Implementation of a Fan-Like Triangulation for the CPA Method to compute Lyapunov Functions

Peter Giesl† and Sigurður Hafstein*

***Abstract*— An integral part of the CPA method to compute Continuous and Piecewise Affine Lyapunov functions for nonlinear systems is the generation of a suitable triangulation. Recently, the CPA method was revised by using more advanced triangulations and it was proved that it can compute a CPA Lyapunov function for any nonlinear system possessing an exponentially stable equilibrium. This more advanced triangulation scheme includes a simplicial fan close to the equilibrium of the system, for which the Lyapunov function is computed. In this paper we prove a result that allows for a simpler, more general and more efficient generation of this simplicial fan and thus improves the implementation of the CPA method. Moreover, the simplicial fan subdivision at the origin is also of great importance in methods relying on a conic decomposition of the state-space.**

I. Introduction

The stability of an equilibrium of a dynamical system can be characterized by so-called Lyapunov functions, i.e. functionals defined on the state-space with a minimum at the equilibrium and decreasing along all system trajectories in a neighbourhood of the equilibrium. In particular, a Lyapunov function, through its sublevel sets, provides a lower bound on the basin of attraction of an equilibrium. On the other hand, if a system possesses an equilibrium that is stable in some sense, then there exists a Lyapunov function for the system characterizing this kind of stability. The latter results are referred to as *converse theorems* in the theory of dynamical systems. The converse theorems are, however, nonconstructive in nature since they assume the knowledge of the system trajectories, cf. [18], [19], [14], [23].

In the last decades, numerous algorithms to numerically compute Lyapunov functions have emerged. Some are concerned with special kinds of systems, e.g. piecewise quadratic Lyapunov functions for piecewise linear systems [10], [11] or Lyapunov-like polynomial functions for polynomial systems [22]. Others tackle general nonlinear systems, e.g. by using collocation [9], [3], by graph theoretic methods [13], [2], or by parameterizing polynomial Lyapunov functions using semidefinite optimization [20], [21].

One method that has been developed to compute Lyapunov functions for nonlinear systems uses linear programming to parameterize Continuous and Piecewise Affine (CPA) Lyapunov functions [12], [17], [8]. This method is referred to as the CPA method. In the CPA method, one first triangulates a compact neighbourhood of the equilibrium of the system, i.e. subdivides it into simplices. This neighbourhood serves as the domain of the CPA Lyapunov function. Then several inequalities are stated, which are linear in the values of the CPA Lyapunov function at the vertices of the simplices. If these inequalities are fulfilled, then the unique CPA function having these values at the vertices is a Lyapunov function for the system.

Recently, the CPA method was revised [4], [5], [6], [7] and a former limitation removed. The limitation was that the negativity of the orbital derivative of the computed CPA Lyapunov function could not be guaranteed on an arbitrary small, a posteriori fixed neighbourhood of the equilibrium. It was proved that the revised CPA method is able to compute a CPA Lyapunov function for any nonlinear system possessing an exponentially stable equilibrium and the extension of the domain of the computed Lyapunov function is only bounded by the equilibrium's basin of attraction. This revision was achieved by resorting to a fan-like triangulation close to the equilibrium, that is a natural generalization of the 3D graphics primitive triangle fan to n-dimensions. We refer to this local triangulation as *simplicial fan*.

The definition of this simplicial fan from [7] is recalled in Definition 3. However, it is difficult to implement its construction by a computer efficiently. In this paper we propose a new definition that leads to more general simplicial fan triangulations, which can thus be adjusted to specific problems, e.g. to conic decompositions of the state-space [15]. Moreover, the new definition can be directly and efficiently implemented on a computer. Further, the new definition has advantages when quantifying the degeneracy of the simplices by so-called *shape-matrices*.

The structure of this paper is as follows: In Section 2 we briefly review the CPA method and recall the definition of a suitable triangulation. Then we propose a simpler, more general and more constructive definition of a simplicial fan and discuss its advantages over the former definition. In Section 3 we prove in Theorem 1, the main result of this paper, that the construction leads to a triangulation in the sense of Definition 1. In Section 4 we give conclusions.

Notations: We write vectors $\mathbf{x} \in \mathbb{R}^n$ in boldface. The i-th component of a vector $\mathbf{x}$ is written as x_i or $(\mathbf{x})_i$. We use the norm $\|\mathbf{x}\|_\infty := \max_i |x_i|$. For a set $\mathcal{A} \subset \mathbb{R}^n$ we denote its closure by $\overline{\mathcal{A}}$, its interior by $\mathcal{A}^\circ$, and its boundary by $\partial\mathcal{A} := \overline{\mathcal{A}} \setminus \mathcal{A}^\circ$. We define the convex hull of k vectors

† P. Giesl is with the Department of Mathematics at the University of Sussex, UK p.a.giesl@sussex.ac.uk and S. Hafstein is with the School of Science and Engineering at Reykjavik University, Iceland sigurdurh@ru.is.

$\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$, $k \in \{0, 1, \ldots, n\}$ by

$$\begin{aligned} &\mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k\} \\ &:= \left\{ \sum_{i=0}^{k} \lambda_i \mathbf{x}_i \,:\, \sum_{i=0}^{k} \lambda_i = 1,\, \lambda_i \geq 0 \text{ for all } i \right\}. \end{aligned}$$

Inequalities for vectors are interpreted componentwise, e.g. $\mathbf{x} \leq \mathbf{y}$ means $x_i \leq y_i$ for all i.

We denote by S_n the set of the permutations of $\{1, 2, \ldots, n\}$, by $\chi_{\mathcal{J}}(i)$ the characteristic function equal to one if $i \in \mathcal{J}$ and equal to zero if $i \notin \mathcal{J}$, and by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ the standard orthonormal basis of $\mathbb{R}^n$.

II. CPA SUITED TRIANGULATIONS

In the CPA method to compute CPA Lyapunov functions $V : \mathcal{D} \to \mathbb{R}$ for nonlinear systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ possessing an exponentially stable equilibrium one first triangulates the domain $\mathcal{D} \subset \mathbb{R}^n$, i.e. subdivides $\mathcal{D}$ into n-simplices, on which the CPA Lyapunov function is to be computed. Every simplex $\mathfrak{S}$ of the triangulation can be represented as the convex hull of its vertices $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^n$, i.e.

$$\mathfrak{S} = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}.$$

Since $\mathfrak{S}$ is an n-simplex, its vertices $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ are necessarily affinely independent vectors, i.e. the vectors $\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. This means that every $\mathbf{x} \in \mathfrak{S}$ can be written uniquely as the convex combination of the vertices of $\mathfrak{S}$ and therefore we can define an affine function $f : \mathfrak{S} \to \mathbb{R}$ uniquely by fixing its values at the vertices. In formula: given numbers $f_0, f_1, \ldots, f_n \in \mathbb{R}$ we can define a function $f : \mathfrak{S} \to \mathbb{R}$ through

$$\begin{aligned} f(\mathbf{x}_i) &= f_i \text{ for } i = 0, 1, \ldots, n \text{ and} \\ f\left(\sum_{i=0}^{n} \lambda_i \mathbf{x}_i\right) &:= \sum_{i=0}^{n} \lambda_i f_i \end{aligned}$$

for all convex combinations of the vertices of $\mathfrak{S}$.

To define a continuous function, affine on each of the simplices of the triangulation, a nonempty intersection of two different simplices must be a common face. Such triangulations are often referred to as simplicial complexes or, in the theory of finite element methods, shape-regular triangulations.

Definition 1 (Triangulation): Let $\mathcal{T}$ be a collection of n-simplices $\mathfrak{S}_\nu$ in $\mathbb{R}^n$. $\mathcal{T}$ is called a triangulation of the set $\mathcal{D} := \bigcup_{\mathfrak{S}_\nu \in \mathcal{T}} \mathfrak{S}_\nu$ if for every $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$, $\nu \neq \mu$, either $\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \emptyset$ or $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ intersect in a common face. The latter means, with

$$\begin{aligned} &\mathfrak{S}_\nu = \mathrm{co}\left\{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\right\} \quad \text{and} \\ &\mathfrak{S}_\mu = \mathrm{co}\left\{\mathbf{x}_0^\mu, \mathbf{x}_1^\mu, \ldots, \mathbf{x}_n^\mu\right\}, \end{aligned}$$

that there are permutations α and β of the numbers $0, 1, 2, \ldots, n$ such that

$$\mathbf{z}_i := \mathbf{x}_{\alpha(i)}^\nu = \mathbf{x}_{\beta(i)}^\mu,$$

for $i = 0, 1, \ldots, k$, where $0 \leq k < n$, and

$$\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \mathrm{co}\left\{\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_k\right\}.$$

□

Note that in Definition 1, in contrast to similar definitions elsewhere, the simplices in $\mathcal{T}$ are pairwise distinct, so only counted once.

Definition 2 (CPA function): Let $\mathcal{T}$ be a triangulation of a set $\mathcal{D} \subset \mathbb{R}^n$ in the sense of Definition 1. Then we can define a continuous, piecewise affine function $P : \mathcal{D} \to \mathbb{R}$ by fixing its values at the vertices of the simplices of the triangulation $\mathcal{T}$. More exactly, assume that for every vertex $\mathbf{x}$ of every simplex $\mathfrak{S}_\nu \in \mathcal{T}$ we are given a unique real number $P_\mathbf{x}$. In particular, if $\mathbf{x}$ is a vertex of $\mathfrak{S}_\nu \in \mathcal{T}$ and $\mathbf{y}$ is a vertex of $\mathfrak{S}_\mu \in \mathcal{T}$ and $\mathbf{x} = \mathbf{y}$, then $P_\mathbf{x} = P_\mathbf{y}$. Then we can uniquely define a function $P : \mathcal{D} \to \mathbb{R}$ through:

i) $P(\mathbf{x}) := P_\mathbf{x}$ for every vertex $\mathbf{x}$ of every simplex $\mathfrak{S}_\nu \in \mathcal{T}$.

ii) P is affine on every simplex $\mathfrak{S}_\nu \in \mathcal{T}$.

The set of such continuous, piecewise affine functions $\mathcal{D} \to \mathbb{R}$ fulfilling i) and ii) is denoted by $\mathrm{CPA}[\mathcal{T}]$. □

The CPA method uses linear programming to parameterize CPA Lyapunov functions for nonlinear systems. Further, it always succeeds in computing a CPA Lyapunov $V : \mathcal{D} \to \mathbb{R}$ for a system possessing an exponentially stable equilibrium if $\mathcal{D}$ is in the equilibrium's basin of attraction and the simplices are small and not degenerated. This fact can be used to give an algorithm to compute CPA Lyapunov functions for such systems [7].

Note that we can, without loss of generality, always assume that the equilibrium of the system is at the origin. We can thus place the simplicial fan at the origin in Definition 3. Indeed, for the equilibrium $\mathbf{x}_0$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ consider the transformation $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$, which transforms the system into $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y} + \mathbf{x}_0) =: \mathbf{g}(\mathbf{y})$ with equilibrium at the origin.

In the next remark we introduce some useful notations and then we define the triangulations that serve as a basis for computing CPA Lyapunov functions in [7].

Remark 1: For the construction of our triangulations we use the functions $\mathbf{R}^\mathcal{J} : \mathbb{R}^n \to \mathbb{R}^n$, defined for every $\mathcal{J} \subset \{1, 2, \ldots, n\}$ by

$$\mathbf{R}^\mathcal{J}(\mathbf{x}) := \sum_{i=1}^{n} (-1)^{\chi_\mathcal{J}(i)} x_i \mathbf{e}_i.$$

$\mathbf{R}^\mathcal{J}(\mathbf{x})$ puts a minus in front of the coordinate x_i of $\mathbf{x}$ whenever $i \in \mathcal{J}$.

Further, we denote by $\mathcal{N}$ the set of all subsets $\mathcal{D} \subset \mathbb{R}^n$ such that:

i) $\mathcal{D}$ is compact.

ii) The interior $\mathcal{D}^\circ$ of $\mathcal{D}$ is a connected open neighborhood of the origin.

iii) $\mathcal{D} = \overline{\mathcal{D}^\circ}$.

□

For illustration of Definition 3, cf. Figures 1, 2, and 3.

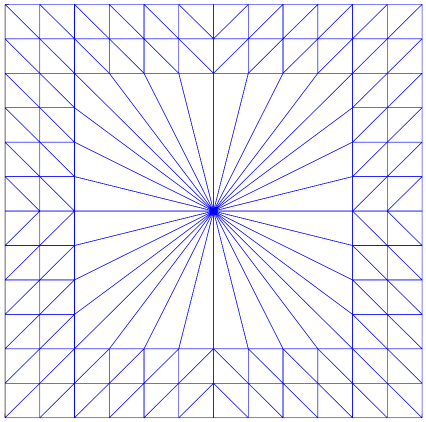

Fig. 1. The triangulation $\mathcal{T}_{2,b}^{[-6b,6b]^2}$ in two dimensions. Note the simplicial fan at the origin.

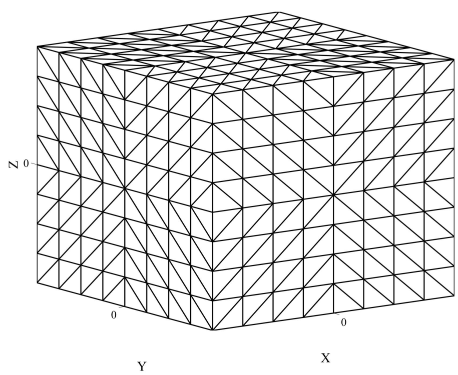

Fig. 2. The simplicial fan $\mathcal{T}_2^{\rm std}$ in three dimensions. By adding the origin as a vertex to all the simplices in the simplicial 2-complex subdividing the boundary of the hypercube we get a fan-like simplicial 3-complex (tetrahedra) locally at the origin.

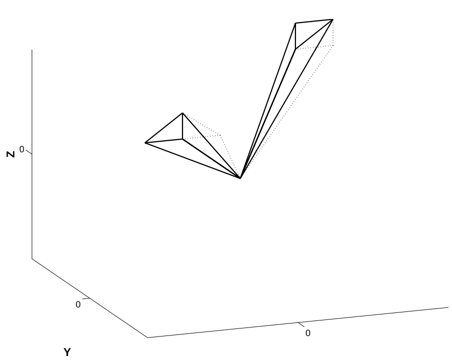

Fig. 3. A few exemplary simplices of the simplicial fan from Figure 2.

Definition 3: Let $\mathcal{C} \in \mathcal{N}$ be a given subset of $\mathbb{R}^n$. We will define a triangulation $\mathcal{T}_{K,b}^{\mathcal{C}}$ of a $\mathcal{D} \in \mathcal{N}$, $\mathcal{D} \supset \mathcal{C}$. To construct the triangulation $\mathcal{T}_{K,b}^{\mathcal{C}}$, we first define the triangulations $\mathcal{T}^{\rm std}$, $\mathcal{T}_K^{\rm std}$, and $\mathcal{T}_{K,b}^{\rm std}$ as intermediate steps.

(A) The standard triangulation $\mathcal{T}^{\rm std}$ consists of the simplices

$$\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \mathrm{co}\left\{\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \dots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right\},$$

where

$$\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma} := \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{i=1}^{j} \mathbf{e}_{\sigma(i)}\right), \tag{1}$$

for all $\mathbf{z} \in \mathbb{N}_0^n$, all $\mathcal{J} \subset \{1, 2, \dots, n\}$, all $\sigma \in S_n$, and $j = 0, 1, \dots, n$.

(B) Choose a $K \in \mathbb{N}_0$, define the hypercube $\mathcal{H}_K := [-2^K, 2^K]^n$, and consider the intersections of the n-simplices $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$ in $\mathcal{T}^{\rm std}$ with the boundary of $\mathcal{H}_K$. We are only interested in those intersections that are $(n-1)$-simplices, i.e. we take every simplex with vertices $\mathbf{x}_j := \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{i=1}^{j} \mathbf{e}_{\sigma(i)}\right)$, $j \in \{0, 1, \dots, n\}$, where exactly one vertex $\mathbf{x}_{j^*}$ satisfies $\|\mathbf{x}_{j^*}\|_\infty \neq 2^K$ and the other n of the $n+1$ vertices satisfy $\|\mathbf{x}_j\|_\infty = 2^K$, i.e. for $j \in \{0, 1, \dots, n\} \setminus \{j^*\}$. Then we replace the vertex $\mathbf{x}_{j^*}$ by $\mathbf{0}$. The collection of such vertices triangulates $\mathcal{H}_K$. We eliminate duplicates and denote this new triangulation of $\mathcal{H}_K$ by $\mathcal{T}_K^{\rm std}$.

(C) Now choose a constant $b > 0$ and scale the triangulation $\mathcal{T}_K^{\rm std}$ of the hypercube $\mathcal{H}_K$ and the triangulation $\mathcal{T}^{\rm std}$ outside of the hypercube $\mathcal{H}_K$ with the mapping $\mathbf{x} \mapsto \rho\mathbf{x}$, where $\rho := 2^{-K}b$. We denote by $\mathcal{T}_{K,b}^{\rm std}$ the resulting set of n-simplices, i.e.

$$\mathcal{T}_{K,b}^{\rm std} = \rho\mathcal{T}_K^{\rm std} \cup \rho\left\{\mathfrak{S} \in \mathcal{T}^{\rm std} : \mathfrak{S} \cap \mathcal{H}_K^\circ = \emptyset\right\}.$$

(D) As a final step define

$$\mathcal{T}_{K,b}^{\mathcal{C}} := \left\{\mathfrak{S}_\nu \in \mathcal{T}_{K,b}^{\rm std} : \mathfrak{S}_\nu \cap \mathcal{C}^\circ \neq \emptyset\right\}$$

and set

$$\mathcal{D} := \bigcup_{\mathfrak{S}_\nu \in \mathcal{T}_{K,b}^{\mathcal{C}}} \mathfrak{S}_\nu.$$

$\square$

Remark 2: The two parameters b and K of the triangulation $\mathcal{T}_{K,b}^{\rm std}$ refer to the size of the hypercube $[-b, b]^n$ covered by its simplicial fan at the origin and to the fineness of the triangulation, respectively. In algorithms to compute CPA Lyapunov functions, values are assigned systematically to K and b to generate increasingly refined triangulations [7]. $\square$

For the implementation of the triangulation $\mathcal{T}_{K,b}^{\mathcal{C}}$ by a computer and to simplify some theoretical consideration it is advantageous to define a simplex as the convex combination of the elements of an ordered tuple rather than a set. This implies that we define

$$\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \mathrm{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \dots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right) \tag{2}$$

in (A) above, where the order of the vertices matters. We can thus refer to $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}$ as the first vertex of the simplex, to $\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}$ as the second vertex of the simplex, etc. The first advantage is that then the so-called shape matrix of an n-simplex is uniquely defined, i.e. the matrix $X_{\nu,y}$ in the proof of Theorem 4.6, part (ii), in [1]. The spectral norm of the shape matrix of a simplex quantifies its degeneracy and is needed for the proof that the CPA method always succeeds in computing a CPA Lyapunov function for a system possessing an exponentially stable equilibrium. This would, e.g., shorten and simplify the argumentation in part (ii) of the proof of the main theorem of [1] considerably.

More importantly, we additionally suggest replacing (B) by:

(B*) Choose a $K \in \mathbb{N}_0$. For every simplex $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} = \mathrm{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right) \in \mathcal{T}^{\mathrm{std}}$, such that $\|\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}\|_\infty = 2^K - 1$ and $\|\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma}\|_\infty = 2^K$ for $j = 1, 2, \ldots, n$ consider the n-simplex $\mathfrak{S}_{\mathbf{0},\mathbf{z}\mathcal{J}\sigma} := \mathrm{co}\left(\mathbf{0}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$. The set of all such simplices $\mathfrak{S}_{\mathbf{0},\mathbf{z}\mathcal{J}\sigma}$ is denoted by $\mathcal{T}_K^{\mathrm{std}}$.

The main advantage of (B*) over (B) is that its implementation is straightforward and much more efficient algorithmically than any implementation of (B). Moreover, we are able to show that the set of simplices $\mathfrak{S}_{\mathbf{0},\mathbf{z}\mathcal{J}\sigma}$ is pairwise distinct, so we do not need to check in an algorithm whether simplices are identical. That is, the removal of duplicates in $\mathcal{T}_K^{\mathrm{std}}$ is unnecessary.

It is clear, that if $\mathcal{T}_K^{\mathrm{std}}$ from (B*) is actually a properly defined triangulation, then it must be the same one as $\mathcal{T}_K^{\mathrm{std}}$ from (B). In the next section we will prove that $\mathcal{T}_K^{\mathrm{std}}$ from (B*) is a triangulation in the sense of Definition 1.

III. Main Results

In this section we prove a slightly more general result than described in the previous section. The triangulation $\mathcal{T}_K^{\mathrm{std}}$ from (B*) is obtained by choosing in Theorem 1 the vectors $\mathbf{K}^m = (-2^K, -2^K, \ldots, -2^K)^T$ and $\mathbf{K}^p = (2^K, 2^K, \ldots, 2^K)^T$, which implies $\mathcal{K} = \mathcal{H}_K = [-2^K, 2^K]^n$, cf. Definition 3.

Theorem 1: Let $\mathbf{K}^m, \mathbf{K}^p \in \mathbb{Z}^n$ be vectors of negative and positive integers respectively, i.e. $\mathbf{K}^m < \mathbf{0} < \mathbf{K}^p$, and define $\mathcal{K} := \{\mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{K}^m \leq \mathbf{x} \leq \mathbf{K}^p\}$.

Consider the simplices $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$ as in (2), but only those such that the first vertex $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}$ is in the interior $\mathcal{K}^\circ$ of $\mathcal{K}$ and the others are at the boundary $\partial\mathcal{K}$ of $\mathcal{K}$. Denote by $\mathcal{T}$ the set of all such simplices, but with the first vertex replaced by $\mathbf{0}$, i.e. $\mathcal{T}$ is the set of all n-simplices $\mathfrak{S} = \mathrm{co}\left(\mathbf{0}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$, such that (cf. formula (1))

$$\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma} \in \mathcal{K}^\circ \text{ and } \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} \in \partial\mathcal{K} \text{ for } 1 \leq i \leq n. \tag{3}$$

Then $\mathcal{T}$ is a triangulation of $\mathcal{K}$ in the sense of Definition 1.

Proof:

We split the proof into four parts. In parts 1 and 2 we show that two different simplices in $\mathcal{T}$ intersect in a common face and in parts 3 and 4 we show that every $\mathbf{x} \in \mathcal{K}$ is contained in a simplex of $\mathcal{T}$.

1. Intersection of simplices

First, we show that the intersection of two different simplices in $\mathcal{T}$ is the convex combination of their common vertices. For this let $\mathfrak{S}_1, \mathfrak{S}_2 \in \mathcal{T}$ be arbitrary. Then there are $\mathbf{z}, \mathbf{z}^* \in \mathbb{N}_0^n$, $\mathcal{J}, \mathcal{J}^* \subset \{1, 2, \ldots, n\}$, and $\sigma, \sigma^* \in S_n$ such that $\mathfrak{S}_1 = \mathrm{co}\left(\mathbf{0}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$ and $\mathfrak{S}_2 = \mathrm{co}\left(\mathbf{0}, \mathbf{x}_1^{\mathbf{z}^*\mathcal{J}^*\sigma^*}, \mathbf{x}_2^{\mathbf{z}^*\mathcal{J}^*\sigma^*}, \ldots, \mathbf{x}_n^{\mathbf{z}^*\mathcal{J}^*\sigma^*}\right)$. Because $\mathcal{T}^{\mathrm{std}}$ is a triangulation, see e.g. Corollary 4.12 in [16], we have

$$\begin{aligned}\mathfrak{S}_1 \cap \mathfrak{S}_2 \cap \partial\mathcal{K} &= \mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} \cap \mathfrak{S}_{\mathbf{z}^*\mathcal{J}^*\sigma^*} \cap \partial\mathcal{K} \\ &= \mathrm{co}(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k),\end{aligned}$$

where $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ are the common vertices of $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$ and $\mathfrak{S}_{\mathbf{z}^*\mathcal{J}^*\sigma^*}$ in $\partial\mathcal{K}$. If $0 \leq k < n$, then we have $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \mathrm{co}(\mathbf{0}, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k)$.

2. Case $k = n$

We consider the more involved case $k = n$. We will show that $\mathbf{z} = \mathbf{z}^*$, $\mathcal{J} = \mathcal{J}^*$, and $\sigma = \sigma^*$, i.e. we only count simplices once, as required in Definition 1.

We first show $\mathcal{J} = \mathcal{J}^*$; indeed, as $\mathbf{z} + \sum_{j=1}^n \mathbf{e}_{\sigma(j)} > \mathbf{0}$ and there is an $l \in \{0, \ldots, n\}$ such that

$$\mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{j=1}^n \mathbf{e}_{\sigma(j)}\right) = \mathbf{R}^{\mathcal{J}^*}\left(\mathbf{z}^* + \sum_{j=1}^l \mathbf{e}_{\sigma^*(j)}\right),$$

and $\mathbf{z}^* + \sum_{j=1}^l \mathbf{e}_{\sigma^*(j)} \geq \mathbf{0}$, we must have $\mathcal{J} = \mathcal{J}^*$.

By (3), we have $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_0^{\mathbf{z}^*\mathcal{J}^*\sigma^*} \notin \partial\mathcal{K}$ and $\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_i^{\mathbf{z}^*\mathcal{J}^*\sigma^*} \in \partial\mathcal{K}$ for all $i = 1, 2, \ldots, n$. Now consider $\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \mathbf{e}_{\sigma(1)}\right) \in \partial\mathcal{K}$; hence, there is an $n^* \in \{1, 2, \ldots, n\}$ such that (i) $(\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma})_{n^*} = (\mathbf{K}^p)_{n^*}$ and $n^* \notin \mathcal{J}$ or (ii) $(\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma})_{n^*} = (\mathbf{K}^m)_{n^*}$ and $n^* \in \mathcal{J}$. We only consider case (i); case (ii) can be dealt with similarly. Since $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma} \notin \partial\mathcal{K}$, we have $(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma})_{n^*} < (\mathbf{K}^p)_{n^*}$; in particular $\sigma(1) = n^*$. By assumption there is an $i^* \in \{1, 2, \ldots, n\}$ such that

$$\begin{aligned}\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma} &= \mathbf{x}_{i^*}^{\mathbf{z}^*\mathcal{J}^*\sigma^*} \\ &= \mathbf{R}^{\mathcal{J}^*}\left(\mathbf{z}^* + \sum_{j=1}^{i^*} \mathbf{e}_{\sigma^*(j)}\right).\end{aligned} \tag{4}$$

This implies $n^* \notin \mathcal{J}^*$, since $(\mathbf{x}_{i^*}^{\mathbf{z}^*\mathcal{J}^*\sigma^*})_{n^*} = (\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma})_{n^*} > 0$.

There are three cases, either (a) $\sigma(1) = \sigma^*(1)$, (b) $\sigma(1) \in \{\sigma^*(2), \sigma^*(3), \ldots, \sigma^*(i^*)\}$ or (c) $\sigma(1) \in \{\sigma^*(i^*+1), \sigma^*(i^*+2), \ldots, \sigma^*(n)\}$. We need to exclude cases (b) and (c).

In case (c), the $\sigma(1) = n^*$-th component of

$$\mathbf{R}^{\mathcal{J}^*}\left(\mathbf{z}^* + \sum_{j=1}^n \mathbf{e}_{\sigma^*(j)}\right)$$

is equal to $(\mathbf{K}^p)_{n^*} + 1$, i.e. the point is not in $\partial\mathcal{K}$ – a contradiction.

In case (b), let $\sigma(1) = \sigma^*(j^*)$ with $2 \leq j^* \leq i^*$, then the $\sigma(1) = n^*$-th component of

$$\mathbf{x}_{j^*-1}^{\mathbf{z}^*\mathcal{J}^*\sigma^*} = \mathbf{R}^{\mathcal{J}^*}\left(\mathbf{z}^* + \sum_{j=1}^{j^*-1} \mathbf{e}_{\sigma^*(j)}\right)$$

is equal to $(\mathbf{K}^p)_{n^*} - 1$. The point is in $\partial\mathcal{K}$ (as it is not $\mathbf{x}_0^{\mathbf{z}^*\mathcal{J}^*\sigma^*}$), hence, there is an $m^* \neq n^*$ such that (i) $\left(\mathbf{x}_{j^*-1}^{\mathbf{z}^*\mathcal{J}^*\sigma^*}\right)_{m^*} = (\mathbf{K}^p)_{m^*}$ with $m^* \notin \mathcal{J}^*$ or (ii) $\left(\mathbf{x}_{j^*-1}^{\mathbf{z}^*\mathcal{J}^*\sigma^*}\right)_{m^*} = (\mathbf{K}^m)_{m^*}$ with $m^* \in \mathcal{J}^*$. Let us restrict ourselves to the first case, the second is dealt with similarly. Then, as $i^* \geq j^*$, we have

$$\left(\mathbf{x}_{i^*}^{\mathbf{z}^*\mathcal{J}^*\sigma^*}\right)_{m^*} \geq \left(\mathbf{x}_{j^*-1}^{\mathbf{z}^*\mathcal{J}^*\sigma^*}\right)_{m^*} = (\mathbf{K}^p)_{m^*} \tag{5}$$

Also, since $n^* \neq m^*$ and $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma} \in \mathcal{K}^\circ$, we have

$$\left(\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}\right)_{m^*} = \left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}\right)_{m^*} < (\mathbf{K}^p)_{m^*}$$

which is in contradiction to (4) and (5).

This leaves case (a) as the only possibility, i.e. $\sigma(1) = \sigma^*(1)$.

Next, we show that $i^* = 1$ in (4). Indeed, assuming that $i^* \geq 2$, there is an $l \in \mathbb{N}$ such that

$$\begin{aligned}\mathbf{z} + \mathbf{e}_{\sigma(1)} - \mathbf{e}_{\sigma^*(i^*)} &= \mathbf{z}^* + \sum_{j=1}^{i^*-1} \mathbf{e}_{\sigma^*(j)} \\ &= \mathbf{z} + \sum_{j=1}^{l} \mathbf{e}_{\sigma(j)}\end{aligned}$$

which implies $-\mathbf{e}_{\sigma^*(i^*)} = \sum_{j=2}^{l} \mathbf{e}_{\sigma(j)}$, which is a contradiction.

Altogether, we have shown that

$$\mathbf{z} + \mathbf{e}_{\sigma(1)} = \mathbf{z}^* + \mathbf{e}_{\sigma^*(1)},$$

which implies $\mathbf{z} = \mathbf{z}^*$. Further, this implies that for every $i \in \{2, 3, \ldots, n\}$ there is an $i^* \in \{2, 3, \ldots, n\}$ such that $\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{x}_{i^*}^{\mathbf{z}^*\mathcal{J}^*\sigma^*}$ and then

$$\sum_{j=2}^{i} \mathbf{e}_{\sigma(j)} = \sum_{j=2}^{i^*} \mathbf{e}_{\sigma^*(j)}$$

follows for $i = 2, 3, \ldots, n$. Clearly, this is only possible if $i^* = i$ and $\sigma^* = \sigma$.

3. *Express boundary points as convex combinations*

We show that for every $\mathbf{x} \in \partial\mathcal{K}$ there is a $\mathbf{z} \in \mathbb{N}_0^n$, $\mathcal{J} \subset \{1, 2, \ldots, n\}$, and $\sigma \in S_n$ such that $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma} \in \mathcal{K}^\circ$, $\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} \in \partial\mathcal{K}$ for $i = 1, 2, \ldots, n$, and $\mathbf{x} \in \operatorname{co}\left(\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$. We do this by explicitly deriving appropriate $\mathbf{z}, \mathcal{J}$, and σ for $\mathbf{x}$.

Define $\mathbf{y} = (|x_1|, |x_2|, \ldots, |x_n|)^T$ and let $\mathcal{J}$ be such that $\mathbf{R}^{\mathcal{J}}(\mathbf{x}) = \mathbf{y}$, and then also $\mathbf{R}^{\mathcal{J}}(\mathbf{y}) = \mathbf{x}$. Since $\mathbf{x} \in \partial\mathcal{K}$, there is an $n^* \in \{1, 2, \ldots, n\}$ such that (i) $x_{n^*} = (\mathbf{K}^p)_{n^*}$ with $n^* \notin \mathcal{J}$ or (ii) $x_{n^*} = (\mathbf{K}^m)_{n^*}$ with $n^* \in \mathcal{J}$.

Define $\mathbf{z} = (z_1, z_2, \ldots, z_n)^T \in \mathbb{N}_0^n$ by

$$\begin{cases} z_i := 0, & \text{if } y_i = 0, \\ y_i - 1 \leq z_i < y_i, & \text{if } y_i > 0. \end{cases}$$

for all $i \in \{1, 2, \ldots, n\}$. In particular $z_{n^*} := y_{n^*} - 1 = |x_{n^*}| - 1$ and $\mathbf{K}^m < \mathbf{R}^{\mathcal{J}}(\mathbf{z}) < \mathbf{K}^p$, i.e. $\mathbf{R}^{\mathcal{J}}(\mathbf{z}) \in \mathcal{K}^\circ$, by the construction of $\mathbf{z}$ and because $\mathbf{K}^m < \mathbf{0} < \mathbf{K}^p$. Finally, set $\mathbf{w} := \mathbf{y} - \mathbf{z}$. Then $0 \leq w_i \leq 1$ for all $i = 1, 2, \ldots, n$. Let $\sigma \in S_n$ such that $\sigma(1) = n^*$ and

$$1 = w_{\sigma(1)} \geq w_{\sigma(2)} \geq \ldots \geq w_{\sigma(n)} \geq 0.$$

We define $\mathbf{x}_k^{\mathbf{z}\mathcal{J}\sigma}$, $k = 0, \ldots, n$, as in (1). We have $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{R}^{\mathcal{J}}(\mathbf{z}) \in \mathcal{K}^\circ$ as shown earlier and $\mathbf{x}_k^{\mathbf{z}\mathcal{J}\sigma} \in \partial\mathcal{K}$ for $k \geq 1$, since $\left(\mathbf{x}_k^{\mathbf{z}\mathcal{J}\sigma}\right)_{n^*} = x_{n^*}$.

To show that $\mathbf{x} \in \operatorname{co}\left(\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$ we define

$$\lambda_k = w_{\sigma(k)} - w_{\sigma(k+1)} \geq 0$$

for $k = 1, 2, \ldots, n-1$ and

$$\lambda_n = w_{\sigma(n)} \geq 0.$$

We have $\sum_{i=1}^n \lambda_i = w_{\sigma(1)} = 1$ and $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}$. Indeed, we show that the k-th component of $\sum_{i=1}^n \lambda_i \left(\mathbf{z} + \sum_{j=1}^i \mathbf{e}_{\sigma(j)}\right)$ is y_k, which shows the statement by applying $\mathbf{R}^{\mathcal{J}}$ on both sides.

$$\begin{aligned}&\left(\sum_{i=1}^n \lambda_i \left(\mathbf{z} + \sum_{j=1}^i \mathbf{e}_{\sigma(j)}\right)\right)_k \\ &= z_k + \left(\sum_{i=1}^n \lambda_i \sum_{j=1}^i \mathbf{e}_{\sigma(j)}\right)_k \\ &= z_k + \left(\sum_{j=1}^n \sum_{i=j}^n \lambda_i \mathbf{e}_{\sigma(j)}\right)_k \\ &= z_k + \sum_{i=\sigma^{-1}(k)}^n \lambda_i \\ &= z_k + w_{\sigma(\sigma^{-1}(k))} \\ &= z_k + w_k \\ &= y_k\end{aligned}$$

where we have used $\sum_{i=1}^n \lambda_i = 1$. This shows the statement.

4. *Express any point as convex combination*

We show that for every $\mathbf{x} \in \mathcal{K}$ there is a simplex $\mathfrak{S} \in \mathcal{T}$ such that $\mathbf{x} \in \mathfrak{S}$. If $\mathbf{x} = \mathbf{0}$, then this is obvious. If $\mathbf{x} \neq \mathbf{0}$ there is a $\gamma \geq 1$ such that $\gamma\mathbf{x} \in \partial\mathcal{K}$. Above we showed that there is a simplex $\mathcal{T} = \operatorname{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$ such that $\gamma\mathbf{x}$ can be written as a convex combination,

$$\gamma\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma},$$

from which

$$\mathbf{x} = (1 - \gamma^{-1})\mathbf{0} + \sum_{i=1}^n (\lambda_i\gamma^{-1})\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} = \sum_{i=0}^n \lambda_i^* \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma},$$

with $\lambda_0^* = 1 - \gamma^{-1}$ and $\lambda_i^* = \lambda_i\gamma^{-1}$ for $i = 1, 2, \ldots, n$, follows. $\square$

Remark 3: In Theorem 1 we have considered simplices as in (2) with one vertex in $\mathcal{K}^\circ$ and all other vertices in $\partial\mathcal{K}$, and we specifically assumed that the vertex inside $\mathcal{K}$ is $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}$.

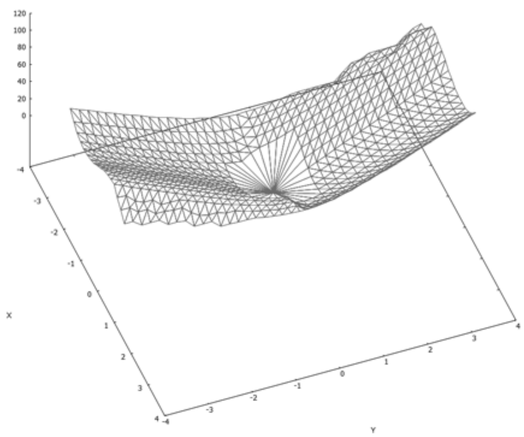

Fig. 4. A computed CPA Lyapunov function for the time-reversed van der Pol oscillator. Note the simplicial fan at the origin.

This assumption is no loss of generality, since if a simplex $\mathfrak{S} := \mathrm{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right) \in \mathcal{T}^{\mathrm{std}}$ has one vertex in $\mathcal{K}^\circ$ and all other vertices in $\partial\mathcal{K}$, then the vertex inside $\mathcal{K}$ is necessarily $\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}$. To see this observe the following :

Let $\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{j=1}^{i} \mathbf{e}_{\sigma(j)}\right) \notin \partial\mathcal{K}$ be the vertex of $\mathfrak{S}$ not lying on the boundary. We want to show that $i = 0$. If $i \neq 0$, then $\mathbf{x}_{i-1}^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} - \mathbf{R}^{\mathcal{J}}\left(\mathbf{e}_{\sigma(i)}\right) \in \partial\mathcal{K}$, so there is an $n^* \in \{1, 2, \ldots, n\}$ such that (i) $\left(\mathbf{x}_{i-1}^{\mathbf{z}\mathcal{J}\sigma}\right)_{n^*} = (\mathbf{K}^p)_{n^*}$ with $n^* \notin \mathcal{J}$ or (ii) $\left(\mathbf{x}_{i-1}^{\mathbf{z}\mathcal{J}\sigma}\right)_{n^*} = (\mathbf{K}^m)_{n^*}$ with $n^* \in \mathcal{J}$. Let us consider the first case, the second case is dealt with similarly. Since $\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} \notin \partial\mathcal{K}$, we have $\left(\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}\right)_{n^*} < (\mathbf{K}^p)_{n^*}$, i.e.

$$(\mathbf{K}^p)_{n^*} = \left(\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}\right)_{n^*} - 1 < (\mathbf{K}^p)_{n^*} - 1$$

if $\sigma(i) = n^*$ or

$$(\mathbf{K}^p)_{n^*} = \left(\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}\right)_{n^*} < (\mathbf{K}^p)_{n^*}$$

if $\sigma(i) \neq n^*$. In both cases we obtain a contradiction.

IV. Conclusions

We gave an improved definition of a suitable triangulation of the domain of a CPA Lyapunov function. We have proven that this new definition leads to a triangulation and, as a special case, we can recover the triangulation of [7]. The simplicial fan of the triangulation can be implemented much more efficiently by a computer using the new definition. This has been incorporated in a software that implements the CPA method to compute Lyapunov functions for nonlinear systems and will be distributed online free of charge shortly. An example of a computed CPA Lyapunov function for a nonlinear system using a triangular fan at the origin computed by this software is given in Figure 4. Further advantages are that it is more general and can thus be adjusted to the particular system, and that some theoretical considerations of the CPA method, such as the shape matrix, are made simpler. Finally, the simplicial fan triangulation can be used for conic decompositions of the state-space and is thus of use for other methods [15] as well.

References

[1] R. Baier, L. Grüne and S. Hafstein, Linear programming based Lyapunov function computation for differential inclusions, Discrete Contin. Dyn. Syst. Ser. B, **17** (2012), 33–56.

[2] H. Ban and W. Kalies, A computational approach to Conley's decomposition theorem, J. Comput. Nonlinear Dynam., **1-4** (2006), 312–319.

[3] P. Giesl, "Construction of Global Lyapunov Functions Using Radial Basis Functions", Lecture Notes in Mathematics **1904**, Springer, 2007.

[4] P. Giesl and S. Hafstein, Existence of piecewise affine Lyapunov functions in two dimensions. J. Math. Anal. Appl., **371** (2010), 233–248.

[5] P. Giesl and S. Hafstein, Construction of Lyapunov functions for nonlinear planar systems by linear programming, J. Math. Anal. Appl., **388** (2012), 463–479.

[6] P. Giesl and S. Hafstein, Existence of piecewise linear Lyapunov functions in arbitary dimensions, Discrete Contin. Dyn. Syst., **32** (2012), 3539–3565.

[7] P. Giesl and S. Hafstein, Revised CPA method to compute Lyapunov function for nonlinear systems, J. Math. Anal. Appl., **410**, (2014), 292–306.

[8] S. Hafstein, "An algorithm for constructing Lyapunov functions", Electron. J. Differential Equ. Monogr., **8** (2007).

[9] T. Johansen, Computation of Lyapunov Functions for Smooth Nonlinear Systems using Convex Optimization, Automatica, **36** (2000), 1617–1626.

[10] M. Johansson and A. Rantzer. On the computation of piecewise quadratic Lyapunov functions. In *Proceedings of the 36th IEEE Conference on Decision and Control*, 1997.

[11] M. Johansson. *Piecewise Linear Control Systems*. Ph.D. Thesis: Lund Institute of Technology, Lund, Sweden, 1999.

[12] P. Julian, J. Guivant and A. Desages, A parametrization of piecewise linear Lyapunov function via linear programming, Int. Journal of Control, **72** (1999), 702–715.

[13] W. Kalies, K. Mischaikow and R. VanderVorst, An algorithmic approach to chain recurrence, Found. Comput. Math., **5-4** (2005), 409–449.

[14] J. Kurzweil, On the inversion of Ljapunov's second theorem on stability of motion. Chechoslovakian Mathematics Journal, **81** (1956), 217—259, 455—484, 1956. English translation in American Mathematical Society Translations (2), v. 24, pp. 19–77.

[15] M. Lazar and A. Jokic, On Infinity Norms as Lyapunov Functions for Piecewise Affine Systems. In: Proceedings of the 13th ACM International Conference on Hybrid Systems: Computation and Control (2010), 131–140.

[16] S. Marinosson, "Stability analysis of nonlinear systems with linear programming: A Lyapunov functions based approach", Ph.D. Thesis: Gerhard-Mercator-University, Duisburg, Germany, 2002.

[17] S. Marinosson, Lyapunov function construction for ordinary differential equations with linear programming. Dynamical Systems, **17** (2002), 137–150.

[18] J. Massera. On Liapunoff's conditions of stability. *Annals of Mathematics*, **50**:705–721, 1949.

[19] J. Massera. Contributions to stability theory. *Annals of Mathematics*, **64-1**:182–206, 1956.

[20] A. Papachristodoulou and S. Prajna, The construction of Lyapunov functions using the sum of squares decomposition. In: Proceedings of the 41st IEEE Conference on Decision and Control (2002), 3482–3487.

[21] M. Peet and A. Papachristodoulou, A converse sum-of-squares Lyapunov result: An existence proof based on the Picard iteration. In: Proceedings of the 49th IEEE Conference on Decision and Control (2010), 5949–5954.

[22] S. Ratschan and Z. She Providing a Basin of Attraction to a Target Region of Polynomial Systems by Computation of Lyapunov-like Functions, SIAM J. Control and Optimization **48-7** (2010), 4377–4394.

[23] T. Yoshizawa. Stability Theory by Liapunov's Second Method. Mathematical Society of Japan, 1966.