Lyapunov Functions for Almost Sure Exponential Stability

Hjortur Björnsson and Sigurdur Freyr Hafstein

Abstract We present a generalization of results obtained by X. Mao in his book "Stochastic Differential Equations and Applications" (2008). When studying what Mao calls "almost sure exponential stability", essentially a negative upper bound on the almost sure Lyapunov exponents, he works with Lyapunov functions that are twice continuously differentiable in the spatial variable and continuously differentiable in time. Mao gives sufficient conditions in terms of such a Lyapunov function for a solution of a stochastic differential equation to be almost surely exponentially stable. Further, he gives sufficient conditions of a similar kind for the solution to be almost surely exponentially unstable. Unfortunately, this class of Lyapunov functions is too restrictive. Indeed, R. Khasminskii showed in his book "Stochastic Stability of Differential Equations" (1979/2012) that even for an autonomous stochastic differential equation with constant coefficients, of which the solution is stochastically stable and such that the deterministic part has an unstable equilibrium, there cannot exists a Lyapunov function that is differentiable at the origin. These restrictions are inherited by Mao's Lyapunov functions. We therefore consider Lyapunov functions that are not necessarily differentiable at the origin and we show that the sufficiency conditions Mao proves can be generalized to Lyapunov functions of this form.

Keywords Almost sure exponential stability · Lyapunov function · Almost sure Lyapunov exponent

1 Introduction

Lyapunov methods, as first described in [1], have been widely used to study the behaviour of various dynamical systems, both real-world examples or purely theo-

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retical ones. This is a very active field due to the complicated dynamics exhibited in several real-world systems, as for example the wobblestone model presented in [2]. Other specific examples include the dynamics of the double [3] or triple pendulum [4, 5], where Lyapunov exponents were used to study the chaotic behavior of the systems. Often it is necessary to modify a dynamical system to include either an unknown force, or to consider the perturbation of the system by some noise, and that is where stochastic differential equations (SDEs) are commonly used. Here in this paper, we are concerned with applying Lyapunov methods for classical dynamical systems to the stochastic framework, as done by Khasminskii [8].

We work in a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$ with a right continuous filtration $\{\mathscr{F}_t\}_{t\geq 0}$ and such that $\mathscr{F}_0$ contains all $\mathbf{P}$ null sets. In this paper we consider strong solutions of the d-dimensional stochastic differential equation

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t) \quad \text{on } t \geq t_0 \tag{1}$$

where $B(t)$ is an m-dimensional Brownian motion. For a more detailed description of the setting cf. [9, Sec. 2.1]. We assume that for any given initial value $x(t_0) = x_0 \in \mathbf{R}^d$ there exists a unique global solution, denoted by $t \mapsto x(t, t_0, x_0)$, with continuous sample paths. Furthermore, we assume that

$$f(0, t) = 0 \quad \text{and} \quad g(0, t) = 0 \quad \text{for all } t \geq t_0.$$

Sufficient condition for the existence of such solutions are, for example, given by the following statement, cf. [9, Thm. 2.3.6].

For any real number $T > 0$ and integer $n \geq 1$, the following hold true:

1. There exists a positive constant $K_{T,n}$, such that for all $t \in [t_0, T]$ and all $x, y \in \mathbf{R}^d$ with $|x| \vee |y| \leq n$,

$$|f(x,t) - f(y,t)|^2 \bigvee |g(x,t) - g(y,t)|^2 \leq K_{T,n}|x-y|^2.$$

2. There exists a positive constant K_T, such that for all $(x, t) \in \mathbf{R}^d \times [t_0, T]$

$$x^\top f(x,t) + \frac{1}{2}|g(x,t)|^2 \leq K_T(1 + |x|^2).$$

Here $|\cdot|$ is the Euclidean norm and the symbols $\wedge$ and $\vee$ are defined to be the minimum and the maximum respectively:

$$a \wedge b := \min(a, b) \quad \text{and} \quad a \vee b := \max(a, b).$$

Corresponding to the initial value $x(t_0) = 0$, we have the solution $x(t) = 0$ for all t. This solution is called the trivial solution. In this paper we are studying the stability of the trivial solution and, more specifically, when it is almost surely exponentially stable. This definition is taken from Mao's book [9, Def. 4.3.1], see also e.g. [6, 11].

Definition 1 The trivial solution of (1) is said to be *almost surely exponentially stable* if

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t, t_0, x_0)| < 0$$

almost surely, for all $x_0 \in \mathbf{R}^d$.

First, we clarify some of the notation used in the paper. For our purposes all integrals in this paper of the form $\int \cdot \, \mathrm{d}B(s)$ are to be interpreted in the Itô sense. We write $b_n \uparrow a$ if the sequence b_n is increasing and has limit a. We denote by $\mathscr{L}^2(\mathbf{R}_+, \mathbf{R}^{d\times m})$ the family of all $(d \times m)$-matrix valued measurable $\{\mathscr{F}_t\}_{t\geq 0}$-adapted processes $f = \{f(t)\}_{t\geq 0}$ such that

$$\int_0^T |f(t)|^2 \mathrm{d}t < \infty \quad \text{a.s. for every } T > 0$$

and by $\mathscr{M}^2(\mathbf{R}_+, \mathbf{R}^{d\times m})$ the family of all processes $f \in \mathscr{L}^2(\mathbf{R}_+, \mathbf{R}^{d\times m})$ such that

$$\mathbf{E}\left\{\int_0^T |f(t)|^2 \mathrm{d}t\right\} < \infty \quad \text{for every } T > 0.$$

Here $\mathbf{E}$ denotes the expectation and a.s. is an abbreviation for *almost surely* as usual. Let $f \in \mathscr{M}^2(\mathbf{R}_+, \mathbf{R}^{d\times m})$ and consider the process

$$M_t = \int_0^t f(s)\mathrm{d}B(s)$$

then there exists a t-continuous version of the process M_t. Furthermore the process is $\{\mathscr{F}_t\}$ adapted and is a square integrable martingale [10, Thm. 3.2.5]. By the preceding remark, we will assume that $\int_0^t f(s)\mathrm{d}B(s)$ refers to a t-continuous version of the integral.

A sequence of stopping times $\{\tau_k\}_{k\geq 1}$ is called a *localization* if it is non-decreasing and $\tau_k \uparrow \infty$ almost surely. A right continuous adapted process $M = \{M_t\}_{t\geq 0}$ is called a *local martingale* if there exists a localization $\{\tau_k\}_{k\geq 1}$ such that the process $\{M_{\tau_k \wedge t} - M_0\}_{t\geq 0}$ is a martingale for every $k \geq 1$. We denote the quadratic variation of a continuous local martingale M by $\langle M, M\rangle_t$, which is the unique continuous adapted process of finite variation, such that $\{M_t^2 - \langle M, M\rangle_t\}_{t\geq 0}$ is a continuous local martingale which takes the value 0 at $t = 0$.

Let M_t be a continuous martingale of the form

$$M_t = \int_0^t f(s)\mathrm{d}B(s).$$

Then the quadratic variation $\langle M, M\rangle_t$ is given by

$$\langle M, M\rangle_t = \int_0^t |f(s)|^2 \mathrm{d}s$$

almost surely [9, Thm. 1.5.14].

Let τ be a stopping time and let $[[0, \tau]]$ be the stochastic interval

$$[[0, \tau]] = \{(t, \omega) \in \mathbf{R}_+ \times \Omega : 0 \leq t \leq \tau(\omega)\}.$$

We now list a few facts needed to give rigid proofs of our results. For any $f \in \mathscr{L}^2(\mathbf{R}_+, \mathbf{R}^{d\times m})$ we can define a sequence of stopping times

$$\tau_n := n \wedge \inf\{t \geq 0 : \int_0^t |f(s)|^2 \mathrm{d}s \geq n\}.$$

It is easy to see that $\tau_n \uparrow \infty$ almost surely. Let I_A, for $A \subset \mathbf{R}_+ \times \Omega$, be the indicator function, that is $I_A(x) = 1$ if $x \in A$ and zero otherwise. Then we can define the process $g_n(t) = f(t)I_{[[0,\tau_n]]}(t)$. We see that $g_n \in \mathscr{M}^2(\mathbf{R}_+, \mathbf{R}^{d\times m})$ so the integral

$$J_n(t) = \int_0^t g_n(s)\mathrm{d}B(s)$$

is a martingale. That is to say, the process

$$J(t) := \int_0^t f(s)\mathrm{d}B(s)$$

is a local martingale with localization $\{\tau_n\}$, since for any $n \geq 1$

$$J(t \wedge \tau_n) = \int_0^{t\wedge\tau_n} f(s)\mathrm{d}B(s) = \int_0^t f(s)I_{[[0,\tau_n]]}(s)\mathrm{d}B(s) = \int_0^t g_n(s)\mathrm{d}B(s) = J_n(t)$$

is a martingale.

In his book [9], Mao considers Lyapunov functions $V(x, t) \in C^{2,1}(\mathbf{R}^d \times [t_0, \infty[; \mathbf{R}_+)$ where $C^{2,1}(\mathbf{R}^d \times [t_0, \infty[; \mathbf{R}_+)$ is the set of all continuous functions $\mathbf{R}^d \times [t_0, \infty[\to \mathbf{R}_+$, which are continuously differentiable twice in the first coordinate x, with $x \in \mathbf{R}^d$, and once in t with $t \in [t_0, \infty[$. Now define a differential operator L associated with (1) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^d f_i(x, t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^d [g(x, t)g^\top(x, t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}, \tag{2}$$

where $[g(x,t)g^\top(x,t)]_{ij}$ is the (i,j)-th component of the $(d \times d)$-matrix $gg^\top$ at (x,t). If $x(t)$ is a solution of (1) then by Itô's formula

$$dV(x(t),t) = LV(x(t),t)\mathrm{d}t + V_x(x(t),t)\mathrm{d}B(t)$$

where $V_x \in \mathbf{R}^{1\times d}$ is the derivative (gradient) of V with respect to x.

Khasminskii showed in his book [8, p. 154–155] that even for SDEs with constant coefficients there cannot exist Lyapunov functions that are differentiable at 0 unless the deterministic part of the SDE is already stable. Therefore we extend the results from Mao's book using the larger class of functions $C_0^{2,1}(\mathbf{R}^d \times [t_0,\infty[\,;\mathbf{R}_+)$ which are continuous, continuously differentiable in t, and twice continuously differentiable in x except at the point $x = 0$.

Below is a theorem taken from Mao's book [9] which we will use in the next chapter. For completeness we give a more worked out proof than in the book.

Theorem 1 [9, Thm. 1.7.4]
Let $g = (g_1,\ldots,g_m) \in \mathscr{L}^2(\mathbf{R}_+,\mathbf{R}^{d\times m})$, *and* T,α,β *be any numbers* ≥ 0. *Then*

$$\mathbf{P}\left\{\sup_{0\leq t\leq T}\left[\int_0^t g(s)\mathrm{d}B(s) - \frac{\alpha}{2}\int_0^t |g(s)|^2\mathrm{d}s\right] > \beta\right\} \leq \mathrm{e}^{-\alpha\beta}. \tag{3}$$

Proof Define the process

$$x(t) = \alpha\int_0^t g(s)\mathrm{d}B(s) - \frac{\alpha^2}{2}\int_0^t |g(s)|^2\mathrm{d}s$$

and for every integer $n \geq 1$, define the stopping time

$$\tau_n = \inf\left\{t \geq 0 : \left|\int_0^t g(s)\mathrm{d}B(s)\right| + \int_0^t |g(s)|^2\mathrm{d}s \geq n\right\}.$$

Then τ_n is a localization, and since

$$\begin{aligned}|x_n(t)| &\leq \alpha\left|\int_0^t g(s)I_{[[0,\tau_n]]}(s)\mathrm{d}B(s)\right| + \frac{\alpha^2}{2}\int_0^t |g(s)|^2 I_{[[0,\tau_n]]}\mathrm{d}s\\ &\leq \alpha n + \frac{\alpha^2}{2}n = n\frac{2\alpha+\alpha^2}{2}\end{aligned}$$

we see that the process $x_n(t) := x(t\wedge\tau_n)$ is bounded.

Apply Itô's formula to $\exp(x_n(t))$ and we obtain

$$\begin{aligned}\exp(x_n(t)) &= 1 + \int_0^t \exp(x_n(s))\mathrm{d}x_n(s) + \frac{\alpha^2}{2}\int_0^t \exp(x_n(s))|g(s)|^2 I_{[[0,\tau_n]]}(s)\mathrm{d}s \\ &= 1 + \Bigg(\alpha \int_0^t \exp(x_n(s))g(s)I_{[[0,\tau_n]]}(s)\mathrm{d}B(s) \\ &\qquad -\frac{\alpha^2}{2}\int_0^t \exp(x_n(s))|g(s)|^2 I_{[[0,\tau_n]]}(s)\mathrm{d}s\Bigg) \\ &\qquad + \frac{\alpha^2}{2}\int_0^t \exp(x_n(s))|g(s)|^2 I_{[[0,\tau_n]]}(s)\mathrm{d}s \\ &= 1 + \alpha \int_0^t \exp(x_n(s))g(s)I_{[[0,\tau_n]]}(s)\mathrm{d}B(s).\end{aligned}$$

The term inside the integral is bounded by $n^2\frac{2\alpha+\alpha^2}{2}$ almost surely, therefore the process $\exp(x_n)$ is a non negative martingale with $\mathbf{E}\{\exp(x_n(T))\} = 1$, for all $n \geq 1$. This construction is known as the Doléans-Dade exponential of the local martingale $Y_t := \int_0^t g(s)\mathrm{d}B(s)$, see [8, Thm. 26.8].
By Doob's martingale inequality [9, Thm. 1.3.8] we get that

$$\mathbf{P}\left\{\sup_{0\leq t\leq T} \exp[x_n(t)] \geq \mathrm{e}^{\alpha\beta}\right\} \leq \mathrm{e}^{-\alpha\beta}\mathbf{E}\{\exp(x_n(T))\} = \mathrm{e}^{-\alpha\beta}.$$

Then it follows that

$$\mathbf{P}\left\{\sup_{0\leq t\leq T} \frac{x_n(t)}{\alpha} > \beta\right\} \leq \mathrm{e}^{-\alpha\beta}.$$

Since this inequality holds for any $n \geq 1$, and

$$\lim_{n\to\infty} x_n(t) = x(t)$$

almost surely, we get by the dominated convergence theorem that

$$\mathbf{P}\left\{\sup_{0\leq t\leq T} \frac{x(t)}{\alpha} > \beta\right\} \leq \mathrm{e}^{-\alpha\beta}$$

and the proof is complete. □

2 The Theorems and Their Proofs

As discussed above, we state two theorems from Mao's book [9], more specifically Theorem 4.3.3 and Theorem 4.3.5, except we allow the Lyapunov functions V to be in the $C_0^{2,1}$ space instead of the too restrictive space $C^{2,1}$, like Mao does. The difference

is that in the former space the functions are not required to be differentiable at the origin, while functions in the latter one are smooth everywhere. As already explained before, this makes the results much more relevant and useful.
First we state and proof Theorem 4.3.3 from [9] with the weaker conditions. Note that, like above, $V_x \in \mathbf{R}^{1\times d}$ is the derivative (gradient) of V with respect to x.

Theorem 2 (advancement of Thm. 4.3.3 in Mao)
Assume there exists a function $V \in C_0^{2,1}(\mathbf{R}^d \times [t_0, \infty); \mathbf{R}_+)$ and constants $p > 0$, $c_1 > 0$, $c_2 \in \mathbf{R}$, $c_3 \geq 0$, such that for all $x \neq 0$ and $t \geq t_0$:

1. $c_1|x|^p \leq V(x, t)$,
2. $LV(x, t) \leq c_2 V(x, t)$,
3. $|V_x(x, t)g(x, t)|^2 \geq c_3 V^2(x, t)$.

Then

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t; t_0, x_0)| \leq -\frac{c_3 - 2c_2}{2p} \quad a.s.$$

for all $x_0 \in \mathbf{R}^d$. In particular, if $c_3 > 2c_2$, the trivial solution of Eq. (1) is almost surely exponentially stable, see Definition 1.

The proof here mostly follows Mao's original argument, but with some modifications, since the process $M(t)$ below isn't necessarily a martingale.

Proof Clearly the inequality holds for $x_0 = 0$ since $x(t, t_0, 0) = 0$ for all t. We only need to show the inequality for all $x_0 \neq 0$. Fix any $x_0 \neq 0$ and write $x(t) := x(t; t_0, x_0)$. It is well known that 0 is an inaccessible point, cf. e.g. [9, Lemma 4.3.2], that is to say, $x(t) \neq 0$ for all $t \geq t_0$ almost surely. Thus one can apply Itô's formula and get

$$\begin{aligned}
&\log V(x(t), t) \\
&= \log V(x_0, t_0) + \int_{t_0}^t \frac{LV(x(s), s)}{V(x(s), s)} \mathrm{d}s + M(t) - \frac{1}{2}\int_{t_0}^t \frac{|V_x(x(s), s)g(x(s), s)|^2}{(V(x(s), s))^2} \mathrm{d}s \\
&\leq \log V(x_0, t_0) + c_2(t - t_0) + M(t) - \frac{1}{2}\int_{t_0}^t \frac{|V_x(x(s), s)g(x(s), s)|^2}{(V(x(s), s))^2} \mathrm{d}s
\end{aligned}$$

where we used condition 2 for the last inequality and

$$M(t) := \int_{t_0}^t \frac{V_x(x(s), s)g(x(s), s)}{V(x(s), s)} \mathrm{d}B(s).$$

We claim the process

$$h(s) := \frac{V_x(x(s), s)g(x(s), s)}{V(x(s), s)}$$

is in $\mathscr{L}^2([t_0, \infty[, \mathbf{R}^d)$. Indeed, for almost all $\omega \in \Omega$, the trajectory of $x(t)(\omega)$, $t_0 \leq t \leq T$, is a compact subset of $\mathbf{R}^d \setminus \{0\}$. Hence, for almost all ω, the function $h(s)(\omega)$

is continuous on the compact set $t_0 \leq s \leq T$ and thus bounded. Since this holds true for all T, we have $h(s) \in \mathscr{L}^2([t_0, \infty[, \mathbf{R}^d)$.

Fix an arbitrary $\varepsilon > 0$. We can now use Theorem 1 and get for all $n \in \mathbf{N}$:

$$\mathbf{P}\left\{\sup_{t_0 \leq t \leq t_0+n}\left[M(t) - \frac{\varepsilon}{2}\int_{t_0}^{t}\frac{|V_x(x(s),s)g(x(s),s)|^2}{(V(x(s),s))^2}\mathrm{d}s\right] > \frac{2}{\varepsilon}\log(n)\right\} \leq \frac{1}{n^2}$$

By the Borel Cantelli theorem, cf. e.g. [7, Thm. 3.18], there exists an $n_0(\omega) > 0$ for almost all ω, such that

$$M(t) \leq \frac{2}{\varepsilon}\log(n) + \frac{\varepsilon}{2}\int_{t_0}^{t}\frac{|V_x(x(s),s)g(x(s),s)|^2}{(V(x(s),s))^2}\mathrm{d}s.$$

for all $t_0 \leq t \leq t_0 + n$ if $n > n_0$. By condition 3,

$$\begin{aligned}
&\log V(x(t),t)\\
&\leq \log V(x_0,t_0) + c_2(t-t_0) + \frac{1}{2}(\varepsilon - 1)\int_{t_0}^{t}\frac{|V_x(x(s),s)g(x(s),s)|^2}{(V(x(s),s))^2}\mathrm{d}s + \frac{2}{\varepsilon}\log(n)\\
&\leq \log V(x_0,t_0) + c_2(t-t_0) - \frac{1}{2}(1-\varepsilon)c_3(t-t_0) + \frac{2}{\varepsilon}\log(n)\\
&= \log V(x_0,t_0) - \frac{1}{2}((1-\varepsilon)c_3 - 2c_2)(t-t_0) + \frac{2}{\varepsilon}\log(n)
\end{aligned}$$

for all $t_0 \leq t \leq t_0 + n$ if $n > n_0$ for almost all ω. Therefore we have for almost all ω, that

$$\frac{1}{t}\log V(x(t),t) \leq -\frac{t-t_0}{2t}[(1-\varepsilon)c_3 - 2c_2] + \frac{\log V(x_0,t_0) + 2\log(n)/\varepsilon}{t_0+n-1}$$

if $t_0 + n - 1 \leq t \leq t_0 + n$ and $n > n_0$.

Fix ω and let $n \to \infty$, then

$$\limsup_{t\to\infty}\frac{1}{t}\log V(x(t),t) \leq -\frac{1}{2}((1-\varepsilon)c_3 - 2c_2)$$

holds point-wise for almost all ω. Finally using condition 1 we have

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \leq -\frac{(1-\varepsilon)c_3 - 2c_2}{2p}$$

for almost all ω. Since $\varepsilon > 0$ was arbitrary we have the conclusion. □

Now we state and proof Theorem 4.3.5 from [9] with the weaker conditions.

Theorem 3 (advancement of Thm. 4.3.5 in Mao)
Assume that there exists a function $V \in C_0^{2,1}(\mathbf{R}^d \times [t_0, \infty); \mathbf{R}_+)$*, and constants* $p > 0$*,* $c_1 > 0$*,* $c_2 \in \mathbf{R}$*,* $c_3 > 0$*, such that for all* $x \neq 0$ *and* $t \geq t_0$*,*

1. $c_1|x^p| \geq V(x, t) > 0$,
2. $LV(x, t) \geq c_2 V(x, t)$,
3. $|V_x(x, t)g(x, t)|^2 \leq c_3 V^2(x, t)$.

Then

$$\liminf_{t\to\infty} \frac{1}{t} \log |x(t; t_0, x_0)| \geq \frac{2c_2 - c_3}{2p} \quad a.s.$$

for all $x_0 \neq 0$ *in* $\mathbf{R}^d$*.*

The proof again follows the same method Mao used in his book, but here it works without modifications for our weaker assumptions on the function V. For completeness we, however, give a more worked out proof than given in [9].

Proof Just like in the proof of Theorem 2 we fix some $x_0 \neq 0$ and we write $x(t) = x(t; t_0, x_0)$. Furthermore we define $M(t)$ and $h(s)$ as in the proof of Theorem 2, and by Itô's formula we have that

$$\begin{aligned} &\log V(x(t), t) \qquad (4)\\ &\quad = \log V(x_0, t_0) + \int_{t_0}^{t} \frac{LV(x(s), s)}{V(x(s), s)} \mathrm{d}s + M(t) - \frac{1}{2}\int_{t_0}^{t} \frac{|V_x(x(s), s)g(x(s), s)|^2}{(V(x(s), s))^2} \mathrm{d}s. \end{aligned}$$

By condition 3, we have that $|h(s)|^2 < c_3$, so $h \in \mathscr{M}^2(\mathbf{R}_+, \mathbf{R}^{1\times m})$ and $M(t) = \int_{t_0}^{t} h(s)\mathrm{d}B(s)$ is a martingale. By Eq. (4) and condition 2

$$\begin{aligned} \log V(x(t), t) &\geq \log V(x_0, t_0) + c_2(t - t_0) - \frac{c_3}{2}(t - t_0) + M(t) \\ &= \log V(x_0, t_0) + \frac{1}{2}(2c_2 - c_3)(t - t_0) + M(t). \end{aligned} \qquad (5)$$

Since $M(t)$ is a martingale with quadratic variation

$$\langle M(t), M(t)\rangle = \int_{t_0}^{t} |h(s)|^2 \mathrm{d}s \leq c_3(t - t_0),$$

we have by the strong law of large numbers, cf. e.g. [9, Thm 1.3.4], that $\lim_{t\to\infty} M(t)/t = 0$ a.s. It therefore follows from (5) that

$$\liminf_{t\to\infty} \frac{1}{t} \log V(x(t), t) \geq \frac{1}{2}(2c_2 - c_3) \quad \text{a.s.}$$

Finally by condition 1 then

$$\liminf_{t\to\infty}\frac{1}{t}\log|x(t;t_0,x_0)|\geq\frac{2c_2-c_3}{2p}.$$

□

Remark If in the last theorem we have $2c_2 > c_3$, then almost all the sample paths of $t \mapsto |x(t; t_0, x_0)|$ will tend to infinity, and in this case the trivial solution of Eq. (1) is said to be *almost surely exponentially unstable*.

Example Consider the 1-dimensional SDE

$$dX(t) = f(X(t),t)dt + g(X(t),t)dB(t) := \frac{1}{4}X(t)dt + X(t)dB(t) \tag{6}$$

Set $V(x,t) = |x|^{1/2}$, then $V \in C_0^{2,1}$ and, by Eq. (2), the function $LV(x)$ is given by

$$LV(x) = \frac{1}{4}x\cdot(1/2)|x|^{-1/2} + \frac{1}{2}x^2\cdot(-1/2)(1/2)|x|^{-3/2} = \frac{1}{8}|x|^{1/2} - \frac{1}{8}|x|^{1/2} = 0.$$

Furthermore we see that

$$|V_x(x)g(x,t)|^2 = |(1/2)|x|^{-1/2}x|^2 = \frac{1}{4}(|x|^{1/2})^2 = \frac{1}{4}V(x)^2.$$

Fixing constants $c_1 = 1$, $p = 1/2$, $c_2 = 0$ and $0 < c_3 < 1/4$, we see by Theorem 2 that for any solution $x(t)$ of Eq. (6) the following inequality holds

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \leq -\frac{c_3-2c_2}{2p} = -c_3 < 0 \quad \text{a.s.}$$

In particular the trivial solution of system (6) is almost surely exponentially stable (in fact the solution is stable in probability, see [8, Thm. 5.3]), and the function V we used is not differentiable at 0. Moreover, as shown by Khasminskii [8, p. 154–155], there cannot exists a Lyapunov function for this system that is differentiable at the origin.

3 Conclusions

In his book [9] X. Mao states and proves two theorems, Theorem 4.3.3 and Theorem 4.3.5, where he shows that the existence of a certain auxiliary function, so-called Lyapunov function, implies the *almost sure exponential stability* or, for a different kind of function, the *almost sure exponential instability* respectively of the zero solution of a SDE. Unfortunately, the class of functions $C^{2,1}(\mathbf{R}^d \times [t_0,\infty);\mathbf{R}_+)$

he considers to serve as the foundation for Lyapunov functions is too restrictive as had already been pointed out in the literature [8, p. 154–155]. The adequate class of functions is given by $C_0^{2,1}(\mathbf{R}^d \times [t_0, \infty); \mathbf{R}_+)$ and we formulate and prove Mao's theorems for this wider class of functions. This renders these theorems much more useful for applications.

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