Robustness of numerically computed contraction metrics

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Abstract

Contraction metrics are an important tool to show the existence of exponentially stable equilibria or periodic orbits, and to determine a subset of their basin of attraction. One of the main advantages is that contraction metrics are robust with respect to perturbations of the system, i.e. a contraction metric for one particular system is also a contraction metric for a perturbed system. In this paper, we discuss numerical methods to compute contraction metrics for dynamical systems, with exponentially stable equilibria or periodic orbits, and perform perturbation analysis. In particular, we prove the robustness of such metrics to perturbations of the system and give concrete bounds. The results imply that a contraction metric, computed for a particular system, remains a contraction metric for the perturbed system. We illustrate our results by computing contraction metrics for systems from the literature, both with exponentially stable equilibria and exponentially stable periodic orbits, and then investigate the validity of the metrics for perturbed systems. Parts of the results are published in [1].

Keywords: Contraction Metric, Radial Basis Functions, Exponentially stable, Equilibrium Point, Periodic Orbit, Dynamical System

1 Introduction

In this paper we are concerned with the autonomous ordinary differential equation (ODE)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

where $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^n$ is a C^s-vector field, $s \geq 1$. The existence, uniqueness, and stability of equilibria and periodic orbits in a given area can be proved by a contraction metric. A Riemannian metric is called contraction metric, if the distance between any two adjacent trajectories of the system is exponentially decreasing with respect to the Riemannian metric. Two types of contraction conditions are considered: if the distance between all adjacent trajectories is decreasing within a compact, positively invariant set, then all solutions converge to a unique equilibrium point in this set, which is exponentially stable. If the distance between all adjacent trajectories in directions perpendicular to the flow is decreasing within a compact, positively invariant set, which does not contain any equilibrium, then all solutions converge to a unique periodic orbit in this set, which is exponentially stable.

Hence, a contraction metric fulfills a local contraction criterion, which can be expressed in differential inequalities, and the location of an equilibrium or a periodic orbit is not required. Moreover, a contraction metric is robust with respect to perturbations of the system. That is, a contraction metric for the original system is also a contraction for the perturbed system, even though the equilibrium or the periodic orbit may be different in the perturbed system. This fact is studied in detail in the paper.

A contraction metric delivers a local criterion for the study of the evolution of the distances between neighbouring trajectories. Two closely related notions are *incremental stability* and *convergent systems*. Incremental stability studies the distance between two adjacent solutions, while in convergent systems all solutions converge to a unique solution as time tends to infinity. These concepts are related, but for general non-autonomous systems, independent of each other, see [2] and [3]. Incremental stability has been, for example, studied in [4] and [5] and the related notion of Finsler-Lyapunov functions was introduced in [6]. The study of contraction metrics in general goes back to [7–9]. The review [10] puts the definition from [4] into historical context. For a review of contraction metrics and their computation see [11].

Contraction metrics for equilibria are studied in [12], [13] and [14] as well as [4]. Converse theorems to prove the existence of a Riemannian contraction metric for a system with an exponentially stable equilibrium are considered in [15]; in particular, the existence of a contraction metric on the entire phase space, which satisfies a certain linear PDE, is shown.

The computation of a contraction metric for an *equilibrium* using meshfree collocation is studied in [16] using generalized interpolation with compactly supported radial basis functions (RBFs) to compute an approximation to the solution of the PDE, see [17]. In [18] the authors present a rigorous verification of the properties of a contraction metric for the metric computed in [16] and show that the combination of a numerically computed approximation to a contraction metric together with a subsequent verification delivers a method that is able to compute and verify a contraction metric for any system with an exponentially stable equilibrium.

Contraction metrics for periodic orbits are studied in [19] with the Euclidean metric and [20] with a general Riemannian metric. Further studies include [12, 13, 21–23].

For periodic orbits, the notion of Zhukovskii stability, see e.g. [24], i.e. stability of solutions after reparameterisation of time, is used to study stability. The reparameterisation or synchronisation of the time of adjacent trajectories is used to show that the existence of a contraction metric implies the existence of a unique, exponentially stable periodic orbit to which all trajectories converge, see [25] or [26]. Converse theorems to prove the existence of a Riemannian contraction metric for a system with an exponentially stable periodic orbit go back to [27] (local result) and [26] (global on compact sets). The latter also discusses the robustness to parameters, using the construction in [28].

[29] contains a global converse theorem, showing the existence of a contraction metric on the entire phase space for systems with an exponentially stable periodic orbit. The existence and uniqueness of the metric is proved by characterizing it as the solution to a linear PDE. In [30] a numerical method using generalized interpolation with compactly supported RBFs to compute an approximation to such a contraction metric is presented. In [31] the authors present a rigorous verification of the properties of a contraction metric for the metric computed in [30] and show that the combination of a numerically computed approximation to a contraction metric together with a subsequent verification delivers a method that is able to compute and verify a contraction metric for any system with an exponentially stable periodic orbit. The main idea of the procedure for periodic orbits and equilibria is similar, however, the case of a periodic orbit involves the restriction to the $(n-1)$-dimensional subspace perpendicular to $\mathbf{f}(\mathbf{x})$ at each point $\mathbf{x}$, which requires a more sophisticated argumentation.

Other numerical methods for the computation of contraction metrics include [32] for periodic orbits in time-periodic systems, where the contraction metric is a continuous piecewise affine (CPA) function and the contraction conditions are transformed into constraints of a semidefinite optimization problem. Moreover, contraction metrics are constructed using Linear Matrix Inequalities and sums of squared polynomials (SOS) in [33] for equilibria and [26] for periodic orbits. While all of these methods also include a rigorous verification, similar to our approach, they are of higher computational complexity because they require solving a semidefinite optimization problem, whereas solving a system of linear equations is computationally the most demanding step in our approach, see [1, 31].

In this paper, we focus on perturbations of the system and prove two theorems in this regard, Theorem 1 and 2. Then we demonstrate the robustness of our numerically computed metrics in examples from the literature. Essentially, the metric is computed using RBF and the verification is not only performed for the original system, but also for a perturbed system, proving that the metric is a contraction metric for it as well.

The paper is organized as follows: In Section 2 we introduce contraction metrics, summarize the methods to compute them using meshfree collocation and prove our main results, asserting the robustness of the computed contraction metric with respect to perturbations of the system. In Section 3 we show the robustness for examples, before we conclude in Section 4. In Appendix A we give an algorithmic description of the method, both for the equilibrium and periodic orbit case.

2 Contraction metrics and Computational methods

Let us give a concise description of contraction metrics for systems with exponentially stable equilibria or exponentially stable periodic orbits, together with the numerical methods from [18] and [31].

We first need a few definitions.

Definition 1 (Riemannian metric) Let G be an open subset of $\mathbb{R}^n$. A Riemannian metric is a locally Lipschitz continuous matrix-valued function $M\colon G \to \mathbb{S}^{n\times n}$, such that $M(\mathbf{x})$ is positive definite for all $\mathbf{x} \in G$, where $\mathbb{S}^{n\times n}$ denotes the symmetric $n \times n$ matrices with real entries.

The metrics we construct with our numerical method are not differentiable and therefore we only demand of a Riemannian metric that it is locally Lipschitz. An appropriate derivative of such a metric along solution trajectories is the forward orbital derivative

$$M'_+(\mathbf{x}) := \limsup_{h\to 0^+} \frac{M(S_h\mathbf{x}) - M(\mathbf{x})}{h} = \limsup_{h\to 0^+} \frac{M\big(\mathbf{x} + h\mathbf{f}(\mathbf{x})\big) - M(\mathbf{x})}{h} \qquad (2)$$

with respect to (1) at $\mathbf{x} \in G$. Here $t \mapsto S_t\mathbf{x}$ is the solution to (1) passing through $\mathbf{x}$ at time $t = 0$. Note that at a point $\mathbf{x}$ where M is differentiable, the formula for $M'_+(\mathbf{x})$ simplifies due to the chain-rule to

$$M'_+(\mathbf{x}) = \frac{d}{dt} M(S_t\mathbf{x})\Big|_{t=0} = \Big(\nabla M_{ij}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})\Big)_{i,j=1,\dots,n}.$$

That is, the (i, j)-th entry in the matrix $M'_+(\mathbf{x})$ is the dot product of the gradient of the (i, j)-th entry of M with $\mathbf{f}(\mathbf{x})$.

Definition 2 (Contraction metric for an equilibrium) A *contraction metric for an equilibrium* is a Riemannian metric $M : G \to \mathbb{S}^{n\times n}$ fulfilling a contraction condition expressed by $L^e_M(\mathbf{x}) \le -\nu < 0$ for all $\mathbf{x} \in K \subset G$, where L^e_M is defined in (3) below and K is a compact subset of the open set $G \subset \mathbb{R}^n$.

$$\begin{aligned} L^e_M(\mathbf{x}) &= \max_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T M(\mathbf{x})\mathbf{v}=1} L^e_M(\mathbf{x};\, \mathbf{v}) \quad \text{where} \qquad (3)\\ L^e_M(\mathbf{x};\, \mathbf{v}) &= \frac{1}{2}\mathbf{v}^T \Big(M'_+(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x}) D\mathbf{f}(\mathbf{x})\Big)\mathbf{v}. \end{aligned}$$

We refer to M as a (Riemannian) contraction metric on K for an equilibrium or a metric contracting in K with respect to an equilibrium.

Definition 3 (Contraction metric for a periodic orbit) A *contraction metric for a periodic orbit* is a Riemannian metric $M : G \to \mathbb{S}^{n\times n}$ fulfilling a contraction condition expressed by

$L_M^p(\mathbf{x}) \leq -\nu < 0$ for all $\mathbf{x} \in K \subset G$, where L_M^p is defined in (5) below and K is a compact subset of the open set $G \subset \mathbb{R}^n$ such that $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ holds for all $\mathbf{x} \in K$. For the definition of L_M^p we first define

$$V(\mathbf{x}) \;=\; D\mathbf{f}(\mathbf{x}) - \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T(D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T)}{\|\mathbf{f}(\mathbf{x})\|_2^2}. \tag{4}$$

for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$; then

$$L_M^p(\mathbf{x}) \;=\; \max_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}^T M(\mathbf{x})\mathbf{v}=1, \mathbf{v}^T\mathbf{f}(\mathbf{x})=0} L_M(\mathbf{x};\, \mathbf{v}) \quad \text{where} \tag{5}$$

$$L_M^p(\mathbf{x};\, \mathbf{v}) \;=\; \frac{1}{2}\mathbf{v}^T\left(M'_+(\mathbf{x}) + V(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x})V(\mathbf{x})\right)\mathbf{v}.$$

We refer to M as a (Riemannian) contraction metric on K for a periodic orbit or a metric contracting in K with respect to a periodic orbit.

The function $L_M^e(\mathbf{x};\, \mathbf{v})$ in Definition 2 is negative, if for small $\delta > 0$ the distance between solutions through $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{v}$ decreases with respect to the metric $M(\mathbf{x})$. This implies that the distance converges to zero and the trajectories have the same long-term behaviour, which turns out to be an equilibrium. Similarly, the function $L_M^p(\mathbf{x};\, \mathbf{v})$ in Definition 3 is negative for $\mathbf{v}$ with $\mathbf{v}^T\mathbf{f}(\mathbf{x}) = 0$, if for small $\delta > 0$ the distance between time-synchronized solutions $S_t\mathbf{x}$ and $S_{\theta(t)}(\mathbf{x} + \delta\mathbf{v})$, such that the difference satisfies $\left(S_{\theta(t)}(\mathbf{x} + \delta\mathbf{v}) - S_t\mathbf{x}\right)^T \mathbf{f}(S_t\mathbf{x}) = 0$ for all t, decreases with respect to the metric $M(\mathbf{x})$. For a heuristic explanation of this fact, see, e.g. [30, Section 1]. In this case, they have again the same long-term behaviour, which turns out to be a periodic orbit.

The connection between exponentially stable equilibria and contraction metrics is as follows: In Theorem 2.6 in [18], see also Theorem 3.1 in [15], it is shown that the existence of a contraction metric for an equilibrium on a compact, positively invariant set K asserts the existence of a unique exponentially stable equilibrium $\mathbf{x}^e \in K$ and that K is a subset of the orbit's basin of attraction, i.e. $K \subset \mathcal{A}(\mathbf{x}^e)$. Conversely, Theorems 2.2 and 2.3 in [16] establish the existence of a contraction metric for an equilibrium for a system with an exponentially stable equilibrium and the metric is characterized as the solution to a certain PDE.

Similarly, the connection between exponentially stable periodic orbits and contraction metrics is as follows: In Theorem 2.5 in [31], see also Theorem 2.1 in [29], it is shown that the existence of a contraction metric for a periodic orbit on a compact, positively invariant set K, which does not contain any equilibrium, asserts the existence of a unique exponentially stable periodic orbit $\Omega \subset K$ and that K is a subset of the orbit's basin of attraction, i.e. $K \subset \mathcal{A}(\Omega)$. Conversely, Theorems 3.1 and 4.2 in [29] assert the existence of a contraction metric for a periodic orbit for a system with an exponentially stable periodic orbit; the metric is characterized as the solution to a certain PDE.

Let us explain this in more detail, first for the more demanding periodic orbit case. We first define for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ the linear differential operator L^p, acting on $M\colon \mathbb{R}^n \to \mathbb{S}^{n\times n}$ by

$$L^p M(\mathbf{x}) \;:=\; M'_+(\mathbf{x}) + V(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x})V(\mathbf{x}), \tag{6}$$

where V was defined in (4). Further, we define the projection matrix $P_{\mathbf{x}} \in \mathbb{R}^{n\times n}$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ onto the $(n-1)$-dimensional space perpendicular to $\mathbf{f}(\mathbf{x})$, i.e. $P_{\mathbf{x}}^2 = P_{\mathbf{x}}$, $P_{\mathbf{x}}\mathbf{f}(\mathbf{x}) = \mathbf{0}$ and $P_{\mathbf{x}}\mathbf{v} = \mathbf{v}$ if $\mathbf{v}^T\mathbf{f}(\mathbf{x}) = 0$, by

$$P_{\mathbf{x}} := I_{n\times n} - \frac{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})^T}{\|\mathbf{f}(\mathbf{x})\|_2^2}. \tag{7}$$

Then for $B \in C^{s-1}(\mathcal{A}(\Omega); \mathbb{S}^{n\times n})$ such that $B(\mathbf{x})$ is positive definite for all $\mathbf{x} \in \mathcal{A}(\Omega)$, we define $C \in C^{s-1}(\mathcal{A}(\Omega); \mathbb{S}^{n\times n})$ by

$$C(\mathbf{x}) = P_{\mathbf{x}}^T B(\mathbf{x}) P_{\mathbf{x}}. \tag{8}$$

[29, Theorems 3.1 and 4.2] assert that there exists a unique solution $M \in C^{s-1}(\mathcal{A}(\Omega); \mathbb{S}^{n\times n})$ of the linear system of PDEs

$$L^p M(\mathbf{x}) = -C(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{A}(\Omega) \tag{9}$$

$$\text{satisfying } \mathbf{f}(\mathbf{y}_0)^T M(\mathbf{y}_0)\mathbf{f}(\mathbf{y}_0) = c_0\|\mathbf{f}(\mathbf{y}_0)\|_2^4, \tag{10}$$

where $\mathbf{y}_0 \in \mathcal{A}(\Omega)$ and $c_0 \in \mathbb{R}^+$ are fixed. Let us explain these two conditions: (9) shows that the distance between adjacent trajectories contracts in directions perpendicular to $\mathbf{f}(\mathbf{x})$ and remains constant in direction $\mathbf{f}(\mathbf{x})$. To show that $M(\mathbf{x})$ is positively invariant in $\mathbf{f}$-direction, we can use (10) at the point $\mathbf{x}_0$, together with the fact that the distance in $\mathbf{f}(\mathbf{x})$ direction remains constant along solutions.

A PDE for a contraction metric for an equilibrium $\mathbf{x}^e$ can be constructed similarly with $P_{\mathbf{x}} = I_{n\times n} \in \mathbb{R}^{n\times n}$ and the differential operator L^e acting on $M\colon \mathbb{R}^n \to \mathbb{S}^{n\times n}$ by

$$L^e M(\mathbf{x}) = M'_+(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M(\mathbf{x}) D\mathbf{f}(\mathbf{x}).$$

In this case the PDE

$$L^e M(\mathbf{x}) = -C(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{A}(\mathbf{x}^e) \tag{11}$$

has a unique solution $M \in C^{s-1}(\mathcal{A}(\Omega); \mathbb{S}^{n\times n})$ for every $C \in C^{s-1}(\mathcal{A}(\Omega); \mathbb{S}^{n\times n})$ that is positive definite for every $\mathbf{x} \in \mathcal{A}(\mathbf{x}^e)$. Note that (11) is simply the PDE (9) with $P_{\mathbf{x}} = I_{n\times n} \in \mathbb{R}^n$, i.e. projection to $\mathbb{R}^n$ for all $\mathbf{x}$, and the condition (10) is not needed.

We now discuss how metrics that are solutions to (11) and (9) can be computed numerically using collocation with RBF functions. Again we describe the periodic orbit case in some detail and then discuss the equilibrium point case as a simplification of the periodic orbit case.

2.1 Computing a metric for a periodic orbit

We give a concise description of the method in [31] to rigorously compute a contraction metric in a bounded set $O \subset \mathbb{R}^n$. It consists of two essential steps.

Step one: compute an approximation

First fix a finite set of points $X = \{\mathbf{c}_i \in \mathbb{R}^n\} \subset O$, $\mathbf{c}_i \neq \mathbf{c}_j$ if $i \neq j$, and a $B \in C^{s-1}(O;\, \mathbb{S}^{n\times n})$ such that $B(\mathbf{x})$ is positive definite for all $\mathbf{x} \in O$. The set X is called a collocation grid and usually one fixes $B(\mathbf{x})$ as a constant matrix, e.g. $B(\mathbf{x}) = I_{n\times n}$ for all $\mathbf{x}$.

Then the method from [29] is used to solve the PDE (9) numerically using RBFs. For this the point $\mathbf{y}_0 \in \mathbb{R}^n$ and the constant $c_0 > 0$ must be fixed, together with the particular RBF. Any point $\mathbf{y}_0$ in (the unknown) $\mathcal{A}(\Omega)$ will do the job and c_0 is a simple scaling parameter. The particular RBF plays an important role in practice [34], although in theory all permissible choices result in guaranteed convergence. The method in [29] to solve the PDE is an example of a solution to a *generalized interpolation problem* or an *optimal recovery problem*. The function $S \in C^{s-1}(O;\, \mathbb{S}^{n\times n})$ it computes fulfills $L^P S(\mathbf{c}_i) = -C(\mathbf{c}_i)$ at all collocation points and it is norm minimal with this property in the reproducing kernel Hilbert space (RKHS) corresponding to the RBF, see [17].

The existence of a unique solution to the optimal recovery S is established in Theorem 4.2 in [30] and error estimates are derived in Theorem 4.4 of the same paper. The second theorem delivers a proof that the RBF approximation S to the true solution M to the PDE is a contraction metric if the *fill distance*

$$h_{X,O} := \max_{\mathbf{x}\in O} \min_{\mathbf{c}_i \in X} \|\mathbf{x} - \mathbf{c}_i\|_2$$

of the collocation points X is small enough. However, it does not quantify in a useful way how small $h_{X,O}$ must be; there are constants in the error bound that are extremely hard or impossible to estimate. This is the reason why a verification method is very useful, that checks whether S is truly a contraction metric or not. If it is not, then one can add collocation points to make $h_{X,O}$ smaller and verify again.

Step two: verify the approximation

In the second step of the method from [31] the conditions for a contraction metric are rigorously verified for the CPA interpolation P of the approximation S to M computed in Step one. In particular, it is verified whether $P(\mathbf{x})$ is positive definite and whether $L^P(\mathbf{x})$ is negative definite for all $\mathbf{x}$ in the domain of the triangulation $\mathcal{T}$; see Theorem 4.11 in [31]. Further, error estimates and statements about the CPA interpolation are provided, together with criteria that assert that the interpolation P is a contraction metric. These criteria or constraints can be verified very efficiently by a computer.

In more detail, we define an n-simplex as the convex hull

$$\mathfrak{S} = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) = \left\{ \sum_{k=0}^{n} \lambda_k \mathbf{x}_k \mid \sum_{k=0}^{n} \lambda_k = 1, \lambda_k \geq 0 \right\}$$

of its linearly affine vertices $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$, i.e. the vectors $\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. We consider a finite simplicial complex, i.e. a finite set $\mathcal{T} = \bigcup_\nu \mathfrak{S}_\nu$ of n-simplices such that

$$\mathfrak{S}_\mu \cap \mathfrak{S}_\nu = \mathrm{co}(C_\mu \cap C_\nu),$$

where $C_\nu = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ denotes the vertices of $\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$. Let $\mathcal{V}_\mathcal{T} = \bigcup_\nu C_\nu$ be the set of all vertices and denote by $\mathcal{D}_\mathcal{T} = \bigcup_\nu S_\nu$ the domain of the triangulation.

Given values $V(\mathbf{x}_k)$ for all vertices $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$, we can define the unique CPA (continuous and piecewise affine) function $V \colon \mathcal{D}_\mathcal{T} \to \mathbb{R}$ interpolating these values, which is affine on each simplex, in the following way: for $\mathbf{x} \in \mathcal{D}_\mathcal{T}$, there exists a simplex $\mathfrak{S}_\nu$ with $\mathbf{x} \in \mathfrak{S}_\nu$ and there are $\lambda_k \geq 0$ with $\sum_{k=0}^n \lambda_k = 1$ such that $\mathbf{x} = \sum_{k=0}^n \lambda_k \mathbf{x}_k$. Then

$$V(\mathbf{x}) = \sum_{k=0}^{n} \lambda_k V(\mathbf{x}_k)$$

with the same coefficients λ_k. The function V is affine on $\mathfrak{S}_\nu$ and its gradient ∇V^ν of the restriction to $\mathfrak{S}_\nu$ is constant. In a similar way, a matrix-valued CPA function P can be defined component-wise.

Assume now that we are given a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^3(\mathbb{R}^n; \mathbb{R}^n)$, and a simplicial complex as above, such that $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in \mathcal{D}_\mathcal{T}$. We define the verification problem below:

Verification Problem 1

- (VP1) P **positive definite:**
  For each vertex $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$, $P(\mathbf{x}_k)$ is positive definite i.e.:

$$P(\mathbf{x}_k) \succ 0_{n,n}.$$

- (VP2) $A_\nu - \kappa_\nu^* \mathbf{f}\mathbf{f}^T$ **negative definite:**
  For each simplex $\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0, \ldots, \mathbf{x}_n) \in \mathcal{T}$ and each vertex $\mathbf{x}_k$ of $\mathfrak{S}_\nu$:

$$A_\nu(\mathbf{x}_k) - \kappa_\nu^* \mathbf{f}(\mathbf{x}_k)\mathbf{f}^T(\mathbf{x}_k) + h_\nu^2 E_\nu^p I_{n\times n} \prec 0_{n,n}.$$

  Here

$$A_\nu(\mathbf{x}_k) := P(\mathbf{x}_k)V(\mathbf{x}_k) + V(\mathbf{x}_k)^T P(\mathbf{x}_k) + \left(\nabla P_{ij}^\nu \cdot \mathbf{f}(\mathbf{x}_k)\right)_{i,j=1,2,\ldots,n}, \tag{12}$$

  where $\kappa_\nu^* > 0$ (any will do), and E_ν^p is a system dependent constant for each simplex $\mathfrak{S}_\nu \in \mathcal{T}$, and h_ν is the simplex's diameter $h_\nu := \mathrm{diam}(\mathfrak{S}_\nu) = \max_{\mathbf{x},\mathbf{y} \in \mathfrak{S}_\nu} \|\mathbf{x} - \mathbf{y}\|_2$. For the explicit formulas of the E_ν^p, which include κ_ν^*, upper bounds on the derivatives of $\mathbf{f}$ up to order three, and derivatives of V up to order two, we refer to the Appendix and [31].

Our verification problem is a semidefinite feasibility problem and can in theory be solved as such, i.e. directly for P. However, it is computationally much more efficient to assign values to the variables $P(\mathbf{x}_k)$ of the problem using the optimal recovery S of the solution to (9) and (10), i.e. set $P(\mathbf{x}_k) = S(\mathbf{x}_k)$, and then verify that the constraints of the feasibility problem are fulfilled. We will refer to this feasibility problem as the *verification problem for a periodic orbit.* The CPA interpolation of the matrices $P(\mathbf{x}_k)$ over the triangulation $\mathcal{T}$ is a contraction metric for a periodic orbit for the system (1) if all the constraints are fulfilled.

2.2 Computing a metric for an equilibrium

The equilibrium case is somewhat simpler, see [18] . In Step one we can proceed as in the periodic orbit case, but with $P_{\mathbf{x}} = I_{n\times n}$. Further, $\mathbf{y}_0 \in \mathbb{R}^n$ and $c_0 > 0$ are not needed. In Step two the constraints (VP1) are exactly the same, but the constraints (VP2) become

Verification Problem 2

- (VP2) A_ν **negative definite:**
  For each simplex $\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0,\ldots,\mathbf{x}_n) \in \mathcal{T}$ (convex hull) and each vertex $\mathbf{x}_k$ of $\mathfrak{S}_\nu$:

  $$A_\nu(\mathbf{x}_k) + h_\nu^2 E_\nu^e I_{n\times n} \prec 0_{n,n},$$

  where

  $$A_\nu(\mathbf{x}_k) := P(\mathbf{x}_k)D\mathbf{f}(\mathbf{x}_k) + D\mathbf{f}(\mathbf{x}_k)^T P(\mathbf{x}_k) + \left(\nabla P_{ij}^\nu \cdot \mathbf{f}(\mathbf{x}_k)\right)_{i,j=1,2,\ldots,n}, \tag{13}$$

  where the E_ν^e are system dependent constants for each simplex $\mathfrak{S}_\nu \in \mathcal{T}$ and h_ν is the simplex's diameter.

Note that this is equivalent to setting $P_{\mathbf{x}} = I_{n\times n}$ as in Step one and $\kappa_\nu^* = 0$ for all simplices $\mathfrak{S}_\nu$. The constants E_ν^e include upper bounds on the derivatives of $\mathbf{f}$ up to order three; see [18] for the formulas. We refer to this feasibility problem as the *verification problem for an equilibrium.* The CPA interpolation of the matrices $P(\mathbf{x}_k)$ over the triangulation $\mathcal{T}$ is a contraction metric for an equilibrium for the system (1) if all the constraints are fulfilled.

2.3 Robustness of a contraction metric

A contraction metric M for the system (1) is also a contraction metric for a perturbed system, both in the periodic orbit and equilibrium point case. We will show this in the next two theorem, where we first prove the more involved periodic obit case and then the simpler equilibrium point case.

To quantify perturbations it is convenient to define the C^k-norm for $W \in C^k(\mathcal{D};\,\mathcal{R})$ as

$$\|W\|_{C^k(\mathcal{D};\,\mathcal{R})} := \sum_{|\boldsymbol{\alpha}|\le k} \sup_{\mathbf{x}\in\mathcal{D}} \|D^{\boldsymbol{\alpha}} W(\mathbf{x})\|_2 .$$

Here $\mathcal{D} \subset \mathbb{R}^n$ is a non-empty open set, $\mathcal{R}$ is one of $\mathbb{R}, \mathbb{R}^n, \mathbb{S}^{n\times n}$, or $\mathbb{R}^{n\times n}$, and $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ is a multi-index and $|\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha_i$ is its length.

Theorem 1 (Robustness of the contraction metric)
Assume that $M : G \to \mathbb{S}^{n\times n}$ is a contraction metric for a periodic orbit as in Definition 3 for system (1), where $\mathbf{f}$ is C^1, and contracting in the compact set $K \subset G$.

Then there exists $\varepsilon > 0$ such that for any (perturbed) system $\dot{\mathbf{x}} = \widetilde{\mathbf{f}}(\mathbf{x})$, with $\widetilde{\mathbf{f}}$ in C^1 and $\|\widetilde{\mathbf{f}}-\mathbf{f}\|_{C^1(G;\,\mathbb{R}^n)} < \varepsilon$ the Riemannian metric M is a contraction metric for a periodic orbit of the system $\dot{\mathbf{x}} = \widetilde{\mathbf{f}}(\mathbf{x})$.

Proof By assumption $M(\mathbf{x})$ is symmetric and positive definite for every $\mathbf{x} \in G \subset \mathbb{R}^n$. Hence, it suffices to show that $L_M^p(\mathbf{x}) < 0$ holds true for all $\mathbf{x} \in K$ when $\mathbf{f}$ has been substituted by $\widetilde{\mathbf{f}}$ in (4), (5), and (2).

We choose $\varepsilon > 0$ so small that $\widetilde{\mathbf{f}}(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in K$. Fix an $\mathbf{x} \in K$ and note that the right-hand side of formula (4) is a continuous function of $\mathbf{y} = \mathbf{f}(\mathbf{x})$ and $Z = D\mathbf{f}(\mathbf{x})$. Thus $V(\mathbf{x})$ varies continuously when $\mathbf{f}(\mathbf{x})$ is substituted by $\widetilde{\mathbf{f}}(\mathbf{x})$ as long as $\widetilde{\mathbf{f}}(\mathbf{x}) \neq \mathbf{0}$ and this is the case for $\|\widetilde{\mathbf{f}}-\mathbf{f}\|_{C^1(G;\,\mathbb{R}^n)} < \varepsilon$. Second, to see that (2) is a continuous function of $\mathbf{y} = \mathbf{f}(\mathbf{x})$ recall standard results on the continuous dependence of solutions to ODEs with locally Lipschitz right-hand-sides; see e.g. [35, §12.V]. With $0 < L < \infty$ as a Lipschitz constant for $\mathbf{f}$ in $\overline{O}$ we get from $\|\widetilde{\mathbf{f}}-\mathbf{f}\|_{C^1(G;\,\mathbb{R}^n)} < \varepsilon$ that

$$\|\widetilde{S}_h\mathbf{x} - S_h\mathbf{x}\|_2 \leq \frac{\varepsilon}{L}(e^{Lh}-1),$$

where $S_h\mathbf{x}$ and $\widetilde{S}_h\mathbf{x}$ are the solutions at time $h > 0$, starting at $\mathbf{x} \in K$ to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{x}} = \widetilde{\mathbf{f}}(\mathbf{x})$ respectively, and the solution trajectories are in an open set O, $K \subset O \subset \overline{O} \subset G$, and $\overline{O}$ is compact.

Let $0 < L^* < \infty$ be a Lipschitz constant for M on $\overline{O}$ with respect to the $\|\cdot\|_2$ norm. Then we have

$$\left\|\frac{M(\widetilde{S}_h\mathbf{x})-M(\mathbf{x})}{h} - \frac{M(S_h\mathbf{x})-M(\mathbf{x})}{h}\right\|_2 = \left\|\frac{M(\widetilde{S}_h\mathbf{x})-M(S_h\mathbf{x})}{h}\right\|_2$$
$$\leq \frac{L^*}{h}\|\widetilde{S}_h\mathbf{x}-S_h\mathbf{x}\|_2 \leq L^*\varepsilon\frac{e^{Lh}-1}{Lh} \leq 2L^*\varepsilon$$

for small enough $h > 0$. Thus $M'_+(\mathbf{x})$ also varies continuously when $\mathbf{f}$ is substituted by $\widetilde{\mathbf{f}}$ with $\|\widetilde{\mathbf{f}}-\mathbf{f}\|_{C^1(G;\,\mathbb{R}^n)} < \varepsilon$ and we have established that $L_M^p(\mathbf{x};\,\mathbf{v})$ varies continuously. The assertion now follows from the fact that both the argument of the maximum in (5) and the set maximized over, an intersection of an ellipsoid with a hyper-plane, vary continuously when $\mathbf{f}$ is substituted by $\widetilde{\mathbf{f}}$ with $\|\widetilde{\mathbf{f}}-\mathbf{f}\|_{C^1(G;\,\mathbb{R}^n)} < \varepsilon$, because neither $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ nor $\widetilde{\mathbf{f}}(\mathbf{x}) = \mathbf{0}$ on K. Thus $L_M^p(\mathbf{x}) < 0$ remains true after the substitution if $\varepsilon > 0$ is small enough. □

We now prove an identical theorem for contraction metrics for an equilibrium.

Theorem 2 (Robustness of the contraction metric)
Assume that $M : G \to \mathbb{S}^{n\times n}$ is a contraction metric for an equilibrium as in Definition 3 for system (1), where $\mathbf{f}$ is C^1, and contracting in the compact set $K \subset G$.

Then there exists $\varepsilon > 0$ such that for any (perturbed) system $\dot{\mathbf{x}} = \widetilde{\mathbf{f}}(\mathbf{x})$, with $\widetilde{\mathbf{f}}$ in C^1 and $\|\widetilde{\mathbf{f}}-\mathbf{f}\|_{C^1(G;\,\mathbb{R}^n)} < \varepsilon$ the Riemannian metric M is a contraction metric for an equilibrium of the system $\dot{\mathbf{x}} = \widetilde{\mathbf{f}}(\mathbf{x})$.

Proof By assumption $M(\mathbf{x})$ is symmetric and positive definite for every $\mathbf{x} \in G \subset \mathbb{R}^n$. Hence, it suffices to show that $L^e_M(\mathbf{x}) < 0$ holds true for all $\mathbf{x} \in K$ when $\mathbf{f}$ has been substituted by $\widetilde{\mathbf{f}}$ in (3) and (2).

To see that $M'_+(\mathbf{x})$ varies continuously when $\mathbf{f}$ is substituted by $\widetilde{\mathbf{f}}$ with $\|\widetilde{\mathbf{f}}-\mathbf{f}\|_{C^1(G;\,\mathbb{R}^n)} < \varepsilon$, we proceed as in the proof of Theorem 1. It is then clear that $L^e_M(\mathbf{x};\,\mathbf{v})$ varies continuously as well. The assertion now follows from the fact that the maximum in (3) varies continuously when $\mathbf{f}$ is substituted by $\widetilde{\mathbf{f}}$ with $\|\widetilde{\mathbf{f}}-\mathbf{f}\|_{C^1(G;\,\mathbb{R}^n)} < \varepsilon$. Thus $L^e_M(\mathbf{x}) < 0$ remains true after the substitution if $\varepsilon > 0$ is small enough. □

The robustness property established in Theorems 1 and 2 is a very useful property in dynamical systems, in particular in application where the data and dynamics are never known with absolute certainty. Further note that it is possible to prove that the constraints (VP1) and (VP2) of the verification remain true after a small perturbation, but then one must demand that $\mathbf{f}, \widetilde{\mathbf{f}}$ are C^3 and that $\left\|\widetilde{\mathbf{f}}-\mathbf{f}\right\|_{C^3(G;\,\mathbb{R}^n)} < \varepsilon$. The reason for this is that the constants E^p_ν and E^e_ν in (VP2) depend continuously on upper bounds on the third derivatives of the components of $\mathbf{f}$ in (1).

We state this as a theorem for future reference; its proof is obvious from the proofs of Theorems 1 and 2 and the formulas for E^p_ν and E^e_ν.

Theorem 3 (Robustness of the CPA approximation)
Assume that the verification problem for a periodic orbit is fulfilled for certain values $P(\mathbf{x}_k) \in \mathbb{S}^{n\times n}$, $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$, for the system (1) where $\mathbf{f}$ is C^3. Then there is an $\varepsilon > 0$ such that the verification problem is fulfilled with the same values for any system $\dot{\mathbf{x}} = \widetilde{\mathbf{f}}(\mathbf{x})$ fulfilling $\widetilde{\mathbf{f}}$ is C^3 and $\|\widetilde{\mathbf{f}}-\mathbf{f}\|_{C^3(G;\,\mathbb{R}^n)} < \varepsilon$. In particular the CPA interpolation of the values $P(\mathbf{x}_k) \in \mathbb{S}^{n\times n}$, $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$, is a contraction metric for a periodic orbit for this system.

Analogous statements apply for values $P(\mathbf{x}_k) \in \mathbb{S}^{n\times n}$, $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$, that fulfill the verification problem for an equilibrium.

In the next section we demonstrate the applicability of our theoretical results to two examples. Note that the periodic orbit is displayed in the figures through a numerical approximation for comparison in orange, but the method verifies rigorously that it exists and is exponentially stable and, moreover, determines a subset of its basin of attraction.

3 Examples

We implemented our method in C++ and ran the examples on an AMD Ryzen 2700X processor with 8 cores at 3.7 GHz and with 64GB RAM. In order to compute a positively invariant set K for the dynamical systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ we use a procedure motivated by [36]. First we solve numerically the PDE

$$\sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(\mathbf{x}) f_i(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = -\sqrt{\delta^2 + \|\mathbf{f}(\mathbf{x})\|_2^2}, \tag{14}$$

with $\delta = 10^{-8}$, using RBF. Then we use CPA interpolation V_P of the numerical solution V and verify for which simplices the condition $\nabla V_P^\nu \cdot \mathbf{f}(\mathbf{x}_k) + h_\nu^2 E_\nu < 0$ holds true for all vertices $\mathbf{x}_k$, similar to the verification problem for contraction metrics. In this area the function V_P is decreasing along solution trajectories and a sublevel set $\{\mathbf{x} \in \mathbb{R}^n : V_P(\mathbf{x}) \leq c\}$ is necessarily positively invariant, if its boundary is fully contained in this area. Hence, we only need $\nabla V_P(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) < 0$ on the level set $\{\mathbf{x} \in \mathbb{R}^n : V_P(\mathbf{x}) = c\}$, not on the whole sublevel set. We refer to V_P as *Lyapunov-like* function.
The *failing points* of the Lyapunov-like function (see for example Figure 2) are the points where the function V_P does not satisfy the decrease condition mentioned above. In order to obtain a positively invariant set, we need to find a sublevel set of the function such that its boundary does contain any of these points.

3.1 Van der Pol System

Let us consider the classical Van der Pol equation with reversed time

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x - 3(1-x^2)y \end{cases} \tag{15}$$

and denote the right-hand side by $\mathbf{f}(x,y)$. The system has an exponentially stable equilibrium at the origin with basin of attraction bounded by an unstable periodic orbit. We demonstrate the applicability of our method to this well known example.
The kernel given by Wendland's function ($x_+^k := \max\{0,x\}^k$)

$$\psi_{6,4}(r) = (1-cr)_+^{10}\left(2145(cr)^4 + 2250(cr)^3 + 1050(cr)^2 + 250cr + 25\right)$$

with $c = 0.9$ is used with corresponding RKHS H^σ with $\sigma = 4 + \frac{2+1}{2} = 5.5$. We placed collocation points inside the unstable periodic orbit using the hexagonal grid from [37], with scaling factor $\boldsymbol{\alpha} = \mathrm{diag}(0.1, 0.1)$, in the set

$$\left\{ (x,y) \in [-3,3] \times [-6,6] \;:\; \begin{matrix} (y > 2.5x - 2.7), (y > -1.2x - 5.73), \text{ and } (y > -7x - 14) & \text{if } y < 0 \\ (y < 2.5x + 2.7), (y < -1.2x + 5.73), \text{ and } (y < -7x + 14) & \text{if } y \geq 0 \end{matrix} \right\}$$

which results in $N = 1,926$ collocation points. Then, we calculated the CPA verification over the rectangle $[-2.5,\ 2.5] \times [-5.5,\ 5.5]$ with 2200^2 vertices, see Figure 1.

To establish the existence of a unique exponentially stable equilibrium, we additionally need a positively invariant set within the area where the conditions of the contraction metric are fulfilled. To compute such set we used an approach similar to [36], discussed above, and computed a numerical solution to $\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = -\sqrt{\delta^2 + \|\mathbf{f}(\mathbf{x})\|_2^2}$, $\mathbf{x} \in \mathbb{R}^2$, using the RBF method, with $\mathbf{f}$ from (15) and $\delta^2 = 10^{-8}$. Note that an approximate solution will not have negative orbital derivative near the equilibrium, since at the equilibrium $\mathbf{f}(\mathbf{x}) = (0,0)$, see [38], so is not a Lyapunov function. However, if the approximation is sufficiently good, then it will have

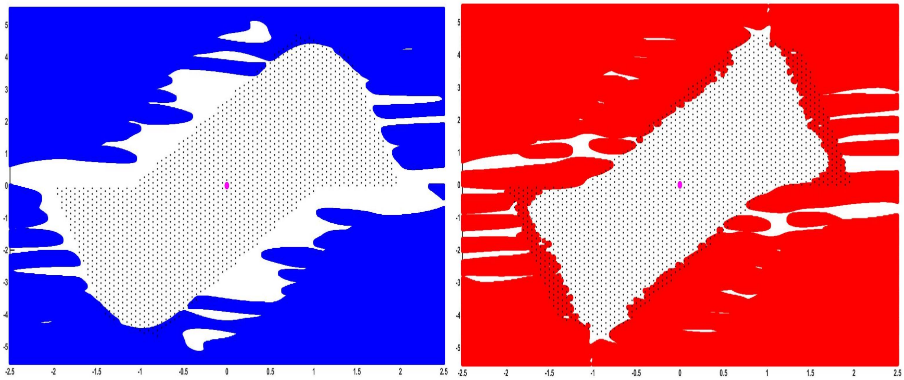

Fig. 1: Example 3.1. The black dots show the 1,926 collocation points. The blue and red dots indicate the vertices where *Constraints* (VP1) and (VP2) of Verification Problem 2 are not satisfied, respectively. The magenta circle indicates the equilibrium of the system at $(0,0)$, and the triangulation is over the area $[-2.5,\ 2.5]\times[-5.5,\ 5.5]$ with 2200^2 vertices.

negative orbital derivative outside a neighborhood of the equilibrium. We thus can use CPA verification to assert that its orbital derivative is truly negative in a large area and then use this information together with level-sets of the computed Lyapunov-like function V to determine a positively invariant set within the area where the metric P is a contraction metric. We used the same collocation points as above, a kernel given by the Wendland's function $\psi_{5,3}(cr) = (1-cr)_+^8\big(32(cr)^3+25(cr)^2+8cr+1\big)$, and $c = 0.2$. We then used a subsequent CPA verification on a regular 601×901 grid on $[-3,3]\times[-5.5,\ 5.5]$.

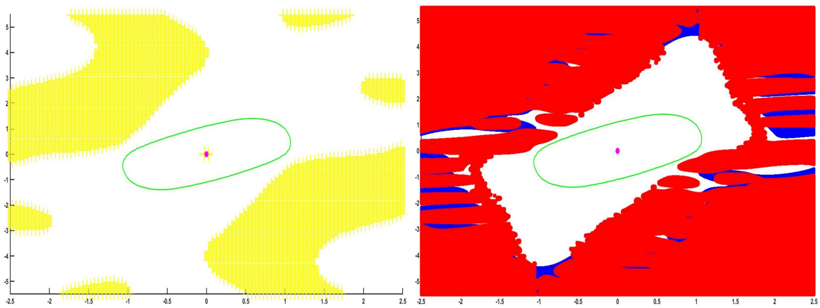

Fig. 2: Example 3.1. The $+$ signs in yellow indicate the points where the Lyapunov-like function does not have negative orbital derivative. The green curve inside the white area is the boundary of a positively invariant set, which is a sublevel-set of a computed Lyapunov-like function. The suitable area suggested by the method for the contraction metric is in white (right).

In Figure 2, we have drawn the largest level set of the computed Lyapunov-like function V that fulfills two conditions: it is inside of the area where P is a contraction metric and the level set is in the area where V has negative orbital derivative. Hence, this sublevel-set is necessarily positively invariant; for more information see [35, Section 10.XV].

In order to show the robustness of the contraction metric with respect to perturbations, we consider the following perturbed system

$$\begin{cases} \dot{x} = (-y+\varepsilon) \\ \dot{y} = (x-\varepsilon) - 3(1-x^2)y \end{cases} \tag{16}$$

with $\varepsilon = 0.1$. This new system has two equilibria at $(0.3609, 0.1)$, and $(-3.694, 0.1)$ where the latter is outside the area we considered.

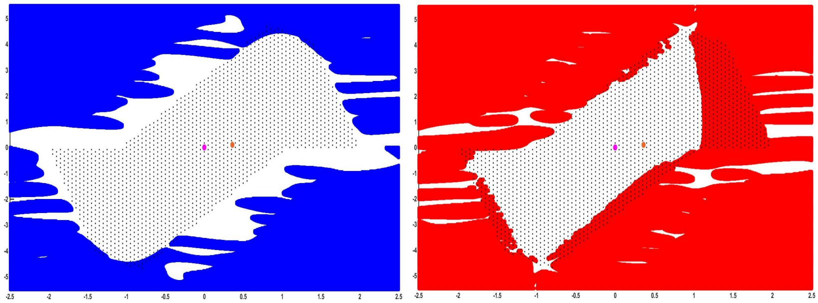

Fig. 3: Example 3.1. The suitable area suggested by the method for the contraction metric is in white. The blue and red dots indicate the vertices where *Constraints* (VP1) and (VP2) of Verification Problem 2 are not satisfied, respectively. The magenta and orange circles indicate the equilibrium of the original system at $(0,0)$, and of the perturbed system at $(0.36, 0.1)$.

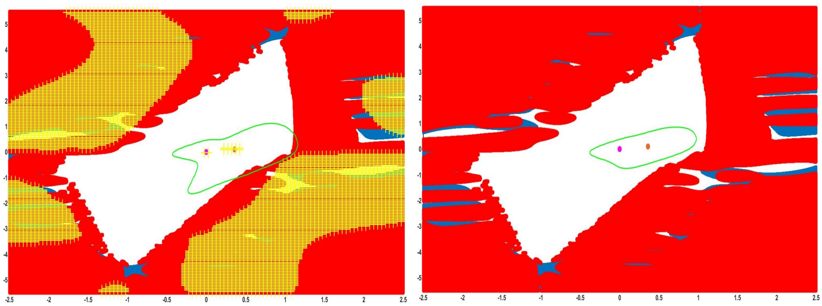

Fig. 4: Example 3.1. The suitable area suggested by the method for the contraction metric is in white. The $+$ signs in yellow indicate the points where the Lyapunov-like function does not have negative orbital derivative. The green curve inside the white area is the boundary of a positively invariant set, which is a sublevel-set of a computed Lyapunov-like function for the original system (left) and for the perturbed system (right).

In Figure 3 one can see that the contraction metric that was calculated for the unperturbed system still works in a reasonable area for the perturbed system, while in

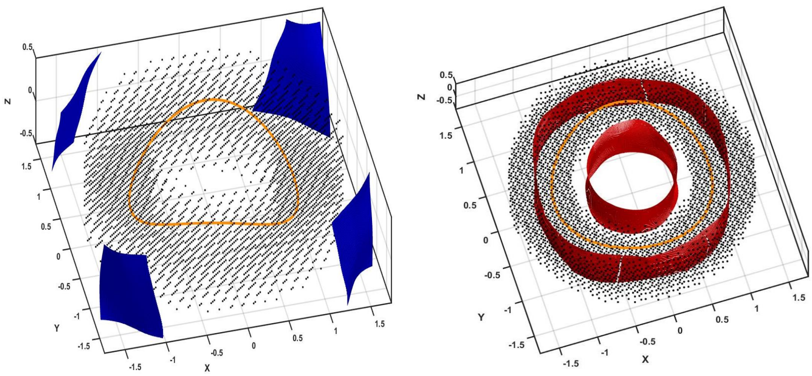

Fig. 5: Example 3.2. The orange curve indicates the periodic orbit for system (17). The black dots show the collocation points. The blue surface indicates the boundary of the area where constraints (VP1) of verification problem 1 are not fulfilled. The red surface indicates the boundary of the area where constraints (VP2) are not satisfied.

Figure 4 the old Lyapunov-like function fails to obtain a sublevel set, whose boundary does not intersect the failing points. Thus a new Lyapunov-like function for the perturbed system was computed.

3.2 Cylindrical absorber

We consider the following three-dimensional system from [30, Section 5.3]

$$\begin{cases} \dot{x} = x(1-x^2-y^2) - y + 0.1yz \\ \dot{y} = y(1-x^2-y^2) + x \\ \dot{z} = -z + xy \end{cases} \tag{17}$$

which has an exponentially stable periodic orbit.

We choose the parameters of the method in the following way: we set $B(\mathbf{x}) = I_{3\times 3}$ and use the hexagonal grid from [37] with scaling factor $\boldsymbol{\alpha} = \mathrm{diag}(0.1398, 0.1398, 0.09)$ in the area $\{(x,y,z) \in \mathbb{R}^3 : 0.75 < \sqrt{x^2+y^2} < 1.55, |z| < 0.45\}$, resulting in $N = 4,458$ collocation points. Further, we set $\mathbf{y}_0 = (1,0,0)$ and $c_0 = 1$. We use the kernel given by the Wendland function $\psi_{6,4}$ with parameter $c = 0.55$, the corresponding Sobolev space is $H^{5.5}(\mathbb{R}; \mathbb{S}^{3\times 3})$. In Figure 5, the black dots are the collocation points, the orange curve is the periodic orbit, the blue surface represents the boundary of the area where the first constraints of Verification problem 1 are not satisfied and the red surface is the boundary of the area where the second constraints are not fulfilled. We triangulated the space $[-1.67, 1.67] \times [-1.67, 1.67] \times [-0.67, 0.67]$ using 601^3 vertices.

For the Lyapunov-like function we use the same set of collocation points, and the kernel given by the Wendland function $\psi_{5,3}$ with parameter $c = 0.6$. In Figure 6, a suitable level set of the Lyapunov-like function is presented in green, while its failing points are in yellow. The second side figure combines all the results, showing that

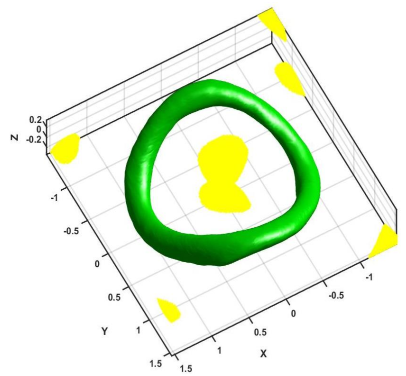

Fig. 6: Example 3.2. The area where the Lyapunov-like function, computed for system (17), is not decreasing is plotted in yellow. The green surface is the level set of the Lyapunov-like function, which thus indicates the boundary of a positively invariant set. Since it is inside the white area where both constraints of the verification problem are satisfied it is a subset of the basin of attraction of a unique periodic orbit within it.

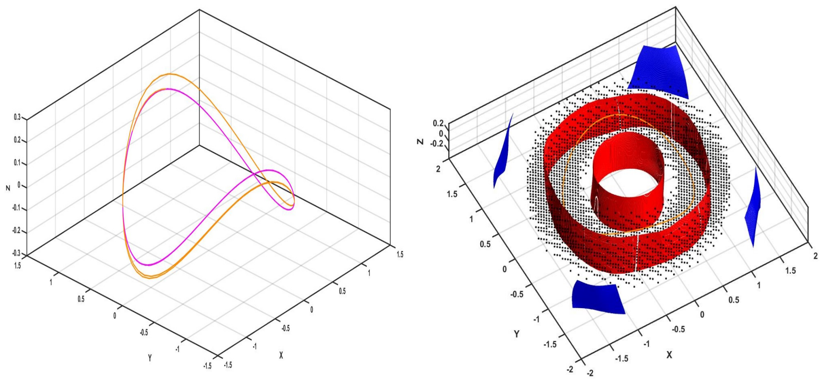

Fig. 7: Example 3.2. The periodic orbit for the original system (17) is the curve in magenta, while the periodic orbit for the perturbed system (18) with $\varepsilon = 0.1$ is depicted in orange. The contraction metric, computed for the unperturbed system, is checked for the perturbed system: the blue surface shows the boundary of the area where the first constraints fail, while the red surface denote the boundary of the area where the second constraints fail.

the conditions of the verification problem are satisfied within a compact, positively invariant set (green).

Now we consider the perturbed system

$$\begin{cases} \dot{x} &= x(1-x^2-y^2)-y+0.1(y+\varepsilon)z \\ \dot{y} &= y(1-x^2-y^2)+x \\ \dot{z} &= -z+x(y+\varepsilon) \end{cases} \tag{18}$$

with $\varepsilon = 0.1$. The periodic orbit for the original and perturbed systems can be seen in Figure 7 in magenta and orange, respectively. It is an interesting observation that

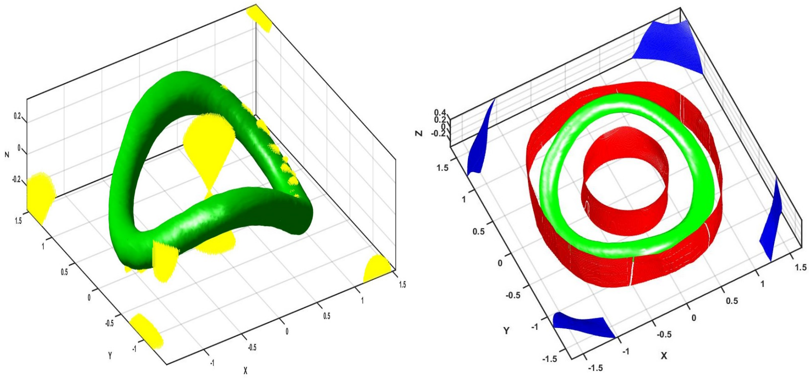

Fig. 8: Example 3.2. The Lyapunov-like function for the unperturbed system (17) cannot be used for the perturbed system (18) with $\varepsilon = 0.1$, because the failing points, where it is not decreasing along solution trajectories, intersects the boundary of the level-set (green). By computing a new Lyapunov-like function for the perturbed system (18) with $\varepsilon = 0.1$, we can verify that the area bounded by the green surface is forward invariant.

while the same contraction metric could be used for the perturbed system, see Figure 7, the Lyapunov-like function fails to give a suitable level set around the periodic orbit and we need to calculate a new one for the perturbed system, see Figure 8. All 3D plots of this example can be accessed on-line here.

4 Conclusions

A contraction metric can be used to determine a subset of the basin of attraction of a periodic orbit or an equilibrium point. Having a PDE characterization of the contraction metric, one can use generalized interpolation with radial basis functions to approximate the solution of the PDE and thus to compute a contraction metric. Subsequently the approximation can be interpolated over a triangulation and it can be rigorously verified that the constructed matrix-valued function truly is a contraction metric.

When compared to other methods to determine the basin of attraction of an exponentially stable periodic orbit, e.g. Lyapunov functions, the computation of a contraction metric is computationally more demanding as we construct a matrix-valued function. The advantage is, however, that we do not need the location of the periodic orbit and that the metric is robust with respect to perturbations of the system.

We have implicitly shown throughout the paper and in Appendix A that finding a set containing an exponentially stable equilibrium point and its basin of attraction can be considered as a special case of the general method we developed for computing a contraction metric for periodic orbits.

Appendix A Description of the algorithm

In this section, we will provide a detailed version of the algorithm both for the periodic orbit case, and the equilibrium point case, presented next to each other for the convenience of the reader. Given is a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, with $\mathbf{f} \in C^{\sigma+1}(\mathbb{R}^n;\, \mathbb{R}^n)$, where $\sigma = \lceil k + \frac{n+1}{2} \rceil$ and $k \geq 2$ if n is odd and $k \geq 3$ if n is even, so that the minimum smoothness needed for the contraction metric M and its optimal recovery S is guaranteed.

The idea is to increase the number of collocation points and vertices gradually so that we obtain a small enough fill distance and fine enough triangulation. The steps of the algorithm are as follows:

- STEP 0. ***setting the constants and parameters***.
  Fix $d \geq 2\sqrt{n}$, $k \geq 2$ if n is odd and $k \geq 3$ if n is even, $c > 0$, and the Wendland function $\psi_0(r) := \psi_{l,k}(cr)$ with $l = \lfloor \frac{n}{2} \rfloor + k + 1$. Denote $\psi_{q+1}(r) = \frac{1}{r}\frac{d\psi_q}{dr}(r)$ for $q = 0, 1$.

  Further, fix a compact set $\mathcal{C} \subset \mathbb{R}^n$ that we triangulate and where we want to compute a contraction metric for the system, and an open set $O \supset \mathcal{C}$. Start with $h_{\text{collo}} > 0$, $h_{\text{triang}} > 0$.

- STEP I. ***RBF subroutine***
  Choose a set of pairwise distinct points $X = \{x_1, \ldots, x_N\}$ in O as collocation points with fill distance $h_{X,O} \leq h_{\text{collo}}$. To obtain a solution of the optimal recovery problem based on RBF approximation we follow these steps:

  (a) Compute the coefficients $b_{(\ell,i,j),(k,\mu,\nu)}$ defined as

$$
\begin{aligned}
b_{(\ell,i,j),(k,\mu,\nu)} \;=\;& \psi_0(\|\mathbf{x}_k - \mathbf{x}_\ell\|_2)\Bigg[\sum_{p=1}^{n} Df_{pi}(\mathbf{x}_\ell)Df_{p\mu}(\mathbf{x}_k)\delta_{\nu j} + Df_{\mu i}(\mathbf{x}_\ell)Df_{j\nu}(\mathbf{x}_k) \\
&+ Df_{i\mu}(\mathbf{x}_k)Df_{\nu j}(\mathbf{x}_\ell) + \delta_{i\mu}\sum_{p=1}^{n} Df_{p\nu}(\mathbf{x}_k)Df_{pj}(\mathbf{x}_\ell)\Bigg] \\
&+ \psi_1(\|\mathbf{x}_k - \mathbf{x}_\ell\|_2)\langle \mathbf{x}_k - \mathbf{x}_\ell, \mathbf{f}(\mathbf{x}_k)\rangle \left[Df_{\mu i}(\mathbf{x}_\ell)\delta_{\nu j} + \delta_{i\mu}Df_{\nu j}(\mathbf{x}_\ell)\right] \\
&+ \psi_1(\|\mathbf{x}_k - \mathbf{x}_\ell\|_2)\langle \mathbf{x}_\ell - \mathbf{x}_k, \mathbf{f}(\mathbf{x}_\ell)\rangle \left[Df_{i\mu}(\mathbf{x}_k)\delta_{\nu j} + \delta_{i\mu}Df_{j\nu}(\mathbf{x}_k)\right] \\
&- \psi_1(\|\mathbf{x}_k - \mathbf{x}_\ell\|_2)\langle \mathbf{f}(\mathbf{x}_\ell), \mathbf{f}(\mathbf{x}_k)\rangle \delta_{i\mu}\delta_{j\nu} \\
&+ \psi_2(\|\mathbf{x}_k - \mathbf{x}_\ell\|_2)\langle \mathbf{x}_k - \mathbf{x}_\ell, \mathbf{f}(\mathbf{x}_k)\rangle\langle \mathbf{x}_\ell - \mathbf{x}_k, \mathbf{f}(\mathbf{x}_\ell)\rangle \delta_{i\mu}\delta_{j\nu}. \qquad \text{(A1)}
\end{aligned}
$$

  for $1 \leq k, \ell \leq N$, and $1 \leq i, j, \mu, \nu \leq n$. Here and below, $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} x_k y_k$ denotes the Euclidean scalar product.

  (a') For the periodic orbit case we need to fix a $c_0 \in \mathbb{R}^+$, and $\mathbf{y}_0 \in \mathcal{A}(\Omega)$, then calculate the coefficients $b_{(\ell,i,j),(k,\mu,\nu)}$, $b_{0,(k,\mu,\nu)}$, $b_{(\ell,i,j),0}$ and $b_{0,0}$ for $1 \leq k, \ell \leq N$, $1 \leq i, j, \mu, \nu \leq n$ using the following formulas

$$
\begin{aligned}
b_{0,(k,\mu,\nu)} \;=\;& \psi_0(\|\mathbf{x}_k - \mathbf{y}_0\|_2)\Bigg[\sum_{p=1}^{n} V_{p\mu}(\mathbf{x}_k)f_p(\mathbf{y}_0)f_\nu(\mathbf{y}_0) + \sum_{p=1}^{n} V_{p\nu}(\mathbf{x}_k)f_p(\mathbf{y}_0)f_\mu(\mathbf{y}_0)\Bigg] \\
&+ \psi_1(\|\mathbf{x}_k - \mathbf{y}_0\|_2)\langle \mathbf{x}_k - \mathbf{y}_0, \mathbf{f}(\mathbf{x}_k)\rangle f_\mu(\mathbf{y}_0)f_\nu(\mathbf{y}_0) \qquad \text{(A2)} \\
b_{0,0} \;=\;& \psi_0(0)\|\mathbf{f}(\mathbf{y}_0)\|_2^4. \qquad \text{(A3)}
\end{aligned}
$$

$$\begin{aligned}
b_{(\ell,i,j),(k,\mu,\nu)} &= \psi_0(\|\mathbf{x}_k-\mathbf{x}_\ell\|_2)\Bigg[\sum_{p=1}^n V_{pi}(\mathbf{x}_\ell)V_{p\mu}(\mathbf{x}_k)\delta_{\nu j}+V_{\mu i}(\mathbf{x}_\ell)V_{j\nu}(\mathbf{x}_k)\\
&\quad +V_{i\mu}(\mathbf{x}_k)V_{\nu j}(\mathbf{x}_\ell)+\delta_{i\mu}\sum_{p=1}^n V_{p\nu}(\mathbf{x}_k)V_{pj}(\mathbf{x}_\ell)\Bigg]\\
&\quad +\psi_1(\|\mathbf{x}_k-\mathbf{x}_\ell\|_2)\langle \mathbf{x}_k-\mathbf{x}_\ell,\mathbf{f}(\mathbf{x}_k)\rangle\left[V_{\mu i}(\mathbf{x}_\ell)\delta_{\nu j}+\delta_{i\mu}V_{\nu j}(\mathbf{x}_\ell)\right]\\
&\quad +\psi_1(\|\mathbf{x}_k-\mathbf{x}_\ell\|_2)\langle \mathbf{x}_\ell-\mathbf{x}_k,\mathbf{f}(\mathbf{x}_\ell)\rangle\left[V_{i\mu}(\mathbf{x}_k)\delta_{\nu j}+\delta_{i\mu}V_{j\nu}(\mathbf{x}_k)\right]\\
&\quad -\psi_1(\|\mathbf{x}_k-\mathbf{x}_\ell\|_2)\langle \mathbf{f}(\mathbf{x}_\ell),\mathbf{f}(\mathbf{x}_k)\rangle\delta_{i\mu}\delta_{j\nu}\\
&\quad +\psi_2(\|\mathbf{x}_k-\mathbf{x}_\ell\|_2)\langle \mathbf{x}_k-\mathbf{x}_\ell,\mathbf{f}(\mathbf{x}_k)\rangle\langle \mathbf{x}_\ell-\mathbf{x}_k,\mathbf{f}(\mathbf{x}_\ell)\rangle\delta_{i\mu}\delta_{j\nu} \qquad \text{(A4)}\\
\text{and } b_{(\ell,i,j),0} &= \psi_0(\|\mathbf{y}_0-\mathbf{x}_\ell\|_2)\Bigg[\sum_{p=1}^n V_{pi}(\mathbf{x}_\ell)f_p(\mathbf{y}_0)f_j(\mathbf{y}_0)+\sum_{p=1}^n V_{pj}(\mathbf{x}_\ell)f_p(\mathbf{y}_0)f_i(\mathbf{y}_0)\Bigg]\\
&\quad +\psi_1(\|\mathbf{y}_0-\mathbf{x}_\ell\|_2)\langle \mathbf{x}_\ell-\mathbf{y}_0,\mathbf{f}(\mathbf{x}_\ell)\rangle f_i(\mathbf{y}_0)f_j(\mathbf{y}_0). \qquad \text{(A5)}
\end{aligned}$$

(*b*) Calculate the coefficients $c_{(\ell,i,j),(k,\mu,\nu)}$ defined as

$$c_{(\ell,i,j),(k,\mu,\nu)} = \frac{1}{4}\left(b_{(\ell,i,j),(k,\mu,\nu)}+b_{(\ell,j,i),(k,\nu,\mu)}+b_{(\ell,i,j),(k,\nu,\mu)}+b_{(\ell,j,i),(k,\mu,\nu)}\right), \qquad \text{(A6)}$$

where we assume $\mu \le \nu$ and $i \le j$. The coefficients $c_{\cdot,\cdot}$ form a symmetric matrix of size $N\frac{n(n+1)}{2}$.

(*b'*) For the periodic orbit case, in addition to the coefficients formula in (*b*), we need one more row and column in the matrix so for all (ℓ,i,j) with $1 \le \ell \le N$, $1 \le i \le j \le n$ we have

$$\begin{aligned}
c_{0,0} &= b_{0,0}, && \text{(A7)}\\
c_{(\ell,i,j),0} &= b_{(\ell,i,j),0}, && \text{(A8)}\\
c_{0,(k,\mu,\nu)} &= \frac{1}{2}\left(b_{0,(k,\mu,\nu)}+b_{0,(k,\nu,\mu)}\right) = b_{0,(k,\mu,\nu)}, && \text{(A9)}
\end{aligned}$$

where we assume $\mu \le \nu$. The coefficients $c_{\cdot,\cdot}$ form a symmetric matrix of size $N\frac{n(n+1)}{2}+1$.

(*c*) Determine $\gamma_k^{(\mu,\nu)}$, by solving the linear system

$$\sum_{k=1}^N \sum_{1\le\mu\le\nu\le n} c_{(\ell,i,j),(k,\mu,\nu)}\gamma_k^{(\mu,\nu)} = (F(S)(\mathbf{x}_\ell))_{i,j} = \lambda_\ell^{(i,j)}(S) = -C_{ij} \qquad \text{(A10)}$$

for $1 \le \ell \le N$, and $1 \le i \le j \le n$. Note that (A10) is a system of $N\frac{n(n+1)}{2}$ equations in $N\frac{n(n+1)}{2}$ unknowns and we usually choose C_{ij} as constants, although the method also works for $C_{ij}(\mathbf{x}_\ell)$.

(c') For the periodic orbit case the $\gamma_k^{(\mu,\nu)}$ solve the modified linear system

$$\sum_{k=1}^{N}\sum_{1\le\mu\le\nu\le n} c_{(\ell,i,j),(k,\mu,\nu)}\gamma_k^{(\mu,\nu)} + c_{(\ell,i,j),0}\gamma_0 = -C_{ij}(\mathbf{x}_\ell) \quad \text{(A11)}$$

$$\sum_{k=1}^{N}\sum_{1\le\mu\le\nu\le n} c_{0,(k,\mu,\nu)}\gamma_k^{(\mu,\nu)} + c_{0,0}\gamma_0 = c_0\|\mathbf{f}(\mathbf{y}_0)\|_2^4 \quad \text{(A12)}$$

for $1\le\ell\le N$, $1\le i\le j\le n$. Note that this is a system of $N\frac{n(n+1)}{2}+1$ equations in $N\frac{n(n+1)}{2}+1$ unknowns.

(d) Compute $\beta_k\in\mathbb{S}^{n\times n}$ from γ_k; recalling that

$$\beta_k^{(j,i)} = \beta_k^{(i,j)} = \frac{1}{2}\gamma_k^{(i,j)} \quad \text{if } i\neq j,$$
$$\beta_k^{(i,i)} = \gamma_k^{(i,i)}.$$

(d') For the periodic orbit case, we need relations in (d) and $\beta_0=\gamma_0$.

(e) We now have a formula for the optimal recovery

$$S(\mathbf{x}) = \sum_{k=1}^{N}\Big[\psi_0(\|\mathbf{x}_k-\mathbf{x}\|_2)\left[D\mathbf{f}(\mathbf{x}_k)\beta_k+\beta_k D\mathbf{f}(\mathbf{x}_k)^T\right] + \psi_1(\|\mathbf{x}_k-\mathbf{x}\|_2)\langle\mathbf{x}_k-\mathbf{x},\mathbf{f}(\mathbf{x}_k)\rangle\beta_k\Big]. \quad \text{(A13)}$$

(e') For the periodic orbit case, we can compute $S(\mathbf{x})$ with

$$S(\mathbf{x}) = \sum_{k=1}^{N}\Big[\psi_0(\|\mathbf{x}_k-\mathbf{x}\|_2)\left[V(\mathbf{x}_k)\beta_k+\beta_k V(\mathbf{x}_k)^T\right] + \psi_1(\|\mathbf{x}_k-\mathbf{x}\|_2)\langle\mathbf{x}_k-\mathbf{x},\mathbf{f}(\mathbf{x}_k)\rangle\beta_k\Big] + \beta_0\psi_0(\|\mathbf{y}_0-\mathbf{x}\|_2)\mathbf{f}(\mathbf{y}_0)\mathbf{f}(\mathbf{y}_0)^T. \quad \text{(A14)}$$

- STEP II. ***CPA subroutine***
  Fix an (h,d)-bounded triangulation $\mathcal{T}$ with $h\le h_{\text{triang}}$ and $\mathcal{D}_\mathcal{T}=\mathcal{C}$. Note that d denotes a bound on the degeneracy of the simplices, for details see [39].

(a) Compute the values $S(\mathbf{y})$ at the vertices of the triangulation $\mathbf{y}\in\mathcal{V}_\mathcal{T}$ and check if they are positive definite. If not, decrease h_{collo} by a factor, e.g. $h_{\text{collo}}\leftarrow h_{\text{collo}}/2$, and go back to STEP I.

(b) Fix constants $B_{2,\nu}$, and $B_{3,\nu}$ as upper bounds on the second- and third-order derivatives of the components f_k of $\mathbf{f}$ on each simplex $\mathfrak{S}_\nu$. Then compute the error term $E_\nu^e := n^2(1+4\sqrt{n})B_{2,\nu}\|\nabla P_{ij}^\nu\|_1+2n^3B_{3,\nu}P_\nu$, where

$$P_\nu := \max_{\mathbf{x}\in\mathfrak{S}_\nu}\|P(\mathbf{x})\|_2 = \max_{k=0,1,\dots,n}\|P(\mathbf{x}_k)\|_2.$$

(b') For the periodic orbit case we need $B_{i,\nu}$ with $i = 0, \cdots, 3$; together with $B_{V_{1,\nu}}$, and $B_{V_{2,\nu}}$ as the upper bounds on the first- and second-order derivatives of the components V_{lj} of V. Then compute

$$E^p_\nu := n^2 \cdot \left((4\sqrt{n} B_{V_{1,\nu}} + B_{2,\nu}) \|\nabla P^\nu_{ij}\|_1 + 2n B_{V_{2,\nu}} P_\nu + 2\kappa^*_\nu B_{0,\nu} B_{2,\nu} + 2\kappa^*_\nu B^2_{1,\nu} \right).$$

(c) Check whether *Constraints* (VP2) of Verification Problem 2 (or 1) are fulfilled. If not, then reduce h_{triang} by a factor, e.g. $h_{\text{triang}} \leftarrow h_{\text{triang}}/8$, and repeat STEP II from the beginning.
If the conditions still do not hold, decrease h_{collo} by a factor, e.g. $h_{\text{collo}} \leftarrow h_{\text{collo}}/2$, and go back to STEP I.

∘ STEP III. ***creating the contraction metric***
Build the metric $P : \mathcal{C} \to \mathbb{S}^{n\times n}$ as the CPA interpolation of the values $P(\mathbf{y})$, $\mathbf{y} \in \mathcal{V}_\mathcal{T}$. P is a contraction metric for the system on any compact $\widetilde{K} \subset \mathcal{C}^\circ$.

Remark 1 Note that in most applications it is more practical to use a relaxed version of the procedure above to compute a contraction metric. If the matrices $P(\mathbf{y})$, $\mathbf{y} \in \mathcal{V}_\mathcal{T}$, in STEP II (a) are positive definite in a reasonably large part of $\mathcal{C}$, then one can proceed to the next sub-steps. Further, if additionally *Constraints* (VP2) of Verification Problem 2 (or 1) are fulfilled in a reasonably large part of $\mathcal{C}$ in STEP II (c), then one can proceed to STEP III.
The CPA interpolation P will then be a contraction metric on any compact subset $\widetilde{K}$ of the interior of the area where P is both positive definite, asserted by *Constraints* (VP1) of Verification Problem 1, and fulfills *Constraints* (VP2) of Verification Problem 2 (or 1). We use this relaxed procedure in our examples.

Declarations

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