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Efficient Lyapunov Function computation for systems with multiple exponentially stable equilibria

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Abstract

Recently a method was presented to compute Lyapunov functions for nonlinear systems with multiple local attractors [5]. This method was shown to succeed in delivering algorithmically a Lyapunov function giving qualitative information on the system's dynamics, including lower bounds on the attractors' basins of attraction. We suggest a simpler and faster algorithm to compute such a Lyapunov function if the attractors in question are exponentially stable equilibrium points. Just as in [5] one can apply the algorithm and expect to obtain partial information on the system dynamics if the assumptions on the system at hand are only partially fulfilled. We give four examples of our method applied to different dynamical systems from the literature.

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Keywords: Lyapunov function, multiple equilibria, nonlinear system, algorithm

1 Introduction

We consider continuous time systems given by ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{f} \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ is two-times continuously differentiable. We denote the solution to (1.1) started at $\boldsymbol{\xi}$ at time $t = 0$ by $t \mapsto \boldsymbol{\phi}(t, \boldsymbol{\xi})$. A so-called complete Lyapunov functions for the system (1.1) is a continuous function from the state-space to the real numbers that characterizes the decomposition of the flow into a gradient-like part and a chain-recurrent part [1, 7, 18]. For a more accessible overview of this fact, sometimes referred to as the *Fundamental Theorem of Dynamical Systems* cf. e.g. [25, 26]. A complete Lyapunov function is decreasing along solution trajectories on the gradient-like part of the flow and constant on the transitive components of the chain-recurrent part.

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Whereas there have been numerous suggestions of how to compute Lyapunov function for systems on a domain containing one stable equilibrium, cf. e.g. [14] for a recent review, there have been much fewer publications on the numerical construction of Lyapunov functions with a more complicated chain-recurrent set.

In [5] a method was presented to compute a function V resembling a complete Lyapunov function for the system (1.1) on a compact subset of its state-space $\mathcal{D} \subset \mathbb{R}^d$, which is allowed to contain multiple attractors. In this method one first computes outer approximations of the attractors using a graph theoretic method [19, 15] followed by a subsequent numerical computation of a Massera-like Lyapunov function candidate [24], see [20] for an overview and classification of the different construction methods. The candidate is then used to parameterize a continuous and piecewise affine (CPA) Lyapunov function, of which the decrease condition along solution trajectories can be verified exactly by checking a certain set of linear inequalities. This set of linear inequalities comes from the so-called CPA method to compute Lyapunov functions, in which linear optimization is used to parameterize a CPA Lyapunov function satisfying these linear inequalities [23, 16, 13]. This method has been adapted to different kinds of systems like differential inclusions [2] and discrete-time systems [12] and to systems with different stability properties like ISS stability [21] and control systems [3]. The main advantage of the CPA method is that it delivers a function that is guaranteed to satisfy the conditions for a Lyapunov function exactly and its main drawback is that as it involves solving a large linear programming problem it is not particularly fast. It has therefore been used in combination with other faster methods to compute Lyapunov functions, the main idea being to compute a Lyapunov function candidate by the faster method and then use the CPA method to verify if the candidate indeed satisfies all conditions of a Lyapunov function. For this methodology cf. e.g. [4, 17, 22, 11] and the paper [5], on which we base this work.

1.1 Notation:

We write vectors $\mathbf{x} \in \mathbb{R}^d$ in boldface, $\|\mathbf{x}\|$ denotes the Euclidian norm of $\mathbf{x}$, and $\mathcal{B}_\varepsilon(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{y}\| < \varepsilon\}$ is an open ball centered at $\mathbf{x}$ with radius $\varepsilon > 0$. We write subsets $\mathcal{K} \subset \mathbb{R}^d$ in calligraphic and its interior is denoted by $\mathcal{K}^\circ$ and its closure by $\overline{\mathcal{K}}$. C^m stands for the set of all m-times continuously differentiable functions, the domain and codomain should always be obvious from the context. We denote by $\mathcal{A}(\mathbf{y}) := \{\mathbf{x} \in \mathbb{R}^d : \limsup_{t\to\infty} \|\boldsymbol{\phi}(t, \mathbf{x}) - \mathbf{y}\| = 0\}$ the *basin of attraction* of a stable equilibrium $\mathbf{y}$. A Lipschitz constant $L > 0$ for $\mathbf{f}$ on a set $\mathcal{K}$ is a constant such that $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \le L\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{K}$. If there exists a Lipschitz constant for $\mathbf{f}$ on every compact set $\mathcal{K} \subset \mathbb{R}^d$, $\mathbf{f}$ is said to be *locally Lipschitz*.

2 The Method

In [5] one first computes outer approximations $\mathcal{F}_i$ of the local attractors Ω_i, $i = 1, 2, \ldots, N$, of the system (1.1) contained in some predefined compact set $\mathcal{D} \subset \mathbb{R}^d$ of interest. Then one defines a sufficiently smooth functions $\gamma : \mathcal{D} \to \mathbb{R}^+$ ($\mathbb{R}^+ := [0, \infty)$) such that $\gamma(\mathbf{x}) = 0$ whenever $\mathbf{x} \in \bigcup_{i=1}^N \mathcal{F}_i$ and $\gamma(\mathbf{x}) > 0$ otherwise. As shown in [5, Theorem 3.2] the function

$$W(\mathbf{x}) := \int_0^T \gamma(\boldsymbol{\phi}(t, \mathbf{x}))dt$$

then has a negative orbital derivative

$$W'(\mathbf{x}) := \limsup_{h\to 0+} \frac{W(\boldsymbol{\phi}(h, \mathbf{x})) - W(\mathbf{x})}{h} \quad \Big(= \nabla W(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \ \text{ if } W \in C^1 \Big)$$

in a neighborhood of each of the $\mathcal{F}_i$. Further, $W(\mathbf{x}) = 0$ for all $\mathbf{x} \in \Omega_i$ and $W(\mathbf{x}) > 0$ for all other $\mathbf{x}$ in this neighborhood of $\mathcal{F}_i$. Thus W resembles a complete Lyapunov function in the sense that connected components of $W^{-1}([0,c])$, $c > 0$, that are compact subsets of $\mathcal{D}^\circ$ and enclose an $\mathcal{F}_i$ completely, are necessary in the basin of attraction of Ω_i. That is, if the system is started in such a set $W^{-1}([0,c])$, then $\boldsymbol{\phi}(t,\mathbf{x})$ ends up in Ω_i as $t \to \infty$.

Clearly $W(\mathbf{x})$ can only be computed in a finite number of points and one of the contributions of [5] is to make the simple idea listed above into a useful algorithm by combining it with the CPA method. Thus one first triangulates the set $\mathcal{D}$ of interest, i.e. subdivides it into a collection $\mathcal{T}$ of d-simplices fulfilling certain properties [5, Definition 4.1]. Then $W(\boldsymbol{\xi})$ is approximated at every vertex $\boldsymbol{\xi}$ of every simplex $\mathfrak{S} \in \mathcal{T}$ by numerically solving the initial-value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \boldsymbol{\xi}$, and numerically integrate $t \to \gamma(\boldsymbol{\phi}(t,\boldsymbol{\xi}))$ over the interval $[0,T]$, where $T > 0$ is a fixed constant.

In what follows we will show the following: If the attractors Ω_i are exponentially stable equilibrium points instead of more complicated attractors, then one can get similar results to [5] by setting $\gamma(\mathbf{x}) := \|\mathbf{f}(\mathbf{x})\|$. This not only simplifies the method but is considerably faster because one does not have to identify the attractors and compute outer approximations.

More exactly, we show in Theorem 1 that if $\mathcal{D}$ contains one or multiple exponentially stable equilibrium points, then the function

$$V(\mathbf{x}) = \int_0^T \|\mathbf{f}(\boldsymbol{\phi}(\tau,\mathbf{x}))\| d\tau \tag{2.1}$$

resembles a complete Lyapunov function in a neighborhood $\mathcal{K}_\mathbf{y} \subset \mathcal{D}$ of each equilibrium $\mathbf{y} \in \mathcal{D}$ if $T > 0$ is large enough. The fact that the CPA approximation of V, defined by computing $V(\mathbf{x})$ at every vertex $\mathbf{x}$ of every simplex of a triangulation of $\mathcal{D}$ and interpolating the values over the simplices, also resembles a complete Lyapunov function is then delivered by [5, Theorem 4.2]. Note that since $\mathbf{f}(\boldsymbol{\phi}(\tau,\mathbf{x})) = \dot{\boldsymbol{\phi}}(\tau,\mathbf{x})$ the formula (2.1) defines $V(\mathbf{x})$ to be the length of the trajectory piece $\{\boldsymbol{\phi}(\tau,\mathbf{x}) : \tau \in [0,T]\}$.

Before we state the main theoretical contribution of this paper, Theorem 1, we prove a useful lemma.

Lemma 1. *Assume $\mathbf{y}$ is an exponentially stable equilibrium of the system* (1.1) *and let $\mathcal{K}_\mathbf{y} \subset \mathcal{A}(\mathbf{y})$ be compact. Then there exist constants $C \geq 1$ and $\lambda > 0$, such that*

$$\|\boldsymbol{\phi}(t,\mathbf{x}) - \mathbf{y}\| \leq Ce^{-\lambda t}\|\mathbf{x} - \mathbf{y}\| \quad \text{for all } \mathbf{x} \in \mathcal{K}_\mathbf{y} \text{ and } t \geq 0. \tag{2.2}$$

Proof. Since $\mathbf{y}$ is exponentially stable there exists a ball $\mathcal{B}_\varepsilon(\mathbf{y})$ and constants $C^* \geq 1$ and $\lambda > 0$ such that

$$\|\boldsymbol{\phi}(t,\mathbf{x}) - \mathbf{y}\| \leq C^* e^{-\lambda t}\|\mathbf{x} - \mathbf{y}\| \quad \text{for all } \mathbf{x} \in \mathcal{B}_\varepsilon(\mathbf{y}) \text{ and } t \geq 0.$$

Pick an arbitrary $\boldsymbol{\xi} \in \mathcal{K}_\mathbf{y} \setminus \mathcal{B}_\varepsilon(\mathbf{y})$. Then there exists an $s_{\boldsymbol{\xi}} > 0$ such that $\|\boldsymbol{\phi}(s_{\boldsymbol{\xi}},\boldsymbol{\xi}) - \mathbf{y}\| < \varepsilon/2$. Fix $\delta^*_{\boldsymbol{\xi}} > 0$ so small that $\overline{\mathcal{B}_{\delta^*_{\boldsymbol{\xi}}}} \subset \mathcal{A}(\mathbf{y})$. The set $\boldsymbol{\phi}([0,s_\mathbf{x}], \overline{\mathcal{B}_{\delta^*_{\boldsymbol{\xi}}}})$ is the image of the compact set $[0,s_\mathbf{x}] \times \overline{\mathcal{B}_{\delta^*_{\boldsymbol{\xi}}}}$ under the continuous mapping $\boldsymbol{\phi}$ and is thus compact. Since $\mathbf{f}$ is C^2 it is locally Lipschitz and there exists a Lipschitz constant $L_{\boldsymbol{\xi}} > 0$ for $\mathbf{f}$ on $\boldsymbol{\phi}([0,s_\mathbf{x}], \overline{\mathcal{B}_{\delta^*_{\boldsymbol{\xi}}}})$. For $\delta_{\boldsymbol{\xi}}$, $0 < \delta_{\boldsymbol{\xi}} < e^{-Ls_\mathbf{x}}\delta^*_{\boldsymbol{\xi}}$, small enough we can ensure that, cf. e.g. [27, §12.V], $\|\boldsymbol{\phi}(s_{\boldsymbol{\xi}},\mathbf{x}) - \boldsymbol{\phi}(s_{\boldsymbol{\xi}},\boldsymbol{\xi})\| \leq \|\mathbf{x} - \boldsymbol{\xi}\| e^{L_{\boldsymbol{\xi}} s_{\boldsymbol{\xi}}} < \varepsilon/2$ for all $\mathbf{x} \in \mathcal{B}_{\delta_{\boldsymbol{\xi}}}(\boldsymbol{\xi}) \cap [\mathcal{K}_\mathbf{y} \setminus \mathcal{B}_\varepsilon(\mathbf{y})]$. The compactness of $\mathcal{K}_\mathbf{y} \setminus \mathcal{B}_\varepsilon(\mathbf{y})$ now delivers the existence of an $s > 0$ and an $L > 0$ such that $\|\boldsymbol{\phi}(s,\mathbf{x}) - \mathbf{y}\| < \varepsilon$ for all $\mathbf{x} \in \mathcal{K}_\mathbf{y} \setminus \mathcal{B}_\varepsilon(\mathbf{y})$. Similarly we get for these $\mathbf{x}$ that for all t, $0 \leq t \leq s$, that $\|\boldsymbol{\phi}(t,\mathbf{x}) - \mathbf{y}\| = \|\boldsymbol{\phi}(t,\mathbf{x}) - \boldsymbol{\phi}(t,\mathbf{y})\| \leq e^{(L+\lambda)s}e^{-\lambda t}\|\mathbf{x} - \mathbf{y}\|$

and for $t \geq s$ that (recall that $\|\mathbf{x}-\mathbf{y}\| \geq \varepsilon$)

$$\begin{aligned}\|\boldsymbol{\phi}(t,\mathbf{x})-\mathbf{y}\| &= \|\boldsymbol{\phi}(t-s,\boldsymbol{\phi}(s,\mathbf{x}))-\mathbf{y}\| \leq C^* e^{-\lambda(t-s)}\|\boldsymbol{\phi}(s,\mathbf{x})-\mathbf{y}\| \\ &\leq C^* e^{\lambda s}e^{-\lambda t}\varepsilon \leq C^* e^{\lambda s}e^{-\lambda t}\|\mathbf{x}-\mathbf{y}\|.\end{aligned}$$

Hence (2.2) holds for all $\mathbf{x} \in \mathcal{K}_\mathbf{y}$ with $C := e^{\lambda s}\max\{C^*, e^{Ls}\}$. □

Theorem 1. *Assume $\mathbf{y} \in \mathcal{D}$ is an exponentially stable equilibrium of the system (1.1) and let $\mathcal{K}_\mathbf{y} \subset \mathcal{D}^\circ$ be a compact neighborhood of $\mathbf{y}$. Then there exists constants $a, b, c, T > 0$ such that*

$$a\|\mathbf{x}-\mathbf{y}\| \leq V(\mathbf{x}) \leq b\|\mathbf{x}-\mathbf{y}\| \quad \textit{and} \tag{2.3}$$

$$V'(\mathbf{x}) \leq -c\|\mathbf{x}-\mathbf{y}\| \tag{2.4}$$

for all $\mathbf{x} \in \mathcal{K}_\mathbf{y}$, where V is defined by the formula (2.1) on $\mathcal{D}$ using the $T > 0$ above.

Proof. By Lemma 1 there exists constants $C \geq 1$ and $\lambda > 0$, such that

$$\|\boldsymbol{\phi}(t,\mathbf{x})-\mathbf{y}\| \leq Ce^{-\lambda t}\|\mathbf{x}-\mathbf{y}\| \quad \text{for all } \mathbf{x} \in \mathcal{K}_\mathbf{y} \text{ and } t \geq 0.$$

Fix an $\mathbf{x} \in \mathcal{K}_\mathbf{y}$. We calculate

$$\begin{aligned}V(\mathbf{x}) &= \int_0^T \|\mathbf{f}(\boldsymbol{\phi}(\tau,\mathbf{x}))\|d\tau \geq \|\int_0^T \mathbf{f}(\boldsymbol{\phi}(\tau,\mathbf{x}))d\tau\| = \|\int_0^T \dot{\boldsymbol{\phi}}(\tau,\mathbf{x})d\tau\| \\ &= \|\boldsymbol{\phi}(T,\mathbf{x})-\mathbf{x}\| \geq \|\mathbf{x}-\mathbf{y}\| - \|\boldsymbol{\phi}(T,\mathbf{x})-\mathbf{y}\| \geq (1-Ce^{-\lambda T})\|\mathbf{x}-\mathbf{y}\|.\end{aligned} \tag{2.5}$$

Let $L > 0$ be a Lipschitz constant for $\mathbf{f}$ on the compact set $\overline{C\mathcal{K}_\mathbf{y}} := \overline{\{C\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \in \mathcal{K}_\mathbf{y}\}}$. Then we get

$$\begin{aligned}V(\mathbf{x}) &= \int_0^T \|\mathbf{f}(\boldsymbol{\phi}(\tau,\mathbf{x}))\|d\tau \leq \int_0^T L\|\boldsymbol{\phi}(\tau,\mathbf{x})-\mathbf{y}\|d\tau \\ &\leq LC\|\mathbf{x}-\mathbf{y}\|\int_0^T e^{-\lambda\tau}d\tau = \frac{LC}{\lambda}(1-e^{-\lambda T})\|\mathbf{x}-\mathbf{y}\|.\end{aligned} \tag{2.6}$$

Further

$$\begin{aligned}V'(\mathbf{x}) &= \limsup_{h\to 0+}\frac{1}{h}\left(V(\boldsymbol{\phi}(h,\mathbf{x}))-V(\mathbf{x})\right) \\ &= \limsup_{h\to 0+}\frac{1}{h}\left(\int_h^{T+h}\|\mathbf{f}(\boldsymbol{\phi}(\tau,\mathbf{x}))\|d\tau - \int_0^T\|\mathbf{f}(\boldsymbol{\phi}(\tau,\mathbf{x}))\|d\tau\right) \\ &= \limsup_{h\to 0+}\frac{1}{h}\left(\int_T^{T+h}\|\mathbf{f}(\boldsymbol{\phi}(\tau,\mathbf{x}))\|d\tau - \int_0^h\|\mathbf{f}(\boldsymbol{\phi}(\tau,\mathbf{x}))\|d\tau\right) = \|\mathbf{f}(\boldsymbol{\phi}(T,\mathbf{x}))\| - \|\mathbf{f}(\mathbf{x})\| \\ &\leq L\|\boldsymbol{\phi}(T,\mathbf{x})-\mathbf{y}\| - \|\mathbf{f}(\mathbf{x})\| \leq LCe^{-\lambda T}\|\mathbf{x}-\mathbf{y}\| - \|\mathbf{f}(\mathbf{x})\|\end{aligned} \tag{2.7}$$

Because $\mathbf{f}(\mathbf{y}) = \mathbf{0}$ and $\mathbf{f} \in C^2$ we get by Taylor's theorem that there is a compact neighborhood $\mathcal{F}_\mathbf{y}$ of $\mathbf{y}$ and a constant $F > 0$ such that

$$\|\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{y})(\mathbf{x}-\mathbf{y})\| \leq F\|\mathbf{x}-\mathbf{y}\|^2 \quad \text{for all } \mathbf{x} \in \mathcal{F}_\mathbf{y}.$$

Since $\mathbf{y}$ is exponentially stable the matrix $D\mathbf{f}(\mathbf{y})$ is nonsingular and with $\mu > 0$ as the square-root of the smallest eigenvalue of the positive definite matrix $D\mathbf{f}(\mathbf{y})^T D\mathbf{f}(\mathbf{y})$ we get

$$\|\mathbf{f}(\mathbf{x})\| \geq \|D\mathbf{f}(\mathbf{y})(\mathbf{x}-\mathbf{y})\| - F\|\mathbf{x}-\mathbf{y}\|^2 \geq \|\mathbf{x}-\mathbf{y}\|(\mu - F\|\mathbf{x}-\mathbf{y}\|) \geq \frac{1}{2}\mu\|\mathbf{x}-\mathbf{y}\| \qquad (2.8)$$

for all $\mathbf{x} \in \mathcal{F}_{\mathbf{y}}$ with $\|\mathbf{x}-\mathbf{y}\| \leq \mu/(2F)$. Thus, there is an ε-ball $\mathcal{B}_\varepsilon(\mathbf{y})$ around $\mathbf{y}$ such that (2.8) holds true for all $\mathbf{x} \in \mathcal{B}_\varepsilon(\mathbf{y})$. Set

$$\alpha^* := \inf_{\mathbf{x}\in\mathcal{K}_{\mathbf{y}}\setminus\mathcal{B}_\varepsilon(\mathbf{y})} \frac{\|\mathbf{f}(\mathbf{x})\|}{\|\mathbf{x}-\mathbf{y}\|}.$$

Because $\mathcal{K}_{\mathbf{y}} \setminus \mathcal{B}_\varepsilon(\mathbf{y})$ is compact this infimum is indeed a minimum and because $\mathcal{K}_{\mathbf{y}} \setminus \mathcal{B}_\varepsilon(\mathbf{y})$ does not contain equilibrium points of $\mathbf{f}$ we have $\alpha^* > 0$. Set $\alpha := \min\{\alpha^*, \mu/2\}$. Then we have

$$\|\mathbf{f}(\mathbf{x})\| \geq \alpha\|\mathbf{x}-\mathbf{y}\| \quad \text{for all } \mathbf{x} \in \mathcal{K}_{\mathbf{y}}$$

and it follows by (2.7) that

$$V'(\mathbf{x}) \leq -(\alpha - LCe^{-\lambda T})\|\mathbf{x}-\mathbf{y}\|. \qquad (2.9)$$

Thus, for

$$T > \frac{1}{\lambda}\left(\ln(C) + \ln\left(\frac{L}{\alpha}\right)\right)$$

we have $T > 0$ because clearly $L \geq \alpha$, $C \geq 1$, and $\lambda > 0$. From $T > \ln(C)/\lambda$ we get from (2.5) the existence of an $a > 0$, from $T > 0$ and (2.6) the existence of a $b > 0$, and from (2.9) and $T > \ln(LC/\alpha)/\lambda$ the existence of a $c > 0$ such that

$$a\|\mathbf{x}-\mathbf{y}\| \leq V(\mathbf{x}) \leq b\|\mathbf{x}-\mathbf{y}\| \quad \text{and} \quad V'(\mathbf{x}) \leq -c\|\mathbf{x}-\mathbf{y}\| \quad \text{for all } \mathbf{x} \in \mathcal{K}_{\mathbf{y}}.$$

□

3 Examples

In this section we present four examples to demonstrate out method in action. In all the examples we used the Runge-Kutta 4th order method RK4 to estimate the solution $t \mapsto \boldsymbol{\phi}(t, \mathbf{x})$ of the system (1.1) at all vertices $\mathbf{x}$ of all simplices of the triangulation. We use the following notation throughout:

(I) T is the upper limit of integration, that is the Lyapunov function V is calculated at $\mathbf{x}$ by estimating $\int_0^T \|\mathbf{f}(\boldsymbol{\phi}(\tau, \mathbf{x})\| d\tau$.

(II) Δt is the time-step used in the $RK4$ method. We used $\Delta t = 5 \cdot 10^3$ in all the examples.

(III) $\mathcal{D}$ is the domain on which we calculate the Lyapunov function.

(IV) N is the number of simplices used in the triangulation of $\mathcal{D}$. We used the regular triangulations $\mathcal{T}_\rho^{\text{std}}$, $\rho > 0$, cf. [11, Definition 4.8]. Their vertices are given by $\rho\mathbb{Z}^d \cap \mathcal{D}$.

We estimate $V(\mathbf{x})$ at the vertices $\mathbf{x}$ of the triangulation by

$$V(\mathbf{x}) = \int_0^T \|\mathbf{f}(\mathbf{x}(t))\| dt \approx \sum_{i=0}^{\lfloor T/\Delta t \rfloor} \Delta t \|\mathbf{f}(\boldsymbol{\phi}(i\Delta t, \mathbf{x})\|.$$

In all our examples we normalize V such that $\max_{\mathbf{x}\in\mathcal{D}} V(\mathbf{x}) = 1$.

3.1 Example 1

We start by by examining the following equation that was also considered in [5, Example 2].

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= 0.3(4x - x^3 - y). \end{aligned} \tag{3.1}$$

A simple analysis of the system yields that it contains two stable equilibrium points at $(-2, 0)$ and $(2, 0)$, and additionally an unstable equilibrium point at $(0, 0)$. In this example we set $T = 128$, $\mathcal{D} = [-4, 4]^2$, and $N = 819, 200$. The resulting function is depicted in Figure 1, and in Figure 2 we plot a few level curves for the function, as well as marking the simplices where the orbital derivative is non-negative. In total there are 4,315 such simplices, or roughly 0.53% of them. Any closed level curve which does not intersect such a simplex is a boundary of a forward invariant set for the system. The results presented here are practically identical to the results form [5, Example 2] and the omission of computing outer approximations of the attractors did not deteriorate the results in any way.

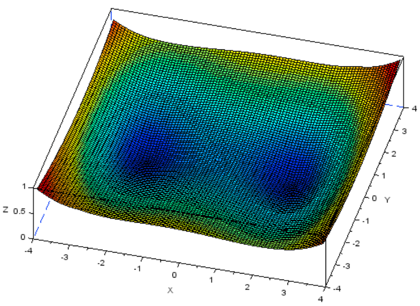

Figure 1: The computed CPA Lyapunov function for the system (3.1).

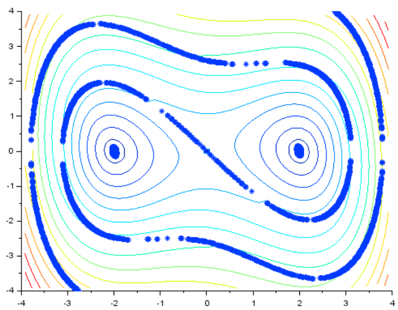

Figure 2: A few level curves for the Lyapunov function in Figure 1 for the system (3.1). The simplices where the orbital derivative is non-negative are depicted with a blue mark.

3.2 Example 2

Next we examine the following system taken from [9] and [8, §6.1].

$$\begin{aligned} \dot{x_1} &= \frac{\alpha_1}{1 + x_2^{\beta}} - x_1 \\ \dot{x_2} &= \frac{\alpha_2}{1 + x_1^{\gamma}} - x_2. \end{aligned} \tag{3.2}$$

This equation describes the genetic toggle switch in Escherichia coli and was constructed in [10]. Here we have selected $\alpha_1 = 1.3$, $\alpha_2 = 1.0$, $\beta = 3.0$, and $\gamma = 10.0$ as in [9, 8] and we refer to these publications for an analysis of the dynamical properties of the system with these parameters.

For our computations we choose $T = 16$, $\mathcal{D} = [-1, 3]^2$ and $N = 204,800$. The resulting function is depicted in Figure 3, and in figure 4 we plot a few level curves for the function, as well as marking the simplices where the orbital derivative is non-negative. As before, any closed level curve which does not intersect such a simplex is a boundary of a forward invariant set for the system. The simplices where the orbital derivative is non-negative are depicted with a blue marks in Figure 4. In total there are 586 such simplices, or roughly 0.28% of them.

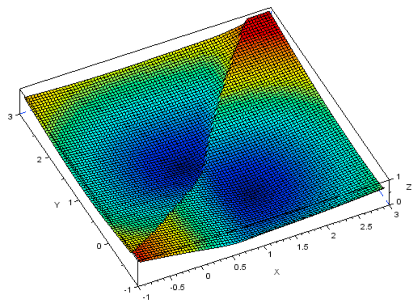

Figure 3: The computed CPA Lyapunov function for the system 3.2.

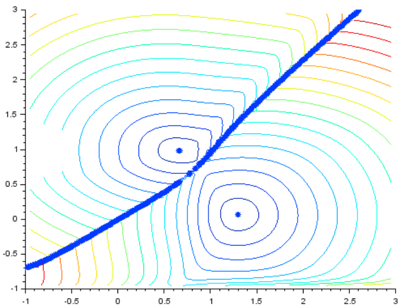

Figure 4: A few level curves for the Lyapunov function in Figure 3 for the system (3.2). The simplices where the orbital derivative is non-negative are depicted with a blue mark.

3.3 Example 3

In our third example, we look at the following system in $\mathbb{R}^3$.

$$\begin{aligned} \dot{x_1} &= -x_1 - x_2 - x_3 \\ \dot{x_2} &= \sin(x_1) - 2x_2(1 + x_1) + x_3 \\ \dot{x_3} &= x_1(1 + x_1) + x_2 - 2\sin(x_3). \end{aligned} \tag{3.3}$$

This system was previously examined in [17]. In this example we have chosen $T = 32$, $\mathcal{D} = [-4, 4]^3$, and $N = 10,368,000$. This system contains a stable equilibrium at the origin, and as we can see in Figure 6 we obtain a forward invariant set containing it. The set of all simplices with non-negative orbital derivative consists of 25,747 simplices, our roughly 0.25% of all the simplices in the triangulation.

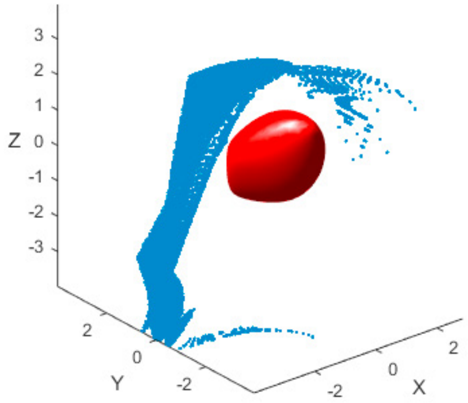

Figure 5: A level-surface for the Lyapunov function computed for the system (3.3). The simplices where the orbital derivative is non-negative are marked with blue dots. Since the level-surface does not intersect the set of simplices with a non-negative orbital derivative, the surface is the boundary of a forward invariant set for the system.

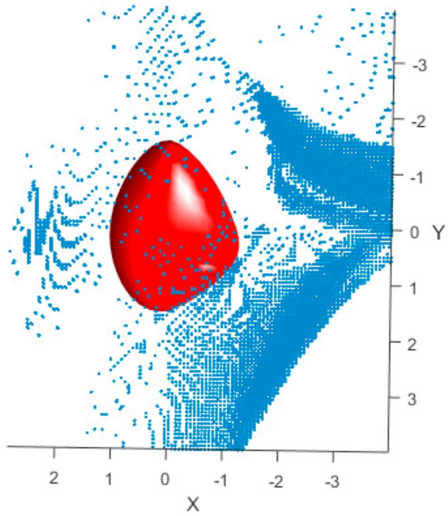

Figure 6: The level surface from Figure 6 viewed from another angle.

3.4 Example 4

Our final example is a simplified model of a repressilator [6] taken from [8, §6.3].

$$\begin{aligned} \dot{x}_1 &= \frac{\alpha}{1+x_2^\beta} - x_1 \\ \dot{x}_2 &= \frac{\alpha}{1+x_3^\beta} - x_2 \\ \dot{x}_3 &= \frac{\alpha}{1+x_1^\beta} - x_3. \end{aligned} \tag{3.4}$$

Here we pick $\alpha = 5.0$ and $\beta = 2.0$ as in [8, §6.3]. Furthermore, we choose $T = 16$, $\mathcal{D} = [-0.484, 3.516]^3$, and $N = 6,000,000$. Level-surfaces of the computed Lyapunov function are depicted in figures 7 and 8. The set of all simplices with possibly positive orbital derivative consists of 526 simplices, our roughly 0.01% of all the simplices in the grid.

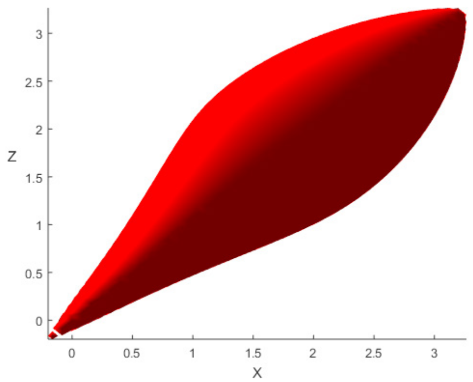

Figure 7: A level-surface for Lyapunov function computed for the system (3.4). The simplices where the orbital derivative non-negative are marked with blue dots. Since the level-surface does not intersect the set of simplices with a non-negative orbital derivative, the surface is the boundary of a forward invariant set for the system.

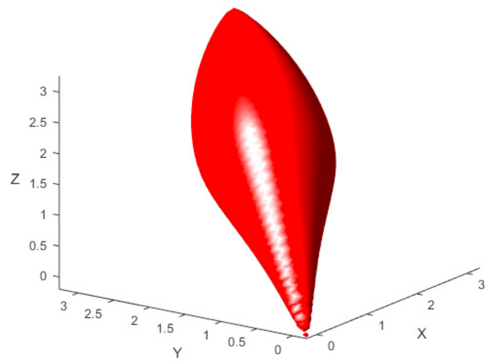

Figure 8: The level surface from Figure 6 viewed from another angle.

4 Conclusions

We presented a novel numerical method to compute Lyapunov functions for non-linear systems on compact domains. We showed in Theorem 1 that the computed Lyapunov functions deliver essential information on the qualitative behavior of the dynamics if the domain includes one or multiple exponentially stable equilibria. The proposed method is inspired by the method presented in [5] for systems with multiple local attractors, but is numerically much less demanding because one does not have to compute outer approximations of the attractors initially. We gave four examples in two- and three dimensions to demonstrate the power and applicability of our method. It delivers a fast, simple, and easy to use analytical tool to extract important and exact information on the system dynamics of non-linear systems with multiple equilibria..

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