Discrete and Continuous Dynamical Systems - Series B
Vol. 27, No. 11, November 2022, pp. 6927–6941
doi:10.3934/dcdsb.2022027

EXISTENCE OF COMPLETE LYAPUNOV FUNCTIONS WITH PRESCRIBED ORBITAL DERIVATIVE

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(Communicated by Peter E. Kloeden)

Abstract. Complete Lyapunov functions for a dynamical system, given by an autonomous ordinary differential equation, are scalar-valued functions that are strictly decreasing along orbits outside the chain-recurrent set. In this paper we show that we can prescribe the (negative) values of the derivative along orbits in any compact set, which is contained in the complement of the chain-recurrent set. Further, the complete Lyapunov function is as smooth as the vector field defining the dynamics. This delivers a theoretical foundation for numerical methods to construct complete Lyapunov functions and renders them accessible for further theoretical analysis and development.

1. Introduction.

Initial value problems of autonomous differential equations arise in many applications and define a dynamical system. Many tools have been developed to study the long-term behaviour of solutions and classify different behaviour depending on the initial conditions. One of the classical and fundamental tools is a Lyapunov function, which is a generalization of the energy in a dissipative system. It is a scalar-valued function, which is non-increasing along orbits of the dynamical system. Complete Lyapunov functions, introduced by [3, 6], are scalar-valued functions, which are strictly decreasing along orbits outside the chain-recurrent set and satisfy additional properties for the values on the chain-recurrent set.

A complete Lyapunov function describes the qualitative behaviour of orbits by separating the phase space into two disjoint areas with fundamentally different behaviour of the flow: the chain-recurrent set and its complement, where the flow

2020 *Mathematics Subject Classification.* Primary: 34D05, 93D30; Secondary: 37C10.
Key words and phrases. Smooth complete Lyapunov function, chain recurrent set.
Suhr is partially supported by the SFB/TRR 191 "Symplectic Structures in Geometry, Algebra and Dynamics", funded by the Deutsche Forschungsgemeinschaft.
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is gradient-like. On the chain-recurrent set the flow is (almost) recurrent, it contains all equilibria, periodic and almost periodic orbits, as well as local attractors and repellers. The flow on the chain-recurrent set is sensitive to arbitrarily small perturbations, while the gradient-like flow is robust to sufficiently small perturbations. Moreover, complete Lyapunov functions reveal stability properties of the chain transitive components of the chain recurrent set as well as the flow between them.

If the complete Lyapunov function is differentiable, then the conditions can be expressed by the derivative along solutions, the orbital derivative: points with vanishing orbital derivative characterize the chain-recurrent set, while the orbital derivative is strictly negative in the area of gradient-like flow.

The existence of complete Lyapunov functions was first shown on compact phase spaces [6] and later on noncompact phase spaces [14, 15, 16, 19]. The existence of C^∞ complete Lyapunov functions on compact state spaces was shown in [8], and in noncompact spaces in [12]. The latter proof used the connection of complete Lyapunov functions to time functions in general relativity [13]; this relation was first noted by [9] and further explored in [5], which gave the first general existence results for C^∞ Lyapunov functions on arbitrary manifolds.

The main condition on complete Lyapunov functions is that the orbital derivative is strictly negative in the gradient-flow part, i.e. the complement of the chain-recurrent set. Hence, complete Lyapunov functions are not unique and a natural question is whether one can prescribe the values of the orbital derivative by a given negative function on the gradient-flow part.

The main result of the paper is that, indeed, the orbital derivative can be prescribed by an arbitrary, sufficiently smooth function on any compact set, which is contained in the complement of the chain-recurrent set, see Theorem 2.6. In the proof we first show that we can reduce the problem to the case where the orbital derivative is fixed to -1 and then we modify an existing C^∞ complete Lyapunov function on the compact set, while preserving it away from it; this is achieved by modifying it on flow boxes. The resulting complete Lyapunov function is as smooth as the vector field defining the system.

This result has implications on the numerical construction of complete Lyapunov functions. There exist a number of numerical approaches to compute complete Lyapunov functions. One approach divides the phase space into cells and computes the flow between these cells to construct a complete Lyapunov function [17, 4]. Other approaches, however, fix the orbital derivative by a prescribed function and use collocation methods to solve the resulting partial differential equation for the complete Lyapunov function [1, 2] – or optimization methods with a mixture of equality and inequality constraints [10]. So far no existence result for these approaches using equations was available. The results of this paper can be used to ensure that numerical methods for constructing complete Lyapunov functions with prescribed orbital derivative are successful, and thus they deliver a theoretical foundation for these methods.

Let us give an overview of the paper: In Section 2 we define complete Lyapunov functions and state the main result. In Section 3 we prove the results, before we conclude the paper in Section 4.

2. **Definition & main result.** Let $U \subset \mathbb{R}^n$ be open and let $X\colon U \to \mathbb{R}^n$ be C^l with $l \in \mathbb{N} \cup \{\infty\}$, where $\mathbb{N} := \{1, 2, 3, 4, 5, \ldots\}$. We consider the dynamical system defined by solutions of the ODE $\dot{x} = X(x)$.

Definition 2.1. The **local flow** of X is the map $\Phi\colon \Omega \to U$, $(t, p) \mapsto \Phi_t(p)$ such that

(i) $\Omega \subset \mathbb{R} \times U$ is open with $\{0\} \times U \subset \Omega$.
(ii) For every $p \in U$ the orbit $t \mapsto \Phi_t(p)$ is the unique maximally extended solution to the initial value problem

$$\begin{cases} \frac{\partial}{\partial t}\Phi_t(p) & = X(\Phi_t(p)) \\ \Phi_0(p) & = p. \end{cases}$$

Remark 1. (1) The attribute "local" for the flow refers to local in time. We do not assume that flowlines of X exist on the whole of $\mathbb{R}$ for all initial values $p \in U$.
(2) With the regularity assumption on X the existence of Φ is implied by the Theorem of Picard-Lindelöf. Note that the local flow enjoys the same regularity as the generator X, i.e. $\Phi \in C^l$.

Let us now define the chain recurrent set. We denote by $\|.\|$ the Euclidian norm on $\mathbb{R}^n$.

Definition 2.2. Let $T > 0$ and $\varepsilon\colon U \to (0, \infty)$ be continuous. A finite collection of points $p_0, \ldots, p_m \in U$ $(m \geq 1)$ is an **(ε, T)-chain** if there exist $t_i \geq T$ with

$$\|\Phi_{t_i}(p_i) - p_{i+1}\| \leq \varepsilon(\Phi_{t_i}(p_i))$$

for all $0 \leq i \leq m - 1$.

Definition 2.3. A point $p \in U$ is **chain recurrent for X** if for all $T > 0$ and all continuous $\varepsilon\colon U \to (0, \infty)$ there exists an (ε, T)-chain $p_0 = p, p_1, \ldots, p_m = p$.

Denote by

$$\mathcal{R}_X$$

the set of chain recurrent points for X.

Recall that the **chain transitive components** of the chain recurrent set are the equivalence classes with respect to the equivalence relation $\sim$, where $p \sim q$ for two points $p, q \in \mathcal{R}_X$ if there exists $T > 0$ such that for all continuous $\varepsilon\colon U \to (0, \infty)$ there is an (ε, T)-chain $p = p_0, p_1, \ldots, p_m = p$ containing q.

The following definition of Lyapunov functions is very closely related to [5, Definition 1.4]. Here we omit the smoothness of the functions in favor of a lower regularity and consider only the case of vector fields.

Definition 2.4. Let $X\colon U \to \mathbb{R}^n$ be C^l with $l \in \mathbb{N} \cup \{\infty\}$. The function $\tau : U \to \mathbb{R}$ is a **Lyapunov function for X** if it is C^l regular,

$$\dot{\tau}(p) := \nabla\tau(p) \cdot X(p) \leq 0$$

for each $p \in U$, and if, at each regular point p of τ, we have $\dot{\tau}(p) < 0$. Recall that a point $p \in U$ is regular for $\tau\colon U \to \mathbb{R}$ if $\nabla\tau(p) \neq 0$.

In the following we sometimes write τ **is Lyapunov** if the vector field X is clear from the context, to indicate that τ is a Lyapunov function in the sense of Definition 2.4 for X.

Remark 2. This definition of a Lyapunov function is not the usual one, but particulary useful when studying numerical methods for the computation of Lyapunov functions; c.f. [11] where similar Lyapunov functions are referred to as *complete Lyapunov function candidates. Note that a usual strict C^1 Lyapunov function for an equilibrium point (or an invariant set I), i.e. a function satisfying $\dot{\tau}(p) < 0$ for all $p \notin I$ and $\dot{\tau}(p) = 0$ for all $p \in I$, attaining its strict minimum at I, is also a Lyapunov function in the sense above. On the other hand, a Lyapunov function as above with the additional assumption that it attains its strict minimum at the equilibrium (or the invariant set where it is constant), is a usual non-strict Lyapunov function, i.e. a function satisfying $\dot{\tau}(p) \leq 0$ for all p, attaining its strict minimum at I.*

In order to state the theorem we adopt the notion of complete Lyapunov function from [6, II.§6.4], see also [12, Definition 4.5]:

Definition 2.5. A Lyapunov function $\tau\colon U \to \mathbb{R}$ for the vector field X is **complete** if it is strictly decreasing along orbits outside of $\mathcal{R}_X$ and such that (1) $\tau(\mathcal{R}_X)$ is nowhere dense and (2) for $t \in \tau(\mathcal{R}_X)$ the set $\tau^{-1}(t) \cap \mathcal{R}_X$ is a chain transitive component.

Remark 3. The original definition in [6, II.§6.4] of a complete Lyapunov function requires for $t \in \tau(\mathcal{R}_X)$ the preimage $\tau^{-1}(t)$ to be a chain transitive component. In general we cannot expect the critical levels of τ to be equal to chain transitive components. As an example consider $U = \mathbb{R}^2$ and a vector field $X = \chi \cdot e_1$ with $\chi \geq 0$ and $\chi(x, y) = 0$ iff $(x, y) = (0, 0)$. It is obvious that the chain recurrent set $\mathcal{R}_X$ consists only of the origin $(0, 0)$ although the critical level $\{\tau = \tau((0, 0))\}$ of any complete Lyapunov function $\tau\colon \mathbb{R}^2 \to \mathbb{R}$ is strictly larger than $\{(0, 0)\}$. To see this note that $\tau(x, 0) > \tau(0, 0)$ for $x < 0$ and $\tau(x, 0) < \tau(0, 0)$ for $x > 0$, because τ is continuous and strictly decreasing along solution trajectories. Thus the critical level $\{\tau = \tau((0, 0))\}$ divides the plane into at least two connected components. Since a single point does not divide the plane we arrive at the conclusion that the critical level is strictly larger than $\{(0, 0)\}$.

Remark 4. Our definition of a complete Lyapunov function is stricter than that of Conley: τ is C^1, whereas Conley's function is merely continuous, and in our definition $p \in \mathcal{R}_X$ implies $\nabla\tau(p) = 0$, which is not necessarily the case in Conley's work, even for a differentiable τ. The advantage of this stricter definition is that the decrease condition can be written $\nabla\tau(p) \cdot X(p) < 0$ for every $p \in U \setminus \mathcal{R}_X$, which is much more accessible for numerical methods. Note that a complete Lyapunov function from Definition 2.5 is also a complete Lyapunov function in the sense of Conley [6] and it was proved in [12] that such a complete Lyapunov function always exists, i.e. our definition is not more restrictive.

Now we are ready to state our main result:

Theorem 2.6. *Let $U \subset \mathbb{R}^n$ be open and let $X\colon U \to \mathbb{R}^n$ be C^l with $l \in \mathbb{N} \cup \{\infty\}$. Then for every compact set $K \subset U \setminus \mathcal{R}_X$ and every C^l-function $g\colon U_K \to (-\infty, 0)$ defined on a neighborhood U_K of K there exists a complete C^l-Lyapunov function*

$$\tau_K\colon U \to \mathbb{R}$$

with $\dot{\tau}_K|_K \equiv g$ and $\dot{\tau}_K < 0$ on $U \setminus \mathcal{R}_X$.

Remark 5. (a) In the proof we will w.l.o.g. assume that the local flow is complete. Note that for every continuous function $f\colon U \to (0,\infty)$ the chain recurrent sets of X and fX coincide.

Further we can choose a C^∞-function $f\colon U \to \mathbb{R}$ with $f|_K \equiv 1$ such that the local flow Ψ of fX is complete, i.e.

$$\Psi\colon \mathbb{R}\times U \to U, \quad (t,p) \mapsto \Psi_t(p)$$

is well defined. Note that Ψ coincides with the local flow of X on K. Further the set $\mathcal{R}_X = \mathcal{R}_{fX}$ is Ψ-invariant. Thus proving Theorem 2.6 for fX instead of X yields the claim for the initial vector field as well. We will continue to use the notation X for the vector field under consideration.

(b) Note that the regularity of τ_K is in general optimal. As an example consider a vector field $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form $X(x,y) = \chi(y)e_1$ for some C^l-function $\chi\colon \mathbb{R} \to (0,\infty)$ which is nowhere C^{l+1}; e.g. the lth derivative might be the Weierstrass function. Let $K := [0,1]\times[0,1] \subset \mathbb{R}^2$ and $g \equiv -1$. By Theorem 2.6 we have a complete C^l-Lyapunov function $\tau_K\colon \mathbb{R}^2 \to \mathbb{R}$ with $\dot\tau_K|_K \equiv -1$. The flow of X is given by $\Phi_t(x,y) = (x + t\chi(y), y)$ and

$$\tau_K(x + t\chi(y), y) - \tau_K(x,y) = -t \tag{1}$$

as long as $\{(x + s\chi(y), y) \mid s \in [0,t]\} \subset K$. Assume that τ_K is C^{l+1} on an open set $V \subset K$. Choose $0 < t_0$, (x_0, y_0) and $\delta > 0$ such that

$$(x + t\chi(y), y) \in V$$

for all $t \in [0,t_0]$ and all $|x - x_0| < \delta$ and $|y - y_0| < \delta$. Since τ_K is C^{l+1} by assumption and for all t and $(x,y) \in V$ in question, we have by (1) for all small enough $h > 0$ that

$$\frac{\tau_K(x+h,y) - \tau_K(x,y)}{h} = \frac{-1}{\chi(y)},$$

in particular

$$\partial_1\tau_K(x,y) = \lim_{h\to 0+} \frac{\tau_K(x+h,y) - \tau_K(x,y)}{h} = \frac{-1}{\chi(y)} \neq 0.$$

Further, the level sets $\{\tau_K = \tau_K(x_0,y_0)\}$ and $\{\tau_K = \tau_K(x_0 + t_0\chi(y_0), y_0)\}$ are graphs

$$\{(\phi_i(y), y) \mid y \in I\}, \quad i = 0,1$$

of two C^{l+1} functions $\phi_0\colon I \to \mathbb{R}$ and $\phi_1\colon I \to \mathbb{R}$ respectively, where $I \subset \mathbb{R}$ is a sufficiently small interval around y_0. Hence

$$\frac{-t_0}{\phi_1(y) - \phi_0(y)} = \frac{\tau_K(\phi_1(y),y) - \tau_K(\phi_0(y),y)}{\phi_1(y) - \phi_0(y)} = \frac{-1}{\chi(y)},$$

i.e. $\phi_1(y) - \phi_0(y) = t_0\chi(y)$ for all $y \in I$, from which $\chi \in C^{l+1}(I)$ follows, a contradiction to the assumption.

3. **Proof of Theorem 2.6.** The proof consists of modifying a sufficiently fast descending Lyapunov function on K while preserving it away from K. This is accomplished in Proposition 1. The proposition in turn relies on the main technical Lemma 3.1, which gives the construction on a single flow box (see definition below). The proof of the proposition is then a repeated application of the lemma.

By [12] we know that X admits a complete C^∞-Lyapunov function, $\tau'\colon U \to \mathbb{R}$. We define **flow boxes** as follows. For $r \in \mathbb{R}$ let $V_{\tau',r}$ be a precompact relatively open subset of $\{\tau' = r\} \setminus \mathcal{R}_X$, i.e. $V_{\tau',r}$ is open in $\{\tau' = r\} \setminus \mathcal{R}_X$ and the closure $\overline{V_{\tau',r}}$ is a compact subset of $U \setminus \mathcal{R}_X$. The map

$$\Phi\colon \mathbb{R} \times V_{\tau',r} \to U \setminus \mathcal{R}_X, \quad (t,q) \mapsto \Phi_t(q)$$

is a diffeomorphism onto its image. For $T > 0$ the set

$$\mathcal{V}_{\tau',r,T} := \Phi((-T,T) \times V_{\tau',r}) \subset U \setminus \mathcal{R}_X$$

is called a *flow box*, cf. Figure 1.

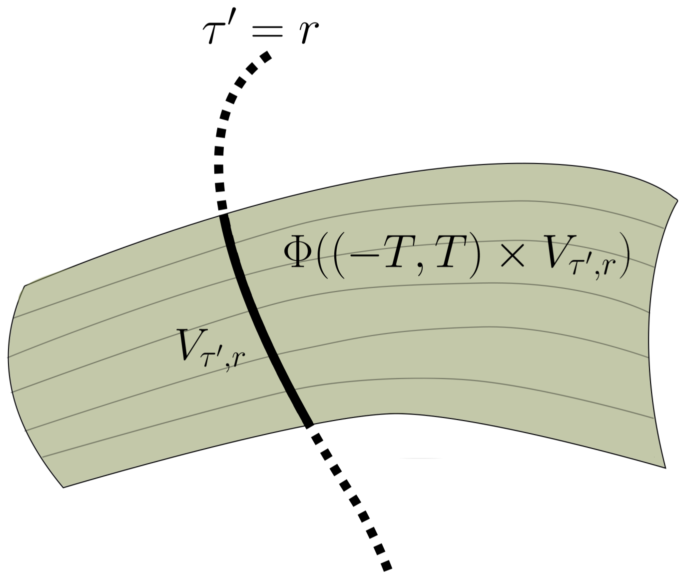

FIGURE 1. Schematic figure of a flow box $\mathcal{V}_{\tau',r,T}$.

Choose $r_1 > \ldots > r_N \in \mathbb{R}$ and $V_{\tau',r_i} \subset \{\tau' = r_i\} \setminus \mathcal{R}_X$ such that the flow boxes

$$\mathcal{V}_{\tau',r_i,1} := \Phi((-1,1) \times V_{\tau',r_i})$$

form an open cover of K, i.e.

$$K \subset \bigcup_{i=1}^{N} \mathcal{V}_{\tau',r_i,1}.$$

Choose precompact relatively open subsets W_{τ',r_i} with $\overline{V_{\tau',r_i}} \subset W_{\tau',r_i} \subset \{\tau' = r_i\}$. Then the flow boxes satisfy

$$\overline{\mathcal{V}_{\tau',r_i,1}} \subset \mathcal{W}_{\tau',r_i,i+1} = \Phi((-(i+1), i+1) \times W_{\tau',r_i}).$$

Choose a constant $0 < C < \infty$ such that $C\dot{\tau}' < -N-3$ on $\bigcup_{i=1}^{N} \mathcal{W}_{\tau',r_i,i+1}$. Set $\tau := C\tau'$, $s_i := Cr_i$, $V_{s_i} := V_{\tau',r_i}$, $W_{s_i} := W_{\tau',r_i}$, $\mathcal{V}_{s_i,1} := \mathcal{V}_{\tau',r_i,1}$, and $\mathcal{W}_{s_i,i+1} := \mathcal{W}_{\tau',r_i,i+1}$. We then have

$$\dot{\tau}(p) < -N-3 \text{ for all } p \in \bigcup_{i=1}^{N} \mathcal{W}_{s_i,i+1}. \tag{2}$$

Recall that a function $\tau : U \to \mathbb{R}$ is a Lyapunov function for the C^l-regular vector field X if it is C^l regular,

$$\dot{\tau}(p) := \nabla\tau(p) \cdot X(p) \leq 0$$

for each $p \in U$, and if, at each regular point p of τ, we have $\dot{\tau}(p) < 0$. Further recall that a Lyapunov function $\tau\colon U \to \mathbb{R}$ for the vector field X is complete if it is strictly decreasing along orbits outside of $\mathcal{R}_X$ and such that (1) $\tau(\mathcal{R}_X)$ is nowhere dense and (2) for $t \in \tau(\mathcal{R}_X)$ the set $\tau^{-1}(t) \cap \mathcal{R}_X$ is a chain transitive component.

We will deduce Theorem 2.6 from the following modification result.

Proposition 1. *Let $X\colon U \to \mathbb{R}^n$ be C^l with $l \in \mathbb{N} \cup \{\infty\}$ and let $K \subset U \setminus \mathcal{R}_X$ compact be given. If (2) holds, then there exists a complete C^l-Lyapunov function $\overline{\tau}_K\colon U \to \mathbb{R}$ such that*

(i) $\overline{\tau}_K$ and τ coincide on $U \setminus \bigcup_i \mathcal{W}_{s_i,i+1}$,
(ii) $\nabla\overline{\tau}_K \cdot X \equiv -1$ on a neighborhood of K, and
(iii) the critical set of $\overline{\tau_K}$ is equal to $\mathcal{R}_X$, i.e. $\{p \in U | \ \nabla\tau_K = 0\} = \mathcal{R}_X$.

Proof of Theorem 2.6. From Proposition 1 we obtain a complete Lyapunov function $\overline{\tau}_K$ which satisfies all claims of Theorem 2.6 except $\nabla\overline{\tau}_K \cdot X|_K \equiv -1$ and not $\nabla\overline{\tau}_K \cdot X|_K \equiv g$ for a given $g\colon U_K \to (-\infty, 0)$. Choose a closed neighborhood V_K of $U \setminus U_K$ disjoint from K. Extend g to a negative C^l-function on U with $g|_{V_K} \equiv -1$ and consider the vector field $X_g := -X/g$. Choose a Lyapunov function τ_K for X_g according to Proposition 1. Then we have $\nabla\tau_K \cdot X_g|_K \equiv -1$ which is equivalent to $\nabla\tau_K \cdot X|_K = \dot{\tau}_K|_K \equiv g$. □

Proposition 1 follows from the next lemma by repeated application.

Lemma 3.1. *Let $\tau\colon U \to \mathbb{R}$ be a complete C^l-Lyapunov function and $M \subset U \setminus \mathcal{R}_X$ be closed and assume that $\nabla\tau \cdot X \equiv -1$ on a neighborhood of M. Let $V_s, W_s \subset \{\tau = s\} \setminus \mathcal{R}_X$ be relatively open precompact sets with $\overline{V_s} \subset W_s$ and $k \in \mathbb{N}$ such that*

$$\dot{\tau}(p) < -k - 3 \tag{3}$$

for all $p \in \Phi([k, k+1] \times \overline{W_s})$. Then there exists a complete C^l-Lyapunov function $\tilde{\tau}\colon U \to \mathbb{R}$ with

(i) $\tilde{\tau} \equiv \tau$ on $U \setminus \mathcal{W}_{s,k+1}$ and
(ii) $\nabla\tilde{\tau} \cdot X \equiv -1$ on a neighborhood of $\overline{\mathcal{V}_{s,1}} \cup M$,

where $\mathcal{V}_{s,1} := \Phi((-1, 1) \times V_s)$ and $\mathcal{W}_{s,k+1} := \Phi((-(k+1), k+1) \times W_s)$ are the flow boxes around V_s and W_s.

Remark 6. It is important for the proof of the lemma (see 3rd step below) as well as for the application to Proposition 1 that M is disjoint from $\Phi([k, k+1] \times \overline{W_s})$. The intersection of M with

$$\Phi(\{-(k+1)\} \times \overline{W_s}) \cup \Phi([-(k+1), k] \times \partial\overline{W_s})$$

is in general nonempty, though.

Proof of Proposition 1. First note that since we assume that τ is a complete Lyapunov function and the critical values of τ and $\overline{\tau}_K$ coincide, it follows trivially that $\overline{\tau}_K$ is also a complete Lyapunov function.

The construction of $\overline{\tau}_K$ proceeds by induction over $k = 1, \ldots, N$; see Figure 2 for a schematic depiction.

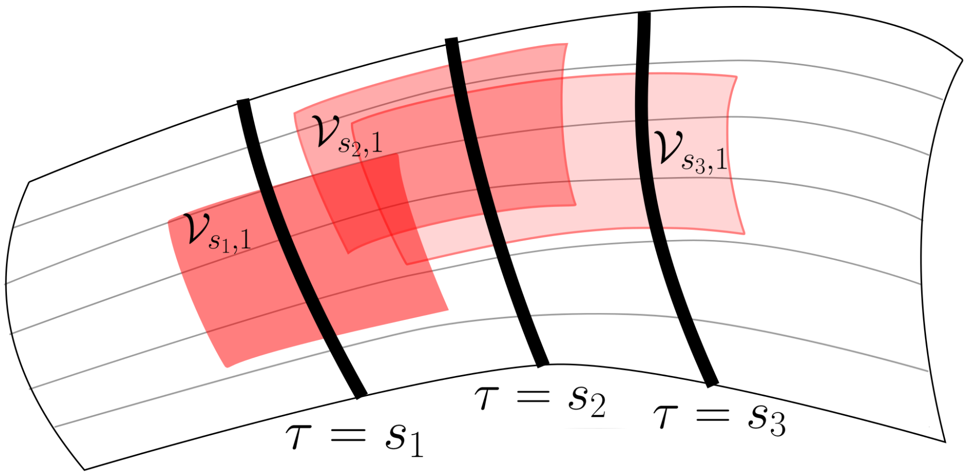

FIGURE 2. Schematic presentation of the sets $\mathcal{V}_{s_i,1}$ in the construction of the functions $\tilde{\tau}_i$.

For $k = 1$ apply Lemma 3.1 to $M = \emptyset$ and $\mathcal{V}_{s,1} = \mathcal{V}_{s_1,1}$ and $\mathcal{W}_{s,2} = \mathcal{W}_{s_1,2}$. Condition (3) is satisfied by assumption (2). This yields a Lyapunov function $\tilde{\tau}_1 \colon U \to \mathbb{R}$ with $\tilde{\tau}_1 \equiv \tau$ on $U \setminus \mathcal{W}_{s_1,2}$ and $\nabla \tilde{\tau}_1 \cdot X \equiv -1$ on a neighborhood of $\overline{\mathcal{V}_{s_1,1}}$.

Now let $k \geq 2$ and assume that a Lyapunov function $\tilde{\tau}_{k-1} \colon U \to \mathbb{R}$ with

$$\tilde{\tau}_{k-1} \equiv \tau \text{ on } U \setminus \bigcup_{i<k} \mathcal{W}_{s_i,i+1}$$

and

$$\nabla \tilde{\tau}_{k-1} \cdot X \equiv -1 \text{ on a neighborhood of } \bigcup_{i<k} \overline{\mathcal{V}_{s_i,1}}$$

has been constructed.

Set $M = \bigcup_{i<k} \overline{\mathcal{V}_{s_i,1}}$. We show that

$$\Phi([k, k+1] \times \overline{W_{s_k}}) \cap \bigcup_{i<k} \mathcal{W}_{s_i,i+1} = \emptyset.$$

This especially implies $\Phi([k, k+1] \times \overline{W_{s_k}}) \cap M = \emptyset$. Assume on the contrary that for a $p \in \overline{W_{s_k}}$ and $t_p \in [k, k+1]$ there exist for an $i < k$ a $q \in W_{s_i}$ and $t_q \in (-(i+1), i+1)$ such that $\Phi(t_q, q) = \Phi(t_p, p)$. Then we have $q = \Phi(t_p - t_q, p)$ with $t_p - t_q > 0$ and it follows that

$$s_k < s_i = \tau(q) = \tau(\Phi(t_p - t_q, p)) < \tau(p) = s_k,$$

a contradiction.

Thus we have $\tilde{\tau}_{k-1} \equiv \tau$ on $\Phi([k, k+1] \times \overline{W_{s_k}})$. With the assumption of (2) we conclude that Condition (3) is satisfied.

Now Lemma 3.1 with $s = s_k$ yields a complete Lyapunov function $\tilde{\tau}_k \colon U \to \mathbb{R}$ with $\tilde{\tau}_k \equiv \tilde{\tau}_{k-1}$ on $U \setminus \mathcal{W}_{s_k,k+1}$, i.e. $\tilde{\tau}_k \equiv \tau$ on $U \setminus \cup_{i \leq k} \mathcal{W}_{s_i,i+1}$ and

$$\nabla \tilde{\tau}_k \cdot X \equiv -1 \text{ on a neighborhood of } \bigcup_{i \leq k} \overline{\mathcal{V}_{s_i,1}}.$$

This finishes the induction. Setting $\overline{\tau}_K = \tilde{\tau}_N$ completes the proof. $\square$

Proof of Lemma 3.1. <u>*Prelude:*</u> The construction will not alter τ on a neighborhood of $U \setminus \mathcal{W}_{s,k+1}$. It therefore suffices to carry out the construction for $\tau|_{\overline{\mathcal{W}_{s,k+1}}}$ ensuring that the new function coincides with τ on a neighborhood of $\partial\mathcal{W}_{s,k+1}$. Recall that $\mathcal{R}_X$ is the set of critical points of the Lyapunov function τ and $\overline{\mathcal{W}_{s,k+1}} \subset U \setminus \mathcal{R}_X$. Therefore the restriction of the flow

$$\Phi\colon [-(k+1), k+1] \times \overline{W_s} \to \overline{\mathcal{W}_{s,k+1}}$$

is a C^l-diffeomorphism. This implies that constructing $\tilde{\tau}$ on $\overline{\mathcal{W}_{s,k+1}}$ is equivalent to constructing the function $\tilde{\tau} \circ \Phi$ on $[-(k+1), k+1] \times \overline{W_s}$ such that it coincides with

$$\tau \circ \Phi\colon [-(k+1), k+1] \times \overline{W_s} \to \mathbb{R}$$

on a neighborhood of the relative boundary

$$\{-(k+1)\} \times W_s \cup \{k+1\} \times W_s \cup [-(k+1), k+1] \times \partial W_s,$$

where $\partial W_s := \overline{W_s} \setminus W_s$. With the same argument we replace M by the preimage of M under Φ.

Note that since Φ is the flow of X we have $X(p) = D\Phi(t,q)e_1$ for $p = \Phi(t,q)$, where e_1 is the direction of the $\mathbb{R}$-factor of $\mathbb{R} \times \overline{W_s}$. Thus $\tau \circ \Phi$ is a Lyapunov function for the constant vector field e_1 on $[-(k+1), k+1] \times \overline{W_s}$, because

$$\dot{\tau}(p) = \nabla\tau(p) \cdot X(p) = \nabla\tau(p) \cdot D\Phi(t,q)e_1 = \nabla(\tau \circ \Phi)(t,q) \cdot e_1.$$

Further, we can equivalently write the orbital derivative $\nabla(\tau \circ \Phi) \cdot e_1$ as $\partial_{e_1}(\tau \circ \Phi)$. The construction will now work with the function $\tau \circ \Phi$ on $[-(k+1), k+1] \times \overline{W_s}$ and the constant vector field e_1. We will drop the notation $\tau \circ \Phi$ for simply τ in the following.

The proof proceeds in several steps. In the first step we construct $\tilde{\tau}$ in a neighborhood of $[-1,1] \times \overline{V_s}$. The second step then carefully interpolates $\tilde{\tau}$ with τ near $\{-(k+1)\} \times W_s$ in order not to destroy the property that $\partial_{e_1}\tau \equiv -1$ on a neighborhood of M. The third step then takes care of the interpolation near $\{k+1\} \times W_s$. Finally the fourth step interpolates $\tilde{\tau}$ with τ near $[-(k+1), k+1] \times \partial W_s$ again carefully in order not to destroy the property that $\partial_{e_1}\tau \equiv -1$ on a neighborhood of $M \cup [-1,1] \times \overline{V_s}$. As a result we obtain a Lyapunov function $\tilde{\tau}$ which coincides with τ outside of $\mathcal{W}_{s,k+1}$ and whose orbital derivative on a neighborhood of $\overline{\mathcal{V}_{s,1}} \cup M$ equals -1.

<u>*1st step:*</u>

In this first step we will construct a Lyapunov function τ_1 on $[-(k+1), k+1] \times W_s$ with $\partial_{e_1}\tau_1 \equiv -1$ on $[-1,1] \times \overline{V_s}$, see Figure 3.

Choose a smooth monotone function $\mu_-\colon \mathbb{R} \to [0,1]$ with:

(1) $\mu_- \equiv 0$ for $t \leq -3/2$ and
(2) $\mu_- \equiv 1$ for $t \geq -5/4$.

Define $\tau_1\colon [-(k+1), k+1] \times W_s \to \mathbb{R}$ by

$$\tau_1(t,q) := (1 - \mu_-(t))\tau(t,q) + \mu_-(t)\left[\tau(-1,q) - \left(t + \frac{3}{2}\right)\right].$$

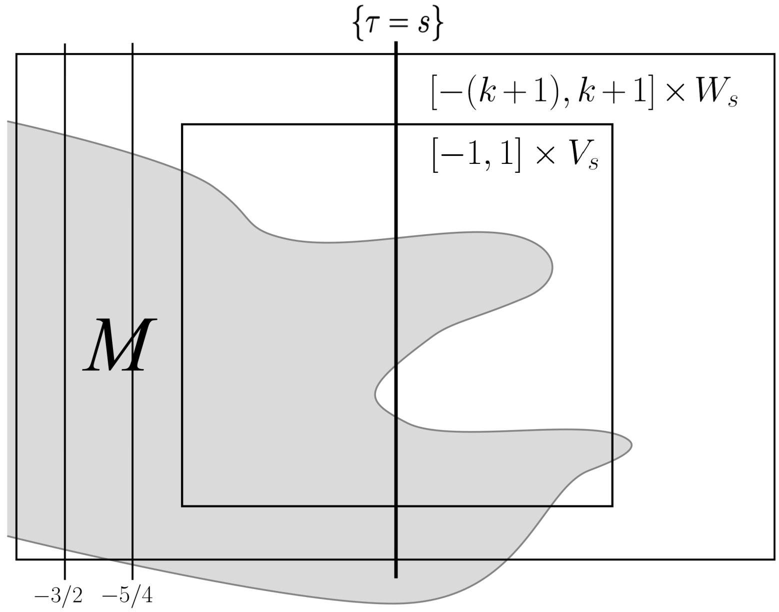

FIGURE 3. The first step. Note that M can intersect the boundary of $[-(k+1), k+1] \times W_s$ at $\{-(k+1)\} \times W_s$ and $[-(k+1), k+1) \times \partial W_s$, but not at $\{k+1\} \times W_s$ (right side).

It is easy to see that τ_1 is C^l-regular with $\partial_{e_1}\tau_1 < 0$ on $[-(k+1), k+1] \times \overline{W_s}$ and $\partial_{e_1}\tau \equiv -1$ on $[-5/4, k+1] \times \overline{W_s}$. Indeed we have

$$\begin{aligned}\partial_{e_1}\tau_1 =&(1-\mu_-(t))\partial_{e_1}\tau(t,q) - \mu_-(t)\\ &+ \mu_-'(t)\left[\tau(-1,q) - \left(t + \frac{3}{2}\right) - \tau(t,q)\right].\end{aligned}$$

The term

$$(1-\mu_-(t))\partial_{e_1}\tau(t,q) - \mu_-(t)$$

is everywhere negative, since τ is a Lyapunov function for e_1 and constant to -1 for $t \geq -5/4$. The term

$$\tau(-1,q) - \tau(t,q) - \left(t + \frac{3}{2}\right)$$

is negative for $-3/2 \leq t \leq -5/4$ because $\tau(t,q) \geq \tau(-1,q)$. Since $\mu_-' \geq 0$ and $\mathrm{supp}\mu_-' \subset [-3/2, -5/4]$ we conclude that τ_1 is Lyapunov. Note that $\tau_1 \equiv \tau$ on $[-(k+1), -3/2] \times \overline{W_s}$.

2^{nd} *step:* In this step, we construct a function τ_2 from τ_1, such that $\partial_{e_1}\tau_2 \equiv -1$ on an appropriate set involving M and $[-1,1] \times \overline{V_s}$, see Claim 3.2 and Figure 4.

Fix a neighborhood $U_M \subset [-(k+1), k+1] \times \overline{W_s}$ of M (in the relative topology), such that $\partial_{e_1}\tau|_{U_M} \equiv -1$. For $(t,p) \in M \cap [-3/2, -5/4] \times \overline{W_s}$ consider the level set

$$\{\tau = \tau(t,p)\} \cap [-7/4, -1] \times \overline{W_s}.$$

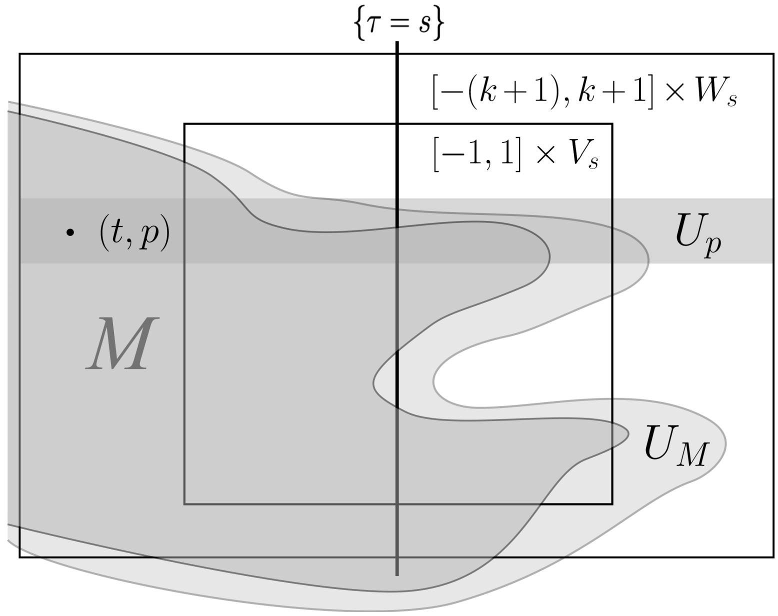

FIGURE 4. The second step.

Choose a neighborhood U_p of p in $\overline{W_s}$ such that there exists an open interval I_p containing $\tau(t,p)$ and

$$\{\tau \in I_p\} \cap [-7/4, -1] \times \overline{U_p} \subset U_M.$$

Note that $\partial_{e_1}\tau \equiv -1$ on $\{\tau \in I_p\} \cap [-7/4,-1] \times \overline{U_p}$ according to the choice of U_M. By the Implicit Function Theorem there exists a C^l-function

$$\phi_p\colon U_p \to [-7/4, -1]$$

with $\tau(u,q) = \tau(t,p)$ iff $u = \phi_p(q)$, i.e. a parameterization of a part of the level set $\{\tau = \tau(t,p)\}$ (if necessary, shrink U_p). Note that we can assume (possibly after further shrinking U_p) that

$$\tau(u,q) = \tau(\phi_p(q), q) - u + \phi_p(q) \tag{4}$$

in a neighborhood of $\{(\phi_p(q), q)|\ q \in U_p\}$ since the points $(\phi_p(q), q)$ belong to U_M and $\partial_{e_1}\tau \equiv -1$ on U_M. Define a function

$$\sigma_p\colon [-(k+1), k+1] \times U_p \to \mathbb{R},$$

$$\sigma_p(u,q) := \begin{cases} \tau(u,q), & \text{for } u \le \phi_p(q) \\ \tau(\phi_p(q), q) - u + \phi_p(q) & \text{for } u \ge \phi_p(q). \end{cases}$$

The function σ_p is C^l-regular by (4). Further we have $\partial_{e_1}\sigma_p < 0$ everywhere with $\equiv -1$ on $\{(u,q)|\ u \ge \phi_p(q)\} \subset [-(k+1), k+1] \times U_p$.

We select a finite subcover $\{U_j := U_{p_j}\}_{j=1,\ldots,R}$ of the compact set

$$M_s := \{p \in \overline{W_s}|\ \exists t \in [-7/4, -5/4] : (t,p) \in M\}.$$

Let $\{\lambda_j\}_j$ be a smooth partition of unity subordinate to $\{U_j\}_j$. Then

$$\sigma\colon [-(k+1),k+1]\times \bigcup_{1\le j\le R} U_j \to \mathbb{R},$$
$$\sigma(t,q) := \sum_j \lambda_j(q)\sigma_{p_j}(t,q)$$

is a C^l-function with $\partial_{e_1}\sigma \equiv -1$ on $[-1,k+1]\times(\cup_j U_j)$. Note that $\sigma\equiv\tau$ on $[-(k+1),-7/4]\times(\cup_j U_j)$.

Let $\mathbb{U}_1$ be a neighborhood of M_s (in the relative topology of $\overline{W_s}$) with closure in $\cup_j U_j$ and let $\nu_1\colon \overline{W_s}\to[0,1]$ be smooth with $\nu_1|_{\mathbb{U}_1}\equiv 1$ and $\mathrm{supp}\nu_1\subset \cup_j U_j$.

Now the function

$$\tau_2\colon [-(k+1),k+1]\times\overline{W_s}\to\mathbb{R},$$
$$\tau_2(t,q) := \nu_1(q)\sigma(t,q) + [1-\nu_1(q)]\tau_1(t,q)$$

is C^l and Lyapunov for e_1.

Claim 3.2. We claim that $\partial_{e_1}\tau_2\equiv -1$ on a neighborhood of

$$M\cup[-1,1]\times\overline{V_s}.$$

Proof of the claim: We have $\partial_{e_1}\sigma\equiv -1$ on a neighborhood of $[-1,k+1]\times\mathrm{supp}\nu_1$ and $\partial_{e_1}\tau_1\equiv -1$ on $[-5/4,k+1]\times\overline{W_s}$. Therefore

$$\partial_{e_1}\tau_2(t,q) = \nu_1(q)\partial_{e_1}\sigma(t,q) + (1-\nu_1(q))\partial_{e_1}\tau_1(t,q)\equiv -1$$

and the claim is obvious on a neighborhood of $[-1,k+1]\times\overline{W_s}$. In particular we obtain $\partial_{e_1}\tau_2\equiv -1$ on a neighborhood of $(M\cap[-1,k+1]\times\overline{W_s})\cup[-1,1]\times\overline{V_s}\subset[-1,k+1]\times\overline{W_s}$.

It only remains to consider the set $M\cap[-(k+1),-1)\times\overline{W_s}$ because $M\subset[-(k+1),k+1]\times\overline{W_s}$.

For $(t,p)\in M$ with $t\in(-5/4,-1)$ there are two cases: If $p\notin\mathrm{supp}\nu_1$ we have $\tau_2\equiv\tau_1$ in a neighborhood of (t,p). It follows that $\partial_{e_1}\tau_2\equiv\partial_{e_1}\tau_1\equiv -1$ in a neighborhood of (t,p). If $p\in\mathrm{supp}\nu_1$ note that $\partial_{e_1}\sigma_{p_j}\equiv -1$ in a neighborhood of (t,p) for all $1\le j\le N$ such that $p\in U_j$. This implies $\partial_{e_1}\sigma\equiv -1$ near (t,p). Since $\partial_{e_1}\tau_1\equiv -1$ in a neighborhood of (t,p) we obtain again $\partial_{e_1}\tau_2\equiv -1$ in a neighborhood of (t,p).

For $(t,p)\in M$ with $t\in[-3/2,-5/4]$ we have $\nu_1\equiv 1$ near p. As in the previous case we have $\partial_{e_1}\sigma\equiv -1$ in a neighborhood of (t,p), i.e. $\partial_{e_1}\tau_2\equiv -1$ near (t,p).

Finally for $(t,p)\in M$ with $t\in[-(k+1),-3/2)$ we again distinguish two cases: First assume $p\notin\mathrm{supp}\nu_1$. Then $\tau_2\equiv\tau_1\equiv\tau$ in a neighborhood of (t,p). Since $(t,p)\in M$ we conclude $\partial_{e_1}\tau_2\equiv -1$ near (t,p). Now assume $p\in\mathrm{supp}\nu_1$. For $1\le i\le N$ such that $\phi_i(p)$ is defined, i.e. $p\in U_{p_i}$, and $t<\phi_i(p)$ we have $\sigma_{p_i}\equiv\tau$ near $(t,p)\in M$, i.e. $\partial_{e_1}\sigma_{p_i}\equiv -1$ near (t,p). For $1\le i\le N$ such that $\phi_i(p)$ is defined and $t\ge\phi_i(p)$ we have $\partial_{e_1}\sigma_{p_i}\equiv -1$ in a neighborhood of (t,p) trivially by construction. Since $\tau_1\equiv\tau$ near (t,p) and $(t,p)\in M$ we also have $\partial_{e_1}\tau_1\equiv -1$ near (t,p). Summing up we conclude $\partial_{e_1}\tau_2\equiv -1$ in a neighborhood of (t,p).

This concludes the proof of the claim.

3rd step: Next we modify τ_2 on $[k,k+1]\times\overline{W_s}$ so that it coincides with τ near $\{k+1\}\times\overline{W_s}$, see Figure 5. We start with estimating $\tau_2(k+1,q)$ from below. Note that by construction

$$\tau_1(k+1,q) = \tau(-1,q) - k - 5/2$$

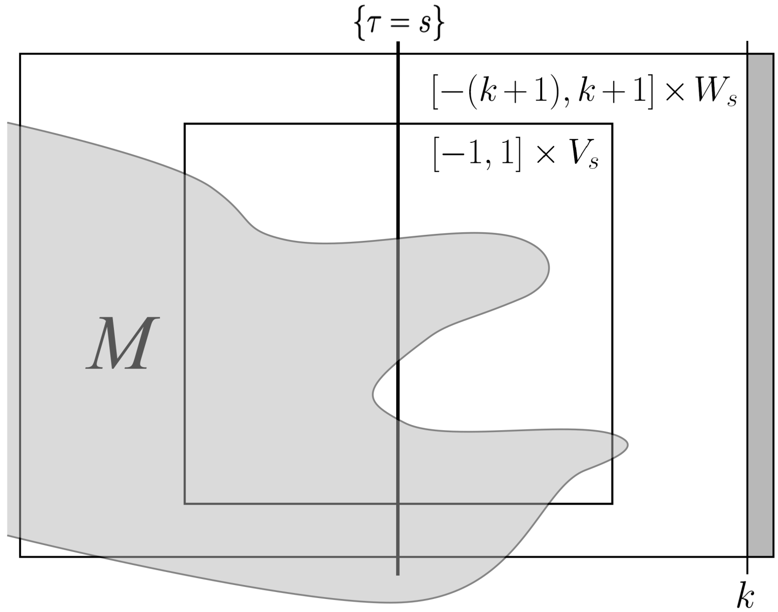

FIGURE 5. The third step

and for $q \in U_j$ we have

$$\begin{aligned}\sigma_{p_j}(k+1,q) &= \tau(\phi_{p_j}(q),q) - (k+1) + \phi_{p_j}(q)\\ &\geq \tau(-1,q) - (k+1) - 7/4,\end{aligned}$$

because $\phi_{p_j}(q) \in [-7/4,-1]$. Combining both, the definition of τ_2 implies

$$\tau_2(k+1,q) \geq \tau(-1,q) - k - 11/4$$

for all $q \in \overline{W_s}$. By (3) we have

$$\tau(k+1,q) \leq \tau(k,q) - k - 3 \leq \tau(-1,q) - k - 3$$

and therefore there exists $\varepsilon \in (0,1/2)$ such that $\tau < \tau_2$ on $[k+1-2\varepsilon, k+1] \times \overline{W_s}$. Choose a smooth monotone function $\mu_+ \colon \mathbb{R} \to [0,1]$ with:

(1) $\mu_+ \equiv 0$ for $t \leq k+1-2\varepsilon$ and
(2) $\mu_+ \equiv 1$ for $t \geq k+1-\varepsilon$

Define $\tau_3 \colon [-(k+1), k+1] \times \overline{W_s} \to \mathbb{R}$ by

$$\tau_3(s,q) := (1-\mu_+(s))\tau_2(s,q) + \mu_+(s)\tau(s,q).$$

As before we see that τ_3 is Lyapunov for e_1, using $\tau < \tau_2$ on the support of the derivative of μ_+. Note that by assumption (3) the sets M and $[k,k+1] \times \overline{W_s}$ are disjoint. Since $\tau_3 \equiv \tau_2$ on $[-(k+1), k+1-2\varepsilon] \times \overline{W_s}$ and $k < k+1-2\varepsilon$ we continue to have $\partial_{e_1}\tau_3 \equiv -1$ on a neighborhood of

$$M \cup [-1,1] \times \overline{V_s}.$$

Moreover, $\tau \equiv \tau_3$ near $\{-(k+1), k+1\} \times \overline{W_s}$.

4th step: It remains to interpolate τ_3 with τ near $[-(k+1), k+1] \times \partial W_s$, see Figure 6. Choose a neighborhood $\mathbb{U}_2$ of $\overline{V_s}$ with closure in W_s and a smooth function

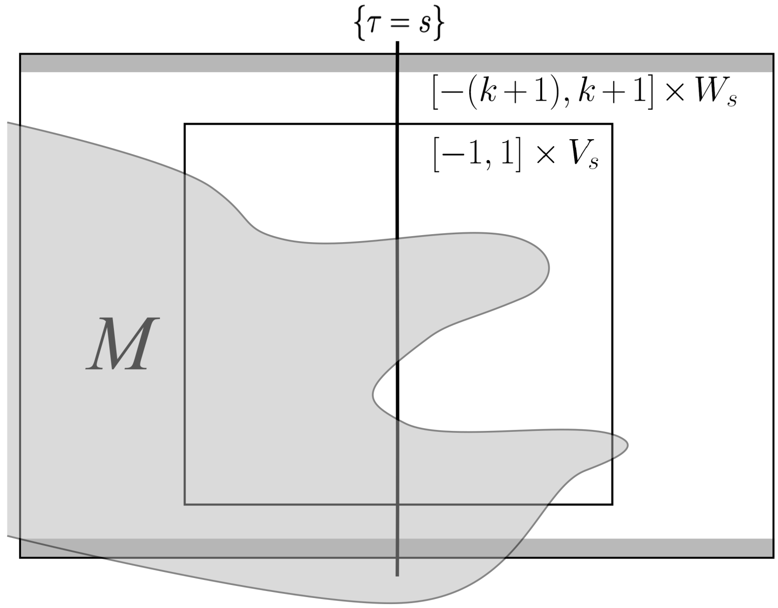

FIGURE 6. The fourth step

$\nu_2\colon \overline{W_s} \to [0,1]$ with $\nu_2 \equiv 1$ on $\mathbb{U}_2$ and $\mathrm{supp}\nu_2 \subset W_s$. Then

$$\tau_4\colon [-(k+1), k+1] \times \overline{W_s} \to \mathbb{R},$$
$$\tau_4(s,q) := \nu_2(q)\tau_3(s,q) + (1-\nu_2(q))\tau(s,q)$$

is a C^l-function which coincides with τ near the boundary of $[-(k+1), k+1] \times \overline{W_s}$ and $\partial_{e_1}\tau_4 \equiv -1$ on a neighborhood of $[-1,1] \times \overline{V_s} \cup M$. Indeed the property holds for τ_3, and for τ outside of $[-1,1] \times \overline{V_s}$. Since $\nu_2|_{[-1,1]\times\overline{V_s}} \equiv 1$ the claim follows immediately. Setting $\tilde{\tau} := \tau_4$ concludes the proof. $\square$

4. **Conclusions.**

We consider a dynamical system, given by the flow of a C^l-vector field $X\colon U \to \mathbb{R}^n$. For any compact subset K of the complement of the chain recurrent set $U \setminus \mathcal{R}_X$ and any C^l-function $g\colon U \to (-\infty, 0)$, we have established the existence of a C^l-regular complete Lyapunov function τ for the system that fulfills $\dot{\tau}(p) = \nabla\tau(p) \cdot X(p) = g(p)$ for every $p \in K$. These results are of essential importance for methods to numerically compute complete Lyapunov functions. Indeed, they present a major leap forward in analyzing and improving several methods that rely on solving PDEs or convex optimization problems containing equality constraints. Note that there exist efficient numerical methods, e.g. based on set-oriented algorithms [7] or [18], which compute an overestimation of the chain recurrent set. By fixing the values of the complete Lyapunov function on the complement of the overestimation, i.e. where the flow in gradient-like, one can compute a complete Lyapunov function and reduce the overestimation of the chain recurrent set. To compute a complete Lyapunov function, one can fix the values outside the chain-recurrent set and use inequality constraints $\dot{\tau}(p) \le 0$ on the (overestimation of the) chain-recurrent set, see [10].

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Received September 2021; revised January 2022; early access February 2022.

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