^h \gamma(\boldsymbol{\phi}(\tau, \mathbf{x})) d\tau
\end{aligned}$$

so

$$\lim_{h \to 0^+} \frac{1}{h} [V(\boldsymbol{\phi}(h, \mathbf{x})) - V(\mathbf{x})] = \gamma(\boldsymbol{\phi}(T, \mathbf{x})) - \gamma(\mathbf{x})$$

and since V is locally Lipschitz and $\boldsymbol{\phi}(T, \mathbf{x}) \in \mathcal{F}_i$, i.e. $V(\boldsymbol{\phi}(T, \mathbf{x})) = 0$, for all $\mathbf{x} \in \mathcal{D}_i$, $i = 1, 2, \dots, N$, we have $D^+V(\mathbf{x}, \mathbf{f}(\mathbf{x})) = -\gamma(\mathbf{x})$, see [17, Theorem 1.17].

It is well known that if $\mathbf{f}$ is C^r then so is $\boldsymbol{\phi}$ and the last proposition follows immediately. □

[1] Such a function γ can, for example, be constructed by convolution of the characteristic function, as discussed in more detail in Section 6.

The function V of Theorem 3.2 satisfies $D^+V(\mathbf{x},\mathbf{f}(\mathbf{x}))<0$ for all $\mathbf{x}\in\mathcal{D}_i\setminus\mathcal{F}_i$. Choosing $\mathcal{D}=\mathcal{D}_i$ and $\mathcal{F}=\overline{\mathcal{F}_i+\mathcal{B}_\varepsilon}$ with $\varepsilon>0$ small enough such that $\mathcal{F}\subset\mathcal{D}$, we can satisfy the condition on the orbital derivative in Definition 2.3 (ii).

Consider a particular triple $\Omega_i,\mathcal{F}_i,\mathcal{D}_i$ as in Theorem 3.2. A Lyapunov function V_i on $\mathcal{D}_i\setminus\mathcal{F}_i^\circ$ for Ω_i implies the existence of a forward invariant set $\mathcal{P}_i=\mathcal{L}_{V_i,c}$ for some $c>0$, such that $\mathcal{F}_i\subset\mathcal{P}_i^\circ\subset\mathcal{P}_i\subset\mathcal{D}_i$, and therefore, if such a set $\mathcal{P}_i$ does not exist, there can be no such Lyapunov function V_i. If, however, there exists such a forward invariant set $\mathcal{P}_i$, then, by eventually increasing the values of γ close to the boundary of $\mathcal{D}_i$, the construction of $V=V_i$ in (3.1) yields a Lyapunov function on $\mathcal{D}_i\setminus\mathcal{F}_i^\varepsilon$, for every $\mathcal{F}_i^\varepsilon=\mathcal{F}_i+\mathcal{B}_\varepsilon$ with $\varepsilon>0$ small enough. This is stated in the next theorem.

Theorem 3.3. *Given a triple $\Omega_i,\mathcal{F}_i,\mathcal{D}_i$ as in Theorem 3.2, $i\in\{1,2,\dots,N\}$, assume there exists a closed forward invariant set $\mathcal{P}_i$ such that $\mathcal{F}_i\subset\mathcal{P}_i^\circ\subset\mathcal{P}_i\subset\mathcal{D}_i^\circ$. Let $V:\mathbb{R}^n\to\mathbb{R}_{\geq0}$ be defined by (3.1). Then, by eventually increasing the values of γ in a neighborhood of $\partial\mathcal{D}_i$, the function $V_i:\mathcal{D}_i\to\mathbb{R}_{\geq0}$, $V_i(\mathbf{x})=V(\mathbf{x})$ for all $\mathbf{x}\in\mathcal{D}_i$, is a Lyapunov function for the system (1.1) on $\mathcal{D}_i\setminus\mathcal{F}_i^\varepsilon$, for every $\mathcal{F}_i^\varepsilon:=\mathcal{F}_i+\mathcal{B}_\varepsilon\subset\mathcal{P}_i$, $\varepsilon>0$.*

Proof. Increasing the values of γ in $\mathcal{D}_i\setminus\mathcal{F}_i$ does not affect $V(\mathbf{x})>0$ and $D^+V(\mathbf{x},\mathbf{f}(\mathbf{x}))<0$ for all $\mathbf{x}\in\mathcal{D}_i\setminus\mathcal{F}_i$. By defining

$$\alpha:=\min_{\mathbf{x}\in\mathcal{D}\setminus\mathcal{F}_i^\varepsilon}\gamma(\mathbf{x})>0$$

the condition in Definition 2.3 (ii) is fulfilled for all $\mathbf{x}\in\mathcal{D}^\circ\setminus\mathcal{F}_i^\varepsilon$. It remains to prove the condition in Definition 2.3 (iii).

We prove $\mathcal{L}_{V_i}^{\inf}\neq\emptyset$ by showing that we can always achieve

$$\min_{\mathbf{x}\in\partial\mathcal{D}_i}V(\mathbf{x})>\max_{\mathbf{x}\in\partial\mathcal{P}_i}V(\mathbf{x})$$

by increasing the values of γ in a small neighborhood of $\partial\mathcal{D}_i$.

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ so large that $\boldsymbol{\phi}(t,\mathbf{x})\in\mathcal{D}$ for all $\mathbf{x}\in\mathcal{D}_i$ and all $t\in[0,T]$, where T is as in Corollary 1, and let $F,G>0$ be constants such that $F>\sup_{\mathbf{x}\in\mathcal{D}}|\mathbf{f}(\mathbf{x})|$ and $G:=\max_{\mathbf{x}\in\mathcal{P}_i}\gamma(\mathbf{x})$.

Choose $\delta>0$ so small such that

$$\mathcal{P}_i+\mathcal{B}_{2\delta}\subset\mathcal{D}_i^\circ\quad\text{and}\quad\delta<TF.$$

Replace γ in the definition of V in (3.1) with a function $\widetilde{\gamma}$ such that $\widetilde{\gamma}(\mathbf{x})=\gamma(\mathbf{x})$ for all $\mathbf{x}\in\mathcal{P}_i$ and such that

$$\mathbf{x}\in\partial\mathcal{D}_i+\mathcal{B}_\delta\Rightarrow\widetilde{\gamma}(\mathbf{x})>\frac{TFG}{\delta}.$$

Then for every $\mathbf{x}\in\mathcal{D}_i$ and every $0\leq t\leq\delta/F$, we have

$$|\boldsymbol{\phi}(t,\mathbf{x})-\mathbf{x}|=\left|\int_0^t\dot{\boldsymbol{\phi}}(\tau,\mathbf{x})d\tau\right|\leq\int_0^t|\mathbf{f}(\boldsymbol{\phi}(\tau,\mathbf{x}))|d\tau<\frac{\delta}{F}F=\delta,$$

especially $\mathbf{x}\in\partial\mathcal{D}_i$ implies $\boldsymbol{\phi}(t,\mathbf{x})\in\partial\mathcal{D}_i+\mathcal{B}_\delta$ for all $0\leq t\leq\delta/F$. Hence, for every $\mathbf{x}\in\partial\mathcal{D}_i$ we have

$$V(\mathbf{x})=\int_0^T\widetilde{\gamma}(\boldsymbol{\phi}(\tau,\mathbf{x}))d\tau>\int_0^{\frac{\delta}{F}}\frac{TFG}{\delta}d\tau=TG.$$

On the other hand, for every $\mathbf{x} \in \mathcal{P}_i$, because $\mathcal{P}_i$ is forward invariant, we have

$$V(\mathbf{x}) = \int_0^T \widetilde{\gamma}(\boldsymbol{\phi}(\tau, \mathbf{x}))d\tau = \int_0^T \gamma(\boldsymbol{\phi}(\tau, \mathbf{x}))d\tau \leq TG.$$

This proves the theorem. □

This theorem has an obvious corollary regarding all the local attractors Ω_i, $i = 1, 2, \ldots, N$, from Theorem 3.2.

Corollary 2. *Let the triples $\Omega_i, \mathcal{F}_i, \mathcal{D}_i$, $i = 1, 2, \ldots, N$, be as in Theorem 3.2, and assume that for each i there exists a closed forward invariant set $\mathcal{P}_i$ such that $\mathcal{F}_i \subset \mathcal{P}_i^\circ \subset \mathcal{P}_i \subset \mathcal{D}_i^\circ$. Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be defined by (3.1). Then, by eventually increasing the values of γ in neighborhoods of the boundaries of the $\partial\mathcal{D}_i$, we can achieve that for each $i \in \{1, 2, \ldots, n\}$ the function $V_i : \mathcal{D}_i \to \mathbb{R}_{\geq 0}$, $V_i(\mathbf{x}) = V(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{D}_i$, is a Lyapunov function for the system (1.1) on $\mathcal{D}_i \setminus \mathcal{F}_i^\varepsilon$, for every $\mathcal{F}_i^\varepsilon := \mathcal{F}_i + \mathcal{B}_\varepsilon \subset \mathcal{P}_i$, $\varepsilon > 0$.*

Proof. The proof is essentially the same as that of Theorem 3.3 by taking care to not choose $\delta > 0$ for a particular triple $\Omega_i, \mathcal{F}_i, \mathcal{D}_i$ so large that it effects the definition of γ in $\mathcal{D}_j$ for a $j \neq i$, e.g. by additionally demanding $(\mathcal{D}_i + \mathcal{B}_{2\delta}) \cap (\mathcal{D}_j + \mathcal{B}_{2\delta}) = \emptyset$ for all $i, j = 1, 2, \ldots, N$, $i \neq j$. □

4. Continuous and piecewise affine Lyapunov functions.

In the sequel, we will define continuous and piecewise affine (CPA) functions on suitable triangulations. Such functions are particularly advantageous as candidate Lyapunov functions because the negativity of the orbital Dini derivative for the system (1.1) can be guaranteed by verifying a finite number of inequalities as we show in Theorem 4.2 below.

Definition 4.1. We call a finite collection $\mathcal{T} = \{\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_N\}$ of n-simplices in $\mathbb{R}^n$ a **suitable triangulation** if

(i) $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$, $\nu \neq \mu$, intersect in a common face or not at all.
(ii) With $\mathcal{D}_\mathcal{T} := \bigcup_{\mathfrak{S} \in \mathcal{T}} \mathfrak{S}$ the set $\mathcal{D}_\mathcal{T}^\circ$ is connected.

Property (i), often called shape regularity in the theory of finite element methods, is needed so that we can parameterize every continuous function, affine on every simplex, by specifying its values at the vertices, cf. Remark 1. Such a triangulation is also referred to as simplicial complex in the literature.

In what follows, we will define simplices by fixing an ordered set of vertices and considering the closed convex hull of those vertices. While simplices are usually defined by an unordered set of vertices, by insisting on an ordered set we obtain uniqueness of the shape matrix defined below in (4.4). We denote the set of vertices of all simplices in $\mathcal{T}$ by $\mathcal{V}_\mathcal{T}$.

For a given suitable triangulation, $\mathcal{T}$, and with $\mathcal{D}_\mathcal{T} := \bigcup_{\mathfrak{S} \in \mathcal{T}} \mathfrak{S}$, we denote the set of all continuous functions $f : \mathcal{D}_\mathcal{T} \to \mathbb{R}$ that are affine on every simplex $\mathfrak{S} \in \mathcal{T}$ by $\mathrm{CPA}[\mathcal{T}]$.

Remark 1. A function $V \in \mathrm{CPA}[\mathcal{T}]$ is uniquely determined by its values at the vertices of the simplices of $\mathcal{T}$. To see this, let $\mathfrak{S}_\nu = \mathrm{co}\,(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{T}$. Every point $\mathbf{x} \in \mathfrak{S}_\nu$ can be written uniquely as a convex combination of its vertices, $\mathbf{x} = \sum_{i=0}^n \lambda_i^\mathbf{x} \mathbf{x}_i$, $\lambda_i^\mathbf{x} \geq 0$ for all $i = 0, 1, \ldots, n$, and $\sum_{i=0}^n \lambda_i^\mathbf{x} = 1$. The value of V at $\mathbf{x}$ is given by $V(\mathbf{x}) = \sum_{i=0}^n \lambda_i^\mathbf{x} V(\mathbf{x}_i)$. Additionally, V has a representation on $\mathfrak{S}_\nu$

as $V(\mathbf{x}) = \mathbf{w}_\nu^T(\mathbf{x} - \mathbf{x}_0) + a_\nu$ for some $\mathbf{w}_\nu \in \mathbb{R}^n$ and some $a_\nu \in \mathbb{R}$. In what follows, for $V \in \mathrm{CPA}[\mathcal{T}]$ and $\mathbf{x} \in \mathfrak{S}_\nu$ we denote

$$\nabla V_\nu := \nabla V(\mathbf{x})\Big|_{\mathbf{x}\in\mathfrak{S}_\nu^\circ} = \mathbf{w}_\nu.$$

Then, as shown in [10, Remark 9], ∇V_ν is linear in the values of V at the vertices $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$. □

The following theorem and corollary provide a set of linear inequalities involving a given CPA function V and the system (1.1) so that, if the inequalities are fulfilled, then the orbital Dini derivative of V along the trajectories of (1.1) is negative. The proofs of Theorem 4.2 and Corollary 3 are similar to [9, Theorem 2.6].

Theorem 4.2. *Assume that $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T$ defining the system (1.1) is C^2. Let $\mathcal{T}$ be a suitable triangulation and let $V \in \mathrm{CPA}[\mathcal{T}]$. Let $\mathfrak{S}_\nu = \mathrm{co}\,(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu) \in \mathcal{T}$ and let $\mu_\nu \in \mathbb{R}_{\geq 0}$ satisfy*

$$\max_{\substack{i,j,k=1,2,\ldots,n\\ \mathbf{x}\in\mathfrak{S}_\nu}} \left|\frac{\partial^2 f_k}{\partial x_i \partial x_j}(\mathbf{x})\right| \leq \mu_\nu. \tag{4.1}$$

For each $\mathfrak{S}_\nu$, for $i = 0, 1, \ldots, n$ define the constants

$$E_{i,\nu} := \frac{n\mu_\nu}{2}|\mathbf{x}_i - \mathbf{x}_0|\left(|\mathbf{x}_i - \mathbf{x}_0| + \mathrm{diam}(\mathfrak{S}_\nu)\right). \tag{4.2}$$

Then, for every $\mathfrak{S}_\nu$ such that the inequalities

$$\nabla V_\nu^T \mathbf{f}(\mathbf{x}_i^\nu) + |\nabla V_\nu|_1 E_{i,\nu} \leq -\alpha_\nu \tag{4.3}$$

hold for an $\alpha_\nu \in \mathbb{R}$ and all vertices $\mathbf{x}_i^\nu \in \mathfrak{S}_\nu$, $i = 0, 1, \ldots, n$, we have

$$\nabla V_\nu^T \mathbf{f}(\mathbf{x}) \leq -\alpha_\nu$$

for all $x \in \mathfrak{S}_\nu$.

Corollary 3. *Assume that $\Omega \subset \mathbb{R}^n$ is a local attractor for the system (1.1), where $\mathbf{f}$ is C^2, and that $\mathcal{F}, \mathcal{D} \in \mathfrak{N}(\Omega)$, $\mathcal{F} \subset \mathcal{D}^\circ$. Assume that $\mathcal{T}$ is a suitable triangulation such that $\mathcal{D}_\mathcal{T} = \mathcal{D} \setminus \mathcal{F}^\circ$. If a function $V \in \mathrm{CPA}[\mathcal{T}]$ satisfies*

- *$V(\mathbf{x}) > 0$ for every vertex $\mathbf{x} \in \mathcal{V}_\mathcal{T}$,*
- *there is a constant $\alpha > 0$ such that the inequalities (4.3) are fulfilled with $\alpha_\nu = \alpha$ for all $\mathfrak{S}_\nu \in \mathcal{T}$, and*
- *there is an $m > 0$ such that $\mathcal{L}_{V,m} \neq \emptyset$,*

then V is a Lyapunov function for Ω on $\mathcal{D} \setminus \mathcal{F}^\circ$.

Remark 2. The usefulness of Theorem 4.2 and Corollary 3 is that it reduces the verification that a function $V \in \mathrm{CPA}[\mathcal{T}]$ is a Lyapunov function for (1.1) to the verification of a finite number of inequalities (4.3). Finding a candidate CPA Lyapunov function can be done as in [1, 10, 11, 16], via linear programming. Alternatively, as in [5], [12], and this paper, one can define $V \in \mathrm{CPA}[\mathcal{T}]$ by computing suitable values at the vertices of the simplices of $\mathcal{T}$ and then verify the inequalities (4.3). □

We now turn to the question of when a given Lyapunov function can be approximated by a CPA-Lyapunov function. To do this, we require the following definitions.

Definition 4.3. Let $\mathcal{D} \subset \mathbb{R}^n$ be a domain, $f : \mathcal{D} \to \mathbb{R}$ be a function, and $\mathcal{T}$ be a suitable triangulation such that $\mathcal{D}_\mathcal{T} \subset \mathcal{D}$. The $\mathrm{CPA}[\mathcal{T}]$ *approximation g of f on* $\mathcal{D}_\mathcal{T}$ is the function $g \in \mathrm{CPA}[\mathcal{T}]$ defined by $g(\mathbf{x}) = f(\mathbf{x})$ for all vertices $\mathbf{x} \in \mathcal{V}_\mathcal{T}$.

A CPA approximation of a Lyapunov function must approximate its values, as well as its first derivative. A triangulation with simplices of a small diameter is sufficient for approximating the values of a Lyapunov function with arbitrary precision. For approximating the first derivative of a Lyapunov function we additionally need that the simplices in the triangulation $\mathcal{T}$ are not too close to being degenerate; that is, no n-simplex should be close to being of dimension $n-1$. This property can be quantified as follows: For an n-simplex $\mathfrak{S}_\nu := \operatorname{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathcal{T}$ define its **shape-matrix**, X_ν, by writing the vectors $\mathbf{x}_1 - \mathbf{x}_0$, $\mathbf{x}_2 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ in its rows subsequently; i.e.,

$$X_\nu = [(\mathbf{x}_1 - \mathbf{x}_0), (\mathbf{x}_2 - \mathbf{x}_0), \dots, (\mathbf{x}_n - \mathbf{x}_0)]^T. \tag{4.4}$$

The degeneracy of the simplex $\mathfrak{S}_\nu$ is quantified by the value $\operatorname{diam}(\mathfrak{S}_\nu)\|X_\nu^{-1}\|$, where $\|X_\nu^{-1}\|$ is the spectral norm of the inverse of X_ν (see part (ii) in the proof of [1, Theorem 4.6]). To see why this quantity captures a "distance-to-degeneracy" of the n-simplex $\mathfrak{S}_\nu$, observe that degeneracy corresponds to some $\mathbf{x}_i$, $i = 1, \dots, n$ being equal to $\mathbf{x}_0$, which results in a zero row of X_ν, or some $\mathbf{x}_i$ being equal to some $\mathbf{x}_j$, $i, j = 1, \dots, n$, $i \neq j$, which results in X_ν having linearly dependent rows. In both cases, and in fact if X_ν has any linearly dependent rows, then X_ν is singular. If, rather than being equal, two points as described above are close to each other, then the spectral norm of X_ν^{-1} will be large. Of course, we may wish to use very small simplices in order to reduce the error between a given Lyapunov function and its CPA approximation, and hence a reasonable measure of distance-to-degeneracy should also scale the spectral norm of the inverse of X_ν by the diameter of the simplex, leading to the quantity $\operatorname{diam}(\mathfrak{S}_\nu)\|X_\nu^{-1}\|$.

We will only give a local result on the approximation to a C^2 Lyapunov function, i.e., we will use a singleton triangulation $\mathcal{T} = \{\mathfrak{S}_\nu\}$ for a simplex $\mathfrak{S}_\nu$. More general results can be proved, but are rather technical to formulate and the necessary technical burden makes the main idea of the theorem less transparent. The following theorem is general enough to be sufficient for our needs in this paper and we refer to [12] for an example of a similar, but more general theorem in a sightly different setting. Since the proof in [12] can be adapted to our local case in a straightforward manner we omit the proof.

Theorem 4.4. *Assume that* $\mathbf{f}$ *in system* (1.1) *is* C^2*, let* $W : \mathbb{R}^n \to \mathbb{R}$*,* $\mathbf{y} \in \mathbb{R}^n$*, and assume that* W *is* C^2 *in* $\mathcal{N} := \{\mathbf{y}\} + \mathcal{B}_\varepsilon$*,* $\varepsilon > 0$*, and that for a constant* $\alpha > 0$ *we have*

$$\nabla W(\mathbf{x})^T \mathbf{f}(\mathbf{x}) \leq -2\alpha \quad \text{for all } \mathbf{x} \in \mathcal{N}. \tag{4.5}$$

Then for every $R \in \mathbb{R}_{>0}$ *there exists a* $\delta_R > 0$ *so that, for any* n*-simplex* $\mathfrak{S}_\nu := \operatorname{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$ *satisfying*

$$\begin{aligned} &\mathbf{x} \in \mathfrak{S}_\nu \subset \mathcal{N}, \\ &\operatorname{diam}(\mathfrak{S}_\nu) \leq \delta_R, \text{and} \qquad (4.6) \\ &\operatorname{diam}(\mathfrak{S}_\nu)\|X_\nu^{-1}\| \leq R \qquad (4.7) \end{aligned}$$

the CPA$[\{\mathfrak{S}_\nu\}]$ *approximation* V *to* W *on* $\mathfrak{S}_\nu$ *fulfills*

$$\nabla V_\nu^T \mathbf{f}(\mathbf{x}) \leq -\alpha \text{ for all } \mathbf{x} \in \mathfrak{S}_\nu.$$

Theorem 4.4 implies that it is always possible to find a triangulation that admits a CPA Lyapunov function approximating a twice continuously differentiable Lyapunov function. We note that the assumption of twice differentiability and

the bound on $\operatorname{diam}(\mathfrak{S}_\nu)\|X_\nu^{-1}\|$ are required in proving that $|\nabla V_\nu - \nabla W(\mathbf{x}_i)|$ can be made arbitrarily small for every vertex $\mathbf{x}_i$ of $\mathfrak{S}_\nu$ by choosing $\mathfrak{S}_\nu$ with a small diameter. The essential idea is that since

$$\nabla V_\nu = X_\nu^{-1} \begin{pmatrix} W(x_1) - W(x_0) \\ W(x_2) - W(x_0) \\ \vdots \\ W(x_n) - W(x_0) \end{pmatrix},$$

an upper bound on $|\nabla V_\nu - \nabla W(\mathbf{x}_i)|$ can be obtained by

$$|\nabla V_\nu - \nabla W(\mathbf{x}_i)| \le \|X_\nu^{-1}\| \sum_{j=1}^{n} |W(\mathbf{x}_j) - W(\mathbf{x}_0) - (\mathbf{x}_j - \mathbf{x}_0)\nabla W(\mathbf{x}_i)|,$$

which converges to zero with $\operatorname{diam}(\mathfrak{S}_\nu)$, when $\operatorname{diam}(\mathfrak{S}_\nu)\|X_\nu^{-1}\|$ and the second derivative of W are bounded.

5. Approximation of the attractors.

As stated in the introduction we shall now describe a method to approximate the attractors for (1.1) on a compact set $\mathcal{D}$. The method is very similar to those presented in [2] and [15]. We start by giving the main outline of our approach and theoretical justifications, and then we give a short discussion on the approximation algorithm and complications that arise when implementing it. A more detailed discussion on our implementation can be found in [4].

5.1. Main ideas and theory.

Our goal is to work with a discretized approximation to the system (1.1) on a compact set $\mathcal{D} \subset \mathbb{R}^n$ that captures its behavior to a sufficient extent. In particular we need to discretize both the space $\mathcal{D}$ and the continuous dynamics on it. We start by discretizing $\mathcal{D}$ and we do this by constructing a set $\mathfrak{G} \subset P(\mathcal{D})$ having certain nice properties. Recall that $P(\mathcal{D})$ denotes the power set of $\mathcal{D}$.

Definition 5.1. For a given compact set $\mathcal{D} \subset \mathbb{R}^n$, we say that a set $\mathfrak{G} \subset P(\mathcal{D})$ is a **grid** for $\mathcal{D}$ if it fulfills the following conditions:

(i) $\mathcal{D} = \bigcup_{\mathcal{G}\in\mathfrak{G}} \mathcal{G}$.
(ii) $\overline{\mathcal{G}^\circ} = \mathcal{G}$ for all $\mathcal{G} \in \mathfrak{G}$.
(iii) $\mathcal{H} \cap \mathcal{G} = \partial\mathcal{H} \cap \partial\mathcal{G}$ for all $\mathcal{G}, \mathcal{H} \in \mathfrak{G}$.
(iv) $\mathfrak{G}$ consists of a finite number of sets.

In the examples that follow we shall use $\mathcal{D} = [-4, 4]^2$ and $\mathfrak{G}$ will be a uniform rectangular grid on $\mathcal{D}$, which obviously fulfills the conditions above. For a given grid $\mathfrak{G}$ we define its diameter as $\operatorname{diam}(\mathfrak{G}) := \sup\{\operatorname{diam}(\mathcal{H}) : \mathcal{H} \in \mathfrak{G}\}$ and the realization mapping of $\mathfrak{G}$ as

$$|\cdot| \quad : \quad P(\mathfrak{G}) \to \mathcal{D}, \quad |\mathfrak{H}| := \bigcup_{\mathcal{G}\in\mathfrak{H}} \mathcal{G}. \tag{5.1}$$

Our aim is now to encode the dynamics of system (1.1) in a suitable way as a directed graph on $\mathfrak{G}$. We do this by choosing a time-step $t \in \mathbb{R}_{>0}$ and then define the t-advance map $\mathbf{g}_t : \mathbb{R}^n \to \mathbb{R}^n$ by $\mathbf{g}_t(\mathbf{x}) := \boldsymbol{\phi}(t, \mathbf{x})$. This gives us the following discrete-time dynamical system on $\mathbb{R}^n$:

$$\mathbf{x} \mapsto \mathbf{g}_t(\mathbf{x}). \tag{5.2}$$

We now discretize the state space of the system (5.2). For this we define the minimal outer approximation of $\mathbf{g}_t$ on $\mathfrak{G}$ by

$$M_{\min} : \mathfrak{G} \to P(\mathfrak{G}), \quad M_{\min}(\mathcal{G}) = \{\mathcal{H} \in \mathfrak{G} \,:\, \mathbf{g}_t(\mathcal{G}) \cap \mathcal{H} \neq \emptyset\}.$$

The optimal encoding of the dynamics of the system (5.2) with respect to the grid $\mathfrak{G}$ can now be represented as a directed graph on $\mathfrak{G}$. It is naturally defined as $(\mathfrak{G}, \mathbb{A})$ where $(\mathcal{G}, \mathcal{H})$ is an edge from $\mathcal{G}$ to $\mathcal{H}$; i.e., $(\mathcal{G}, \mathcal{H}) \in \mathbb{A}$, if and only if $\mathcal{H} \in M_{\min}(\mathcal{G})$. We say that the graph $(\mathfrak{G}, \mathbb{A})$ represents the map $M_{\min}$. More generally we can for any given map $M : \mathfrak{G} \to P(\mathfrak{G})$ represent it with a graph $(\mathfrak{G}, \mathbb{A})$ in the same way. If such a map satisfies the condition that $M_{\min}(\mathcal{G}) \subset M(\mathcal{G})$ for all $\mathcal{G} \in \mathfrak{G}$ then we say that M covers $M_{\min}$.

In summary, starting from the continuous system (1.1) we first construct the discrete-time system (5.2), given by the t-advance map. Then we construct a spatial discretization $(\mathfrak{G}, \mathbb{A})$ of (5.2) with respect to the grid $\mathfrak{G}$.

In the examples that follow, the discrete-time system (5.2) will be approximated by the classical Runge-Kutta RK4 method. The error of the computed system to $\mathbf{g}_t$ will depend on the time-step used in the Runge-Kutta method. Note, however, that our method includes a verification that the CPA function, obtained in the final step, is a valid Lyapunov function.

The following definition is essential for what follows:

Definition 5.2. Let $(\mathfrak{G}, \mathbb{A})$ be a graph representing the map $M : \mathfrak{G} \to P(\mathfrak{G})$. Then:

- If $M(\mathcal{G}) \neq \emptyset$ for all $\mathcal{G} \in \mathfrak{G}$ we say that the mapping M is **forward closed**.
- If M is forward closed, a set $\mathfrak{A} \subset \mathfrak{G}$ is called a **graph attractor** for M if $M(\mathfrak{A}) = \mathfrak{A}$.

Note that for $M = M_{\min}$ and a graph attractor $\mathfrak{A}$ for M we necessarily have from Definition 5.1 (iii) that

$$\mathbf{x} \in |\mathfrak{A}| \;\Rightarrow \mathbf{g}_t(\mathbf{x}) \in |\mathfrak{A}|^\circ,$$

where $|\mathfrak{A}|^\circ$ denotes the interior of $|\mathfrak{A}|$ in $\mathcal{D}$. Thus, the invariance of $\mathfrak{A}$ under $M_{\min}$ implies attraction if $|\mathfrak{A}| \neq \mathcal{D}$.

Given a graph $(\mathfrak{G}, \mathbb{A})$ representing a forward closed map $M : \mathfrak{G} \to P(\mathfrak{G})$, it is a simple task to identify all graph attractors $\mathfrak{A}$. Indeed, a set $\mathfrak{A} \subset \mathfrak{G}$ is a graph attractor for M, if and only if there exists a $\mathfrak{B} \subset \mathfrak{G}$ such that $\mathfrak{B} \subset M(\mathfrak{B})$ and

$$\mathfrak{A} = \Gamma_+(\mathfrak{B}) := \bigcup_{\ell \in \mathbb{N}} M^\ell(\mathfrak{B}).$$

Let $\mathbf{g}_t : \mathcal{D} \to \mathcal{D}$ and let $M_{\min}$ be its minimal outer approximation. It follows by [15, Proposition 5.5], that for every graph attractor $\mathfrak{A}$ for $M_{\min}$ there exists a set $\mathcal{A} \subset |\mathfrak{A}|$ and a neighborhood $\mathcal{U}$ of $\mathcal{A}$ in $\mathcal{D}$ such that

$$\omega(\mathcal{U}) := \bigcap_{k \in \mathbb{N}} \overline{\bigcup_{\ell \geq k} \mathbf{g}_t^\ell(\mathcal{U})} = \mathcal{A}.$$

Therefore, a local attractor of the system (1.1) is necessarily contained in a graph attractor $\mathfrak{A}$ of $M_{\min}$.

It further follows by [15, Proposition 5.5] that if the grid $\mathfrak{G}$ is fine enough, then every local attractor of the system $\mathbf{g}_t : \mathcal{D} \to \mathcal{D}$, $\mathbf{x} \mapsto \mathbf{g}_t(\mathbf{x})$, has an arbitrarily close outer approximation by a graph attractor of $M_{\min}$. We thus compute outer approximations of the local attractors of our original system (1.1) in $\mathcal{D}$ by choosing

$t \in \mathbb{R}_{>0}$ and our grid $\mathfrak{G}$ sufficiently fine, such that the attractors of $M_{\min}$ give a good approximation to the local attractors of our system. Note, that we are not interested in local attractors that contain other local attractors. We use a well known algorithm of Tarjan [24] to find the graph attractors, which delivers a natural partial ordering of the graph attractors [4].

5.2. **Implementation and complications.** We would now like to compute all the attractors for the system (1.1) by using the following algorithm.

Step 1. Discretize $\mathcal{D}$ with a suitable grid $\mathfrak{G}$ and choose a time-step $t \in \mathbb{R}_{>0}$ to obtain a t-advance map $\mathbf{g}_t$.

Step 2. Construct a map $M : \mathfrak{G} \to \mathcal{P}(\mathfrak{G})$ that covers $M_{\min}$ for $\mathbf{g}_t$ (preferably $M_{\min}$ itself) and construct the graph $(\mathfrak{G}, \mathbb{A})$ representing M.

Step 3. Determine outer approximations of the local attractors of (1.1) by applying the results from Section 5.1.

As it turns out, there are two difficulties in implementing the algorithm above that have to be dealt with in some manner. Firstly, the determination of $M_{\min}$, and consequently finding a map M covering $M_{\min}$ proves to be troublesome. One way to do this is to use Gronwall's Lemma in order to get the inequality

$$|\mathbf{g}_t(\mathbf{x}) - \mathbf{g}_t(\mathbf{y})| \leq |\mathbf{x} - \mathbf{y}|e^{\lambda t}$$

on our domain with

$$\lambda := \sup_{\substack{\tau \in [0,t] \\ \mathbf{x},\mathbf{y} \in \mathcal{D} \\ \mathbf{x} \neq \mathbf{y}}} \frac{|\boldsymbol{\phi}(\tau, \mathbf{x}) - \boldsymbol{\phi}(\tau, \mathbf{y})|}{|\mathbf{x} - \mathbf{y}|}.$$

This estimate, however, turns out to be too conservative in our calculations, since by using it in the examples below yields a graph containing only one strongly connected component covering the whole of $\mathcal{D}$. One way to mend this is to estimate numerically a constant L such that $|\mathbf{g}_t(\mathbf{x}) - \mathbf{g}_t(\mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{D}$. Another way is simply to let $L > 0$ be an input parameter in the algorithm. Obviously we have no guarantee that the algorithm will give us a faithful approximation of the attractors of the system by using these methods, but since we verify the validity of the CPA Lyapunov function once obtained they ought to be reasonable enough, as the examples in Section 6 indeed suggest. This problem has been studied in the literature in more detail, cf. e.g. [14, 7, 26].

The second difficulty stems from the fact that the useful results in Section 5.1 are for a $\mathbf{g}_t$ mapping $\mathcal{D}$ into $\mathcal{D}$. Since $\mathcal{D}$ is usually user defined, this condition is hard to fulfill since, in general, $\mathbf{g}_t(\mathcal{D}) \not\subset \mathcal{D}$. If, however, $\mathcal{D}$ is forward invariant in the long run, i.e., there exists a time T such that for all $\mathbf{x} \in \mathcal{D}$ we have that $\boldsymbol{\phi}(t', \mathbf{x}) \in \mathcal{D}$ if $t' > T$, then this can be mended in the following way for our purposes: If $\mathbf{x} \in \mathcal{D}$ and $\mathbf{g}_t(\mathbf{x}) \notin \mathcal{D}$ then we simply redefine $\mathbf{g}_t(\mathbf{x})$ as $\mathbf{g}_t^\ell(\mathbf{x})$, where $\ell > 0$ is the smallest natural number such that $\mathbf{g}_t^\ell(\mathbf{x}) \in \mathcal{D}$.

6. **Examples.** We present some numerical examples in order to evaluate how our method works in practice. In all our examples we use the classical Runge-Kutta method RK4 when estimating solutions $\boldsymbol{\phi}(t, \mathbf{x})$ to (1.1). In all our numerical integrations we shall denote by Δt the time step chosen in RK4 and by $[0, T]$ the interval integrated over. In order to apply our method we need to choose a function γ which fulfills the condition imposed in Theorem 3.2. In order to do so, let $\mathcal{F}_i$ be the disjoint attractors obtained by the graph algorithm above, that do not contain smaller

attractors. Choose a function $\gamma \in C^\infty(\mathbb{R}^n)$ such that $\gamma(\mathbf{x}) = 0$ for all $\mathbf{x} \in \bigcup_i \mathcal{F}_i$ and $\gamma(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{D} \setminus \bigcup_i \mathcal{F}_i$. The choice of γ is otherwise rather arbitrary. Herein we construct γ by mimicking a standard technique for the smooth partition of unity. First we define the set of all points in $\mathcal{D}$ that have distance greater than one to $\bigcup_i \mathcal{F}_i$ and then we define γ as the convolution of the characteristic function of this set and $\rho(\mathbf{x}) = \exp(-1/(1 - \|\mathbf{x}\|_2^2))$ for $\|\mathbf{x}\|_2 < 1$ and $\rho(\mathbf{x}) = 0$ otherwise. Our γ thus grows from zero on $\bigcup_i \mathcal{F}_i$ to one within a distance of 1.

We shall denote by V the Lyapunov function of (3.1) obtained by using γ. Once calculated, we normalize our function by dividing by $\max_{\mathbf{x}\in\mathcal{D}} V(\mathbf{x})$, which clearly does not change the sign of the orbital derivative of V at any point. Finally we plot the graph of V and mark with an $\times$ all the simplices where the inequality (4.3) does not hold and thus the orbital derivative is not guaranteed to be negative.

In all of our examples we use $\mathcal{D} = [-4, 4]^2$ and start by approximating the attractors of the system using the graph algorithm of Section 5.2 with the t-advance map $\mathbf{g}_1$. We do this with a uniform 250×250 rectangular grid on the specified domain, that is we set

$$\mathfrak{G} := \left\{ \left[-4 + \tfrac{8i}{250}, -4 + \tfrac{8(i+1)}{250}\right] \times \left[-4 + \tfrac{8j}{250}, -4 + \tfrac{8(j+1)}{250}\right] : i, j \in \{0, 1, \ldots, 249\} \right\}.$$

which clearly fulfills the conditions in Definition 5.1. Furthermore, when constructing our map $M : \mathfrak{G} \to P(\mathfrak{G})$ we define $L := \operatorname{diam}(\mathfrak{G})$ and set $\mathcal{H} \in M(\mathcal{G})$ if and only if $\mathcal{H} \cap \{\mathbf{y} \in \mathcal{D} : |\mathbf{y} - \mathbf{g}_1(\mathbf{x}_{\text{mid}})|_\infty \leq \frac{L}{2}\} \neq \emptyset$ where $\mathbf{x}_{\text{mid}}$ is the midpoint of $\mathcal{G}$.

We have chosen this rectangular grid as this was used in [2]. Moreover, the construction of the map for this grid is convenient using the above estimates involving the infinity-norm.

Example 1. Consider the two-dimensional system given by

$$\dot{x}_1 = 2x_1 - x_1x_2,$$
$$\dot{x}_2 = 2x_1^2 - x_2.$$

Using the graph algorithm of Section 5 we obtain three graph attractors: one contained in a ball of radius 0.1 around the point $(-1, 2)$, another contained in a ball of the same radius around the point $(1, 2)$, and the third covering the other attractors and the origin. Simple analysis of the equations yields that $(-1, 2)$ and $(1, 2)$ are local attractors for the system and that $(0, 0)$ is a saddle point, which is in accordance with our estimate. As discussed before, we do not consider the third graph attractor because it contains the smaller attractors.

Next we calculate V. We do this by choosing a uniform grid of 51,200 triangles, a time step of $\Delta t = 0.01$, and an integration horizon of $T = 8$. The graph of V is shown in Figure 1 and some level curves for V are shown in Figure 2. Figure 1 indicates that our method works well outside of the x_2 axis. The reason for this is that the x_2 axis is the stable manifold for the saddle point at $(0, 0)$, as can be seen by direct analysis.

Example 2. Our second example is the following Duffing equation:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = 0.3(4x_1 - x_1^3 - x_2).$$

In this example the graph algorithm of Section 5 finds three graph attractors: one contained in a ball of radius 0.081 around the point $(-2, 0)$, another contained

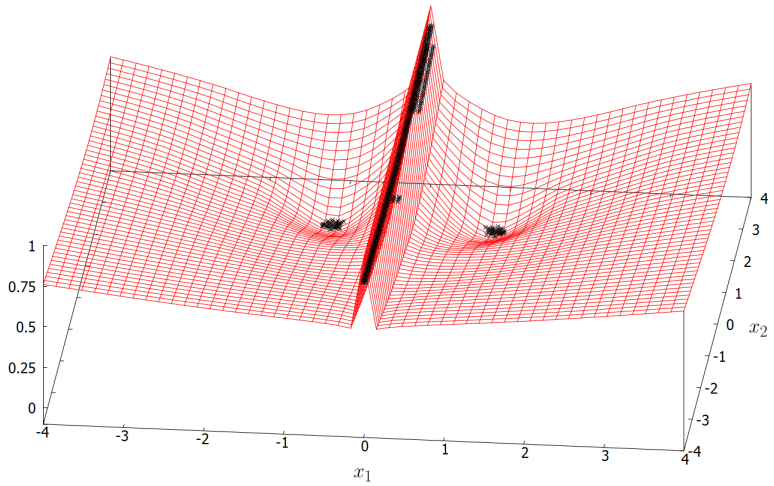

FIGURE 1. The graph of V in Example 1. The orbital derivative of V is negative with exception of the black area.

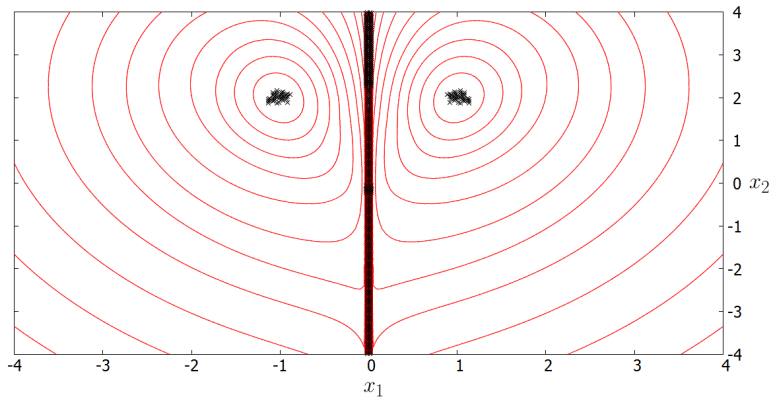

FIGURE 2. Some level curves (red) for V in Example 1. Each closed level curve is a forward invariant set. The orbital derivative of V is negative with exception of the black area as verified by inequalities (4.3).

in a ball of the same radius around the point $(2, 0)$, and a third containing the other two and the origin, cf. Figure 3. For this example, the points $(-2, 0)$ and $(2, 0)$ are known local attractors and at the origin there is a saddle point. As before the graph algorithm gives a quite good estimate of the attractors and we do not consider the larger third attractor containing the other two.

This example requires a considerably finer grid and larger time interval to calculate a reasonable V than in the previous example; most likely because the basins of attraction of the different attractors are intertwined. More specifically, we chose a uniform grid of 819,200 triangles, a time step of $\Delta t = 0.005$, and an integration horizon of $T = 128$. The graph of V is shown in Figure 4 and some level curves for V are shown in Figure 5.

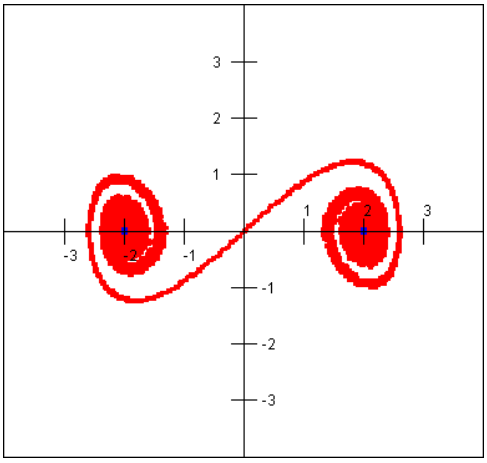

FIGURE 3. The graph attractor found for Example 2. The large attractor (red) contains the smaller attractors at $(-2, 0)$ and $(2, 0)$ and is thus not considered in the computation of the Lyapunov function.

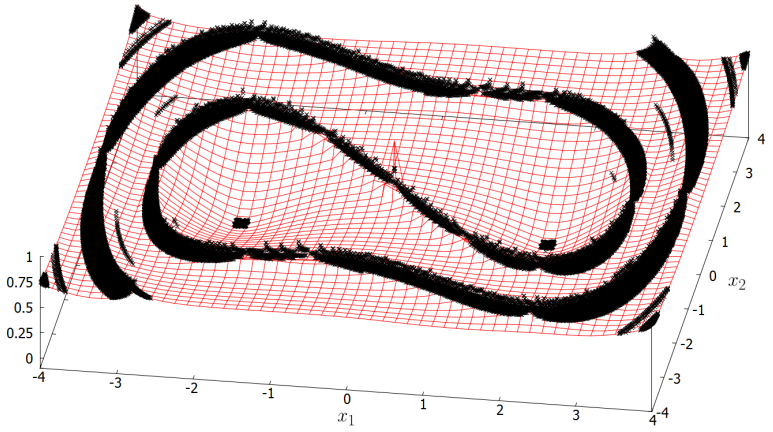

FIGURE 4. The graph of V in Example 2. The orbital derivative of V is negative with exception of the black area as verified by inequalities (4.3).

Example 3. Our final example is the following Van der Pol oscillator:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = (1 - x_1^2)x_2 - x_1.$$

Using the graph algorithm of Section 5 we obtain the graph attractor shown in Figure 6. It is an outer approximation of a stable periodic orbit.

Here we calculate V by choosing a uniform grid of 204,800 triangles, a time step of $\Delta t = 0.005$, and an integration horizon of $T = 8$. The graph of V is shown in Figure 7 and some level curves for V are shown in Figure 8.

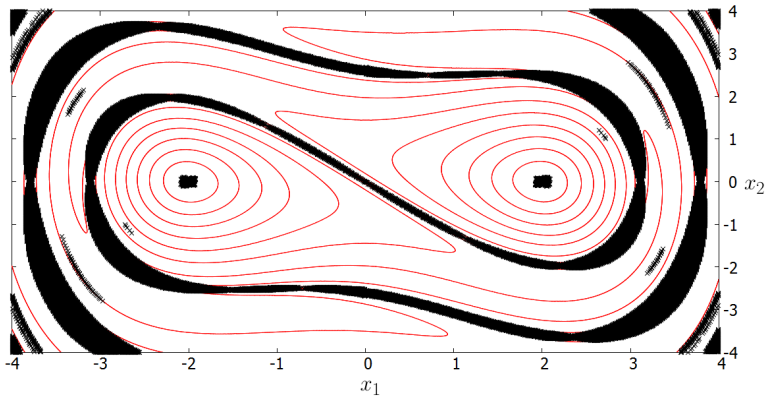

FIGURE 5. Some level curves (red) for V in Example 2. Each closed level curve is a forward invariant set. The orbital derivative of V is negative with exception of the black area.

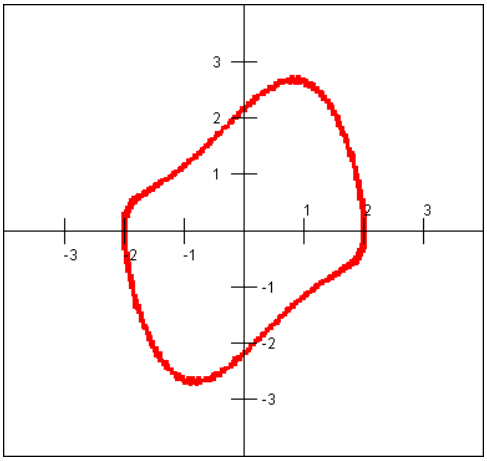

FIGURE 6. An outer approximation of all local attractors for our system in Example 3 obtained by the graph algorithm.

7. **Conclusions.** We developed the theory for a novel method to compute Lyapunov functions for continuous-time systems with multiple attractors and illustrated its applicability to three planar systems. The computed Lyapunov function can be regarded as a rudimentary complete Lyapunov function, giving information on attractors and their basins of attraction. Note especially, that we compute our Lyapunov function without estimating the repellers, i.e., the attractors of the time-reversed dynamics of system (1.1). This is a significant advantage for systems whose state-space is not bounded.

The Lyapunov functions for the systems in the examples were each computed in less than 30 minutes on a PC with an i5-4670 processor. The most expensive computational step in our algorithm is by far the computation of the values of the Lyapunov function at the vertices of the simplicial complex. However, since the computation at a vertex is independent of the computation at every other vertex, this

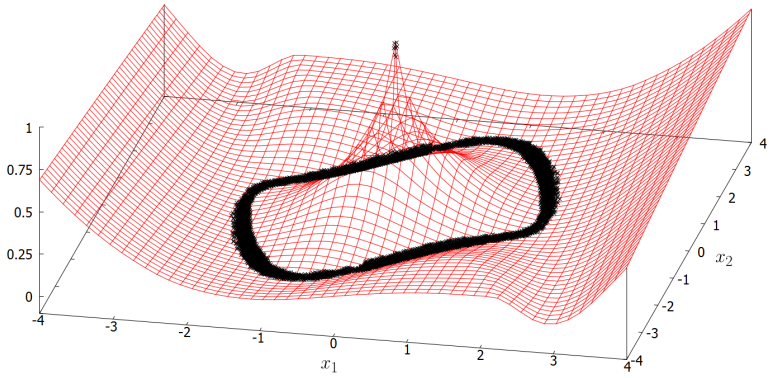

FIGURE 7. The graph of V in Example 3. The orbital derivative of V is negative with exception of the black area as verified by inequalities (4.3).

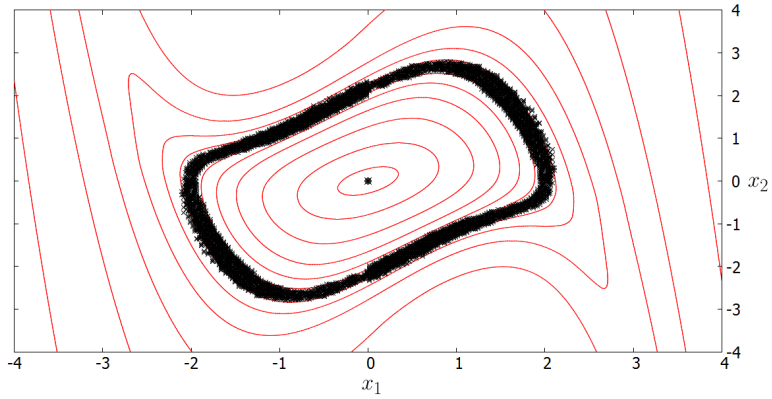

FIGURE 8. Some level curves (red) for V in Example 3. Each closed level curve is a forward invariant set. The orbital derivative of V is negative with exception of the black area.

step can be completely parallelized; i.e., subject to available computing resources, all vertex values can be computed simultaneously. Furthermore, all computational steps of our method can be parallelized. The only non-trivial part to parallelize is the computation of the strongly connected component of the graph $(\mathfrak{G}, \mathbb{A})$ in Step 3 in Section 5. For low-dimensional examples with a reasonably small state-space, this computation can be done quite efficiently without parallelization using Tarjan's Algorithm [24], but larger examples would certainly benefit from a parallel algorithm [3].

REFERENCES

[1] R. Baier, L. Grüne and S. Hafstein, Linear programming based Lyapunov function computation for differential inclusions, *Discrete and Continuous Dynamical Systems Series B*, **17** (2012), 33–56.

[2] H. Ban and W. Kalies, A computational approach to Conley's decomposition theorem, *Journal of Computational and Nonlinear Dynamics*, **1** (2006), 312–319.

[3] J. Barnat, J. Chaloupka and J. van de Pol, Distributed algorithms for SCC decomposition, *Journal of Logic and Computation*, **21** (2011), 23–44.
[4] J. Björnsson, P. Giesl and S. Hafstein, Algorithmic verification of approximations to complete Lyapunov functions, in *Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems*, Groningen, The Netherlands, (2014), 1181–1188 (no. 0180).
[5] J. Björnsson, P. Giesl, S. Hafstein, C. M. Kellett and H. Li, Computation of continuous and piecewise affine Lyapunov functions by numerical approximations of the Massera construction, in *Proceedings of the 53rd IEEE Conference on Decision and Control,* Los Angeles, California, USA, (2014), 5506–5511.
[6] C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS Regional Conference Series no. 38, American Mathematical Society, 1978.
[7] M. Dellnitz, G. Froyland and O. Junge, The algorithms behind GAIO-set oriented numerical methods for dynamical systems, in *Ergodic theory, analysis, and efficient simulation of dynamical systems*, Springer, Berlin, (2001), 145–174, 805–807.
[8] P. Giesl, *Construction of Global Lyapunov Functions Using Radial Basis Functions*, no. 1904 in Lecture Notes in Mathematics, Springer, 2007.
[9] P. Giesl and S. Hafstein, Construction of Lyapunov functions for nonlinear planar systems by linear programming, *Journal of Mathematical Analysis and Applications*, **388** (2012), 463–479.
[10] P. Giesl and S. Hafstein, Revised CPA method to compute Lyapunov functions for nonlinear systems, *Journal of Mathematical Analysis and Applications*, **410** (2014), 292–306.
[11] S. Hafstein, *An Algorithm for Constructing Lyapunov Functions*, Electronic Journal of Differential Equations Mongraphs, 2007.
[12] S. Hafstein, C. M. Kellett and H. Li, Continuous and piecewise affine Lyapunov functions using the Yoshizawa construction, in *Proceedings of the 2014 American Control Conference,* Portland, Oregon, USA, (2014), 548–553 (no. 0170).
[13] M. Hurley, Lyapunov functions and attractors in arbitrary metric spaces, *Proc. Amer. Math. Soc.*, **126** (1998), 245–256.
[14] O. Junge, *Mengenorientierte Methoden zur Numerischen Analyse Dynamischer Systeme*, PhD thesis at the University of Paderborn, Germany, Shaker, 2000.
[15] W. Kalies, K. Mischaikow and R. VanderVorst, An algorithmic approach to chain recurrence, *Foundations of Computational Mathematics*, **5** (2005), 409–449.
[16] S. Marinosson, Lyapunov function construction for ordinary differential equations with linear programming, *Dynamical Systems*, **17** (2002), 137–150.
[17] S. Marinosson, *Stability Analysis of Nonlinear Systems with Linear Programming: A Lyapunov Functions Based Approach*, PhD thesis, Gerhard-Mercator-University, Duisburg, Germany, 2002.
[18] J. L. Massera, On Liapounoff's conditions of stability, *Annals of Mathematics*, **50** (1949), 705–721.
[19] D. Norton, The fundamental theorem of dynamical systems, *Comment. Math. Univ. Carolinae*, **36** (1995), 585–597.
[20] A. Papachristodoulou and S. Prajna, The construction of Lyapunov functions using the sum of squares decomposition, in *Proceedings of the 41st IEEE Conference on Decision and Control*, **3**, Las Vegas, Nevada, USA, (2002), 3482–3487.
[21] M. Patrao, Existence of complete Lyapunov functions for semiflows on separable metric spaces, *Far East Journal of Dynamical Systems*, **17** (2011), 49–54.
[22] M. Peet and A. Papachristodoulou, A converse sum-of-squares Lyapunov result: An existence proof based on the Picard iteration, in *Proceedings of the 49th IEEE Conference on Decision and Control*, Atlanta, Georgia, USA, (2010), 5949–5954.
[23] S. Ratschan and Z. She, Providing a basin of attraction to a target region of polynomial systems by computation of Lyapunov-like functions, *SIAM J. Control and Optimization*, **48** (2010), 4377–4394.
[24] R. Tarjan, Depth-first search and linear graph algorithms, *SIAM J. Comput.*, **1** (1972), 146–160.
[25] A. R. Teel and L. Praly, A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions, *ESAIM Control Optim. Calc. Var.*, **5** (2000), 313–367.
[26] W. Tucker, A rigorous ODE solver and Smale's 14th problem, *Found. Comput. Math.*, **2** (2002), 53–117.

[27] T. Yoshizawa, On the stability of solutions of a system of differential equations, *Memoirs of the College of Science, University of Kyoto, Series A: Mathematics*, **29** (1955), 27–33.

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