

Computation and Verification of Lyapunov Functions*

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Abstract. Lyapunov functions are an important tool to determine the basin of attraction of equilibria in Dynamical Systems through their sublevel sets. Recently, several numerical construction methods for Lyapunov functions have been proposed, among them the RBF (Radial Basis Function) and CPA (Continuous Piecewise Affine) methods. While the first method lacks a verification that the constructed function is a valid Lyapunov function, the second method is rigorous, but computationally much more demanding. In this paper, we propose a combination of these two methods, using their respective strengths: we use the RBF method to compute a potential Lyapunov function. Then we interpolate this function by a continuous piecewise affine (CPA) function. Checking a finite number of inequalities, we are able to verify that this interpolation is a Lyapunov function. Moreover, sublevel sets are arbitrarily close to the basin of attraction. We show that this combined method always succeeds in computing and verifying a Lyapunov function, as well as in determining arbitrary compact subsets of the basin of attraction. The method is applied to two examples.

Key words. Lyapunov function, basin of attraction, mesh-free collocation, radial basis function, continuous piecewise affine interpolation, computation, verification

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1. Introduction. Given an autonomous ordinary differential equation (ODE), we are interested in the determination of the basin of attraction of an exponentially stable equilibrium.

The basin of attraction can be computed using a variety of methods: Invariant manifolds form the boundaries of basins of attraction, and their computation can thus be used to find a basin of attraction [24, 23]. The cell mapping approach [20] or set oriented methods [4] divide the phase space into cells and compute the dynamics between these cells; they have also been used to construct Lyapunov functions [16, 22, 14].

Lyapunov functions [26] are a natural way of analyzing the basin of attraction. A Lyapunov function is a function which is decreasing along solutions of the ODE; sublevel sets of the Lyapunov function are subsets of the basin of attraction. The explicit construction of Lyapunov functions for a given system, however, is a difficult problem.

In the last decades, several numerical methods to construct Lyapunov functions have been developed; for a review, see [12]. These methods include the SOS (sums of squares) method, which is usually applied to polynomial vector fields, introduced in [29] and available as a MATLAB tool box [28]. It can also be applied to more general systems as shown in [30, 31] and it constructs a polynomial Lyapunov function by semidefinite optimization.

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A different method deals with Zubov’s equation and computes a solution of this partial differential equation (PDE) [3]; the corresponding generalized Zubov equation is a Hamilton–Jacobi–Bellmann equation. This equation has a viscosity solution which can be approximated using standard techniques after regularization at the equilibrium; for example, one can use piecewise affine approximating functions and adaptive grid techniques [15].

The Continuous Piecewise Affine (CPA) method constructs a CPA (continuous piecewise affine)¹ Lyapunov function using linear optimization [17, 18, 11]. A simplicial complex is fixed and the space of CPA functions which are affine on each simplex is considered. This space can be parameterized by the values at the vertices. The conditions of a Lyapunov function are transformed into a set of finitely many linear inequalities at the vertices, which include error estimates ensuring that the CPA Lyapunov function has negative orbital derivative inside each simplex. These linear inequalities are used as constraints of a linear programming problem, which can be solved by standard methods.

The RBF (Radial Basis Function) method, a special case of mesh-free collocation, considers a particular Lyapunov function, satisfying a linear PDE, and solves it using mesh-free collocation [7, 13]. For this method, a set of scattered collocation points is used to find an approximation to the solution of the linear PDE. It is computed by solving a linear system of equations. Error estimates show that the method always constructs a (smooth) Lyapunov function if the collocation points are dense enough and placed in the appropriate area. So far, however, the method lacks a verification of whether the points are dense enough, i.e., if the computed function fulfills the properties of a Lyapunov function everywhere.

In this paper, we propose a combination of the RBF and CPA methods: we first compute an approximation V_R to a Lyapunov function using the RBF method. Then we interpolate this function by a continuous piecewise affine (CPA) function V_C , affine on each simplex of a triangulation. To show that V_C is a Lyapunov function, we verify a finite number of inequalities at the vertices of the simplices, ensuring that the function is a Lyapunov function in the whole set.

We will show that these inequalities can always be fulfilled if both the collocation points for the RBF approximation are sufficiently dense and the simplices of the triangulation are sufficiently small. Furthermore, we also show that any given compact subset \mathcal{C} of the basin of attraction can be covered by a sublevel set of V_C . Hence, the method can rigorously establish that \mathcal{C} is a subset of the basin of attraction. The new method requires a much shorter computation time compared to the original CPA method. Examples show that the method works well in practice.

Let us give an overview of this paper: After introducing some notation, we discuss the main facts about Lyapunov functions in section 2. In particular, since our method constructs a function, which is not a Lyapunov function in the classical sense, we introduce the weaker notion of a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$ (cf. Definition 2.5) also known as practical stability in the literature. Such a function is not required to be decreasing along solutions on a small neighborhood \mathcal{F} of the equilibrium; moreover, as the function is only assumed to be Lipschitz,

¹In numerical analysis such functions are often referred to as piecewise linear or linear splines. As there are methods that compute Lyapunov functions that are truly linear in cones with the origin as apex, we refrain from this terminology to avoid confusion.

the decreasing property along solutions is expressed using the Dini orbital derivative. The section additionally introduces sublevel sets of a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$, and recalls the existence of a classical Lyapunov function V satisfying a linear PDE.

In section 3, we discuss mesh-free collocation, in particular using RBFs, to solve a generalized interpolation problem. In particular, this is applied to approximately solve the linear PDE for the Lyapunov function from section 2. Note that an approximation V_R to V , if close enough, is a valid Lyapunov function, i.e., decreases along solutions. Error estimates on the difference between the approximation V_R and the true solution V , both of the functions and their orbital derivatives, are established.

Section 4 deals with the interpolation and verification. After recalling triangulations of a set, CPA functions are introduced, which are characterized by their values at the vertices of the triangulation. The CPA interpolation W_C of a C^2 function W is defined, and error estimates between W and W_C as well as their derivatives are established. Moreover, it is shown that if the CPA function W_C satisfies finitely many inequalities at the vertices of the triangulation, W_C has a negative Dini orbital derivative in the whole set.

In section 5, the RBF approximation and CPA interpolation are finally combined and the main theorem, Theorem 5.1, is proved. Given a (small) compact neighborhood \mathcal{F} of the equilibrium and a (large) compact subset \mathcal{C} of its basin of attraction, the RBF approximation V_R of V and, subsequently, its CPA interpolation V_C are constructed. If both the collocation points of the RBF approximation are dense enough and the CPA triangulation is fine enough and in the appropriate area, then V_C satisfies finitely many inequalities at the vertices, verifying that it is a Lyapunov function. Moreover, we show through the sublevel sets of V_C that solutions starting in \mathcal{C} enter \mathcal{S} in finite time.

Finally, we apply the method to a two- and a three-dimensional example in section 6, and we end the paper with conclusions in section 7.

Remark 1.1. To keep the proof of the main theorem of this paper, Theorem 5.1, at a reasonable length, we prove several new results in the sections before, namely Lemma 2.4, Theorem 2.8, parts of Lemma 3.8, and Corollary 4.12. Other results are essentially known, but are stated in a different form that can be immediately applied in the proof of Theorem 5.1. These are Theorem 3.9, Corollary 4.11, Lemma 4.15, and Theorem 4.16.

1.1. Notation. We denote the natural numbers by $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the positive real numbers by $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$, and the nonnegative real numbers by $\mathbb{R}_0^+ = \{r \in \mathbb{R} : r \geq 0\}$. For a vector $\mathbf{x} \in \mathbb{R}^n$ and $p \geq 1$, we define the norm $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. We also define $\|\mathbf{x}\|_\infty = \max_{i \in \{1, \dots, n\}} |x_i|$. The induced matrix norm $\|\cdot\|_p$ is defined by $\|A\|_p := \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p$. Clearly, $\|A\mathbf{x}\|_p \leq \|A\|_p \|\mathbf{x}\|_p$.

Let \mathcal{B} be a bounded open subset of \mathbb{R}^n . The set of m -times continuously differentiable functions $u: \mathcal{B} \rightarrow \mathbb{R}$, such that the derivatives $D^\alpha u$, $\alpha \in \mathbb{N}_0^n$ a multi-index with $|\alpha| := \sum_{i=1}^n \alpha_i \leq k$, can be continuously extended to $\bar{\mathcal{B}}$ is denoted by $C^m(\bar{\mathcal{B}})$ and we define the norm

$$\|u\|_{C^k(\bar{\mathcal{B}})} = \sum_{|\alpha| \leq k} \sup_{\mathbf{x} \in \bar{\mathcal{B}}} |D^\alpha u(\mathbf{x})|.$$

The space of Hölder continuous functions $u: \bar{\mathcal{B}} \rightarrow \mathbb{R}$ with exponent $0 < \gamma \leq 1$ is denoted

by $C^{k,\gamma}(\overline{\mathcal{B}})$ and the Hölder norm is defined by

$$\|u\|_{C^{k,\gamma}(\overline{\mathcal{B}})} = \sum_{|\alpha| \leq k} \sup_{\mathbf{x} \in \overline{\mathcal{B}}} |D^\alpha u(\mathbf{x})| + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{\mathcal{B}})},$$

where

$$[u]_{C^{0,\gamma}(\overline{\mathcal{B}})} = \sup_{\mathbf{x}, \mathbf{y} \in \overline{\mathcal{B}}, \mathbf{x} \neq \mathbf{y}} \frac{|u(\mathbf{x}) - u(\mathbf{y})|}{\|\mathbf{x} - \mathbf{y}\|_2^\gamma}.$$

For $k \in \mathbb{N}_0$, the Sobolev space $W_2^k(\mathcal{B})$ consists, as usual, of all u with weak derivatives $D^\alpha u \in L_2(\mathcal{B})$, $|\alpha| \leq k$. Fractional order Sobolev spaces $W_2^\tau(\mathcal{B})$, where $\tau \in \mathbb{R}^+$, can be defined, e.g., by interpolation theory.

We denote the closure of a set $\mathcal{M} \subset \mathbb{R}^n$ by $\overline{\mathcal{M}}$, the interior of \mathcal{M} by \mathcal{M}° , and the boundary of \mathcal{M} by $\partial\mathcal{M}$. Moreover, the complement of a set $\mathcal{M} \subset \mathbb{R}^n$ is denoted by $\mathcal{M}^C = \mathbb{R}^n \setminus \mathcal{M}$. Finally, $\mathcal{B}(\delta)$ is defined as the open $\|\cdot\|_2$ -ball with center $\mathbf{0}$ and radius δ , i.e., $\mathcal{B}(\delta) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < \delta\}$. For a compact set $\mathcal{M} \subset \mathbb{R}^n$ and $\varepsilon > 0$ we define the set $\mathcal{M}_\varepsilon := \{\mathbf{x} \in \mathbb{R}^n : \text{dist}(\mathbf{x}, \mathcal{M}) < \varepsilon\}$, where $\text{dist}(\mathbf{x}, \mathcal{M}) := \min_{\mathbf{y} \in \mathcal{M}} \|\mathbf{x} - \mathbf{y}\|_2$.

2. Lyapunov functions. In this paper we consider a general autonomous Ordinary Differential Equation as described in the following definition.

Definition 2.1. *Let*

$$(2.1) \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{f} \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$, $n, \sigma \in \mathbb{N}$. We assume that the origin is an exponentially stable equilibrium of the system and we denote the solution of the system with initial value $\boldsymbol{\xi}$ at time zero by $t \mapsto S_t \boldsymbol{\xi}$. We denote the origin's basin of attraction by $\mathcal{A} := \{\boldsymbol{\xi} \in \mathbb{R}^n : \lim_{t \rightarrow \infty} S_t \boldsymbol{\xi} = \mathbf{0}\}$.

Clearly, $S_t \boldsymbol{\xi}$ is properly defined for all $t \geq 0$ whenever $\boldsymbol{\xi} \in \mathcal{A}$, and thus defines a Dynamical System. The origin is an exponentially stable equilibrium of the system (2.1), if and only if $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and the real parts of all eigenvalues of $D\mathbf{f}(\mathbf{0})$ are strictly negative.

A smooth (strict) Lyapunov function for the equilibrium is a continuously differentiable function $V : \mathcal{M}^\circ \rightarrow \mathbb{R}$, where \mathcal{M} is a neighborhood of the origin, fulfilling $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ as well as

$$(2.2) \quad V'(\mathbf{x}) := \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) < 0$$

for all $\mathbf{x} \in \mathcal{M}^\circ \setminus \{\mathbf{0}\}$. Here $V'(\mathbf{x})$ denotes the orbital derivative, the derivative along solutions of (2.1).

We will relax this definition in two directions, since the construction algorithm will in general only produce such a relaxed Lyapunov function. First, it is well known that the condition “continuously differentiable” can be relaxed to “locally Lipschitz continuous” if the orbital derivative in (2.2) is replaced by the so-called Dini derivative. Second, we will only assume the condition (2.2) on $\mathcal{M}^\circ \setminus \mathcal{F}$, where $\mathcal{F} \subset \mathcal{M}^\circ$ is a neighborhood of the origin.

For $\mathcal{M} \subset \mathbb{R}^n$, the (upper right) orbital Dini derivative of a locally Lipschitz function $V : \mathcal{M} \rightarrow \mathbb{R}$ along the solution trajectories of (2.1) is defined at every $\mathbf{x} \in \mathcal{M}^\circ$ by

$$(2.3) \quad D^+V(\mathbf{x}) := \limsup_{h \rightarrow 0^+} \frac{V(\mathbf{x} + h\mathbf{f}(\mathbf{x})) - V(\mathbf{x})}{h}.$$

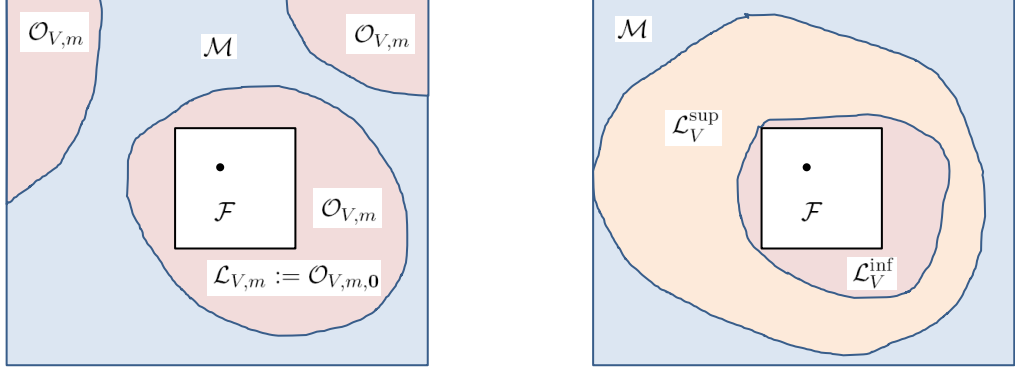


Figure 1. Schematic figures of the sets defined in Definition 2.3. Left: $\mathcal{F} \subset \mathcal{M}^\circ$ contains the origin (black dot) as an interior point. The set $\mathcal{O}_{V,m}$ is the union of $\{\mathbf{x} \in \mathcal{M} \setminus \mathcal{F}^\circ : V(\mathbf{x}) < m\}$ and \mathcal{F} . In the figure it consists of three connected components. The connected component of $\mathcal{O}_{V,m}$ containing the origin is denoted $\mathcal{O}_{V,m,0}$. If $\mathcal{F} \subset \mathcal{O}_{V,m,0}^\circ \subset \overline{\mathcal{O}_{V,m,0}} \subset \mathcal{M}^\circ$, as in the figure, we define $\mathcal{L}_{V,m} := \mathcal{O}_{V,m,0}$. Otherwise, $\mathcal{L}_{V,m} := \emptyset$. Right: $\mathcal{L}_V^{\text{inf}}$ is the intersection of all $\mathcal{L}_{V,m} \neq \emptyset$ and $\mathcal{L}_V^{\text{sup}}$ is the union of all $\mathcal{L}_{V,m}$. For properties of the sets $\mathcal{L}_V^{\text{inf}}$ and $\mathcal{L}_V^{\text{sup}}$ when V is a Lyapunov function in the sense of Definition 2.5, see Theorem 2.6.

Since V is locally Lipschitz, it is differentiable almost everywhere and, where V is differentiable, we have $D^+V(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$. If V is continuously differentiable, then the orbital Dini derivative coincides with the usual orbital derivative everywhere.

Before we give a precise definition of the type of Lyapunov functions which we will use in this paper, a few definitions are useful.

We define the set \mathfrak{N}^n of certain neighborhoods of the origin in \mathbb{R}^n that we will repeatedly use in this paper.

Definition 2.2. Denote by \mathfrak{N}^n the set of all subsets $\emptyset \neq \mathcal{M} \subset \mathbb{R}^n$ that fulfill the following:

- (i) \mathcal{M} is compact.
- (ii) The interior \mathcal{M}° of \mathcal{M} is a simply connected open neighborhood of the origin.
- (iii) $\mathcal{M} = \overline{\mathcal{M}^\circ}$.

For a function V with domain $\mathcal{M} \setminus \mathcal{F}^\circ$, where $\mathcal{F}, \mathcal{M} \in \mathfrak{N}^n$, we define $\mathcal{L}_{V,m}$, which denotes the connected component of the sublevel set with level m that contains the origin. Figure 1 illustrates the sets in the following definition, Definition 2.3.

Definition 2.3. Let $\mathcal{F}, \mathcal{M} \in \mathfrak{N}^n$, $\mathcal{F} \subset \mathcal{M}^\circ$. Let $V : \mathcal{M} \setminus \mathcal{F}^\circ \rightarrow \mathbb{R}$ be a continuous function and $m \in \mathbb{R}$ be a constant. Define the set

$$\mathcal{O}_{V,m} := \mathcal{F} \cup \{\mathbf{x} \in \mathcal{M} \setminus \mathcal{F}^\circ : V(\mathbf{x}) < m\} \subset \mathcal{M}.$$

Denote by $\mathcal{O}_{V,m,0}$ the connected component of $\mathcal{O}_{V,m}$ satisfying $\mathbf{0} \in \mathcal{O}_{V,m,0} \subset \mathcal{O}_{V,m}$. If $\mathcal{F} \subset \mathcal{O}_{V,m,0}^\circ \subset \overline{\mathcal{O}_{V,m,0}} \subset \mathcal{M}^\circ$, then we define the sublevel set $\mathcal{L}_{V,m} := \mathcal{O}_{V,m,0}$; note that in this case $\mathcal{L}_{V,m}$ is open. If no such $\mathcal{O}_{V,m,0}$ exists, then we define $\mathcal{L}_{V,m} := \emptyset$. We further define

$$\mathcal{L}_V^{\text{inf}} := \bigcap_{\substack{m \in \mathbb{R} \\ \mathcal{L}_{V,m} \neq \emptyset}} \mathcal{L}_{V,m} \quad \text{and} \quad \mathcal{L}_V^{\text{sup}} := \bigcup_{m \in \mathbb{R}} \mathcal{L}_{V,m}.$$

Observe that it is possible that $\mathcal{L}_V^{\text{inf}}$ and $\mathcal{L}_V^{\text{sup}}$ are empty if $\mathcal{O}_{V,m,\mathbf{0}} = \emptyset$ for all $m \in \mathbb{R}$. However, when nonempty, $\mathcal{L}_V^{\text{inf}}$ is a closed set, $\mathcal{L}_V^{\text{sup}}$ is an open set, and $\mathcal{L}_V^{\text{inf}} \subset \mathcal{L}_V^{\text{sup}}$ (see [2, Theorem 2.5]), which also holds for general $m \in \mathbb{R}$. Before we define a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$, we prove a lemma, relating the sublevel sets of two functions V and W that are close together.

Lemma 2.4. *Let $\mathcal{F}, \mathcal{M} \in \mathfrak{N}^n$, $\mathcal{F} \subset \mathcal{M}^\circ$. Let $V, W : \mathcal{M} \setminus \mathcal{F}^\circ \rightarrow \mathbb{R}$ be continuous functions and $m, c \in \mathbb{R}$ be constants, $c > 0$. If*

$$|V(\mathbf{x}) - W(\mathbf{x})| \leq c$$

holds for all $\mathbf{x} \in \mathcal{M} \setminus \mathcal{F}^\circ$ and $\mathcal{L}_{V,m-c}, \mathcal{L}_{V,m+c} \neq \emptyset$, then $\mathcal{L}_{W,m} \neq \emptyset$ and

$$\mathcal{L}_{V,m-c} \subset \mathcal{L}_{W,m} \subset \mathcal{L}_{V,m+c}.$$

Proof. Using the notation from Definition 2.3, we have by assumption that

$$\mathcal{O}_{V,m-c} \subset \mathcal{O}_{W,m} \subset \mathcal{O}_{V,m+c}.$$

We want to show, using the notation from Definition 2.3 again, that

$$\mathcal{O}_{V,m-c,\mathbf{0}} \subset \mathcal{O}_{W,m,\mathbf{0}} \subset \mathcal{O}_{V,m+c,\mathbf{0}}.$$

Let $\mathbf{x} \in \mathcal{O}_{V,m-c,\mathbf{0}}$, which is open by assumption. By definition, $\mathcal{O}_{V,m-c,\mathbf{0}}$ is path-connected, so there is a continuous function $\gamma : [0, 1] \rightarrow \mathcal{O}_{V,m-c,\mathbf{0}}$ such that $\gamma(0) = \mathbf{0}$ and $\gamma(1) = \mathbf{x}$. We want to show that $\gamma(\theta) \in \mathcal{O}_{W,m,\mathbf{0}}$ for all $\theta \in [0, 1]$. Indeed, either $\gamma(\theta) \in \mathcal{F} \subset \mathcal{O}_{W,m,\mathbf{0}}$, or $V(\gamma(\theta)) < m - c$ and, thus, by assumption $W(\gamma(\theta)) < m$.

To show that $\mathcal{L}_{W,m} \neq \emptyset$, note that since $\mathcal{F} \subset \mathcal{O}_{V,m-c,\mathbf{0}}^\circ$, we have

$$\mathcal{F} \subset \mathcal{O}_{V,m-c,\mathbf{0}}^\circ \subset \mathcal{O}_{W,m,\mathbf{0}}^\circ.$$

Using a similar argument, we can also show

$$\overline{\mathcal{O}_{W,m,\mathbf{0}}} \subset \overline{\mathcal{O}_{V,m+c,\mathbf{0}}} \subset \mathcal{M}^\circ$$

and thus in particular that $\mathcal{L}_{W,m} \neq \emptyset$. \blacksquare

Now we define a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$ which will have a negative Dini orbital derivative in $\mathcal{M} \setminus \mathcal{F}^\circ$. This is a weaker assumption than a classical Lyapunov function which has a negative Dini orbital derivative in $\mathcal{M} \setminus \{\mathbf{0}\}$ and a minimum at $\mathbf{0}$, usually with value 0. For a classical Lyapunov function, (i) and (ii) in Definition 2.5 follow from these two assumptions; for the weaker notion of a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$ we make the assumptions of (i) a negative Dini orbital derivative, bounded away from zero and (ii) $\mathcal{L}_V^{\text{inf}} \neq \emptyset$, i.e., the existence of a sublevel set $\mathcal{O}_{V,m}$ of V such that the connected component including $\mathbf{0}$ satisfies $\mathcal{F} \subset \mathcal{O}_{V,m,\mathbf{0}}^\circ \subset \overline{\mathcal{O}_{V,m,\mathbf{0}}} \subset \mathcal{M}^\circ$. Since we do not assume $V(\mathbf{0}) = 0$ (in fact, $\mathbf{0}$ is not even necessarily in the domain of V), the level $m \in \mathbb{R}$ may be negative.

Definition 2.5. *Let $\mathcal{F}, \mathcal{M} \in \mathfrak{N}^n$ satisfy $\mathcal{F} \subset \mathcal{M}^\circ$. A Lipschitz continuous function $V : \mathcal{M} \setminus \mathcal{F}^\circ \rightarrow \mathbb{R}$ is called a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$ for (2.1) if there exists a constant $\alpha \in \mathbb{R}^+$ such that*

- (i) $D^+V(\mathbf{x}) \leq -\alpha$ for all $\mathbf{x} \in \mathcal{M}^\circ \setminus \mathcal{F}$ and
- (ii) $\mathcal{L}_V^{\text{inf}} \neq \emptyset$.

A classical Lyapunov function provides information about the basin of attraction through its sublevel sets. For a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$ as in Definition 2.5, similar but slightly weaker statements hold as stated in Theorem 2.6 below. Recall that a set $\mathcal{N} \subset \mathbb{R}^n$ is called *forward invariant* if $\xi \in \mathcal{N}$ implies $S_t \xi \in \mathcal{N}$ for all $t \geq 0$.

Essentially, solutions starting in $\mathcal{L}_V^{\text{sup}}$ enter the forward invariant set $\mathcal{L}_V^{\text{inf}}$ in a finite time; they also either stay in \mathcal{F} or enter infinitely often. If \mathcal{F} is known to be a subset of the basin of attraction, then so is $\mathcal{L}_V^{\text{sup}}$.

Theorem 2.6. *Let $\mathcal{F}, \mathcal{M} \in \mathfrak{N}^n$, $\mathcal{F} \subset \mathcal{M}^\circ$, and let V be a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$ for the system (2.1) as in Definition 2.1. Let $m \in \mathbb{R}$ be a constant such that $\mathcal{L}_{V,m} \neq \emptyset$. Then we have the following propositions:*

- (a) $\mathcal{L}_{V,m}$, $\mathcal{L}_V^{\text{inf}}$, and $\mathcal{L}_V^{\text{sup}}$ are forward invariant sets.
- (b) There is a constant $T > 0$ such that $\xi \in \mathcal{L}_V^{\text{sup}}$ implies $S_T \xi \in \mathcal{L}_V^{\text{inf}}$.
- (c) For every $\xi \in \mathcal{L}_V^{\text{sup}}$ there is a sequence $(t_k)_{k \in \mathbb{N}}$, $t_k \rightarrow +\infty$, such that $S_{t_k} \xi \in \mathcal{F}$ for all k .
- (d) If $\mathcal{F} \subset \mathcal{A}$, then $\mathcal{L}_V^{\text{sup}} \subset \mathcal{A}$, where \mathcal{A} denotes the basin of attraction of the origin.
- (e) Set $a := \max_{\mathbf{x} \in \partial \mathcal{F}} V(\mathbf{x})$. Then $\mathcal{L}_V^{\text{inf}}$ is the connected component of $\mathcal{F} \cup \{\mathbf{x} \in \mathcal{M} \setminus \mathcal{F}^\circ : V(\mathbf{x}) \leq a\}$ that contains $\mathbf{0}$. In particular, $\mathcal{L}_V^{\text{inf}}$ is a closed set.
- (f) With $b := \sup\{c \in \mathbb{R} : \mathcal{L}_{V,c} \neq \emptyset\}$ the set $\mathcal{L}_V^{\text{sup}}$ is the connected component of $\mathcal{F} \cup \{\mathbf{x} \in \mathcal{M} \setminus \mathcal{F}^\circ : V(\mathbf{x}) < b\}$ that contains $\mathbf{0}$.

Proof. See [2, Theorem 2.5], where the result is shown for a general stable compact attractor Ω , which here is the equilibrium $\Omega = \{\mathbf{0}\}$; note that the proof works also for general $m \in \mathbb{R}$. Note also that all propositions of the theorem except (d) hold true for any system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{f} : \mathcal{M} \rightarrow \mathbb{R}^n$ Lipschitz continuous. ■

Now we turn to an existence theorem for a smooth Lyapunov function, following [7]. We first define positive definite functions through class \mathcal{K} functions.

Definition 2.7.

- (i) A continuous function $\alpha : [0, +\infty[\rightarrow [0, +\infty[$ is said to be of class \mathcal{K} if $\alpha(0) = 0$ and α is strictly monotonically increasing.
- (ii) Let \mathcal{U} be a neighborhood of the origin, and let $g : \mathcal{U} \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function. We say that g is a positive definite function if $g(\mathbf{0}) = 0$ and there exists a class \mathcal{K} function α such that $g(\mathbf{x}) \geq \alpha(\|\mathbf{x}\|_2)$ for all $\mathbf{x} \in \mathcal{U}$.

Theorem 2.8 (existence of V). *Consider the system (2.1) and let $p \in C^\sigma(\mathbb{R}^n, \mathbb{R})$, $\sigma \geq 1$, be a positive definite function. Further, let $q \in C^\sigma(\mathbb{R}^n, \mathbb{R})$ be a positive function such that $\sup_{\mathbf{x}} [\|\mathbf{f}(\mathbf{x})\|_2/q(\mathbf{x})] < \infty$. Then there exists a unique positive definite function $V \in C^\sigma(\mathcal{A}, \mathbb{R})$ such that*

$$(2.4) \quad V'(\mathbf{x}) := \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = -p(\mathbf{x})q(\mathbf{x})$$

holds for all $\mathbf{x} \in \mathcal{A}$. In particular, V is a strict Lyapunov function for the system (2.1) on \mathcal{A} in the conventional sense.

Further, for every $m > 0$, the sublevel set $K_{V,m} := \{\mathbf{x} \in \mathcal{A} : V(\mathbf{x}) \leq m\}$ is in \mathfrak{N}^n , is homeomorphic to $\overline{\mathcal{B}(1)}$, and has a C^σ boundary.

Proof. Consider the system

$$(2.5) \quad \dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}), \quad \text{where } \mathbf{g}(\mathbf{x}) := \frac{\mathbf{f}(\mathbf{x})}{q(\mathbf{x})}.$$

This system has the same trajectories in the phase space as the system (2.1), as $\mathbf{g} \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$ is \mathbf{f} multiplied by the positive, scalar-valued function $\mathbf{x} \rightarrow 1/q(\mathbf{x})$.

Note that for a positive definite function $p : \mathbb{R}^n \rightarrow \mathbb{R}$ we have

$$\inf_{\mathbf{x} \in \mathbb{R}^n \setminus \mathcal{B}(\varepsilon)} p(\mathbf{x}) \geq \alpha(\varepsilon) > 0$$

for all $\varepsilon > 0$, where $\alpha \in \mathcal{K}$ is as in Definition 2.7. Further, for $p \in C^1(\mathbb{R}^n, \mathbb{R})$ such that $p(\mathbf{0}) = 0$, we have

$$\lim_{\mathbf{x} \rightarrow \mathbf{0}} \frac{|p(\mathbf{x}) - \nabla p(\mathbf{0}) \cdot \mathbf{x}|}{\|\mathbf{x}\|_2} = 0,$$

i.e., $|p(\mathbf{x})| \leq (1 + \|\nabla p(\mathbf{0})\|_2)\|\mathbf{x}\|_2$ for all \mathbf{x} close enough to the origin.

By [7, Theorem 2.46] the PDE

$$(2.6) \quad V'(\mathbf{x}) := \nabla V(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = -p(\mathbf{x})$$

possesses a solution $V \in C^\sigma(\mathcal{A}, \mathbb{R})$ and by [7, Proposition 2.48] this solution is unique up to a constant. We thus have a unique solution $V \in C^\sigma(\mathcal{A}, \mathbb{R})$ to (2.6) fulfilling $V(\mathbf{0}) = 0$. By the construction of V in the proof of [7, Theorem 2.46] this function is positive definite. Note that (2.6) and (2.4) are equivalent. Thus, there is a unique solution to (2.4) that is positive definite.

To show that $K_{V,m}$ is in \mathfrak{N}^n we follow the arguments in [7]. As $\sup_{\mathbf{x} \in \mathcal{A}} \|\mathbf{g}(\mathbf{x})\|_2 < \infty$, the level set $\Omega = \{\mathbf{x} \in \mathcal{A} : V(\mathbf{x}) = m\}$ is a noncharacteristic hypersurface by [7, Lemma 2.37]; for a definition, see [7, Definition 2.36]. Moreover, Ω^i in [7, Definition 2.41] is the set $\Omega^i = \{\mathbf{x} \in \mathcal{A} : V(\mathbf{x}) < m\}$. Hence, by [7, Proposition 2.42] we have that $K_{V,m} = \Omega^i \cup \Omega$ is homeomorphic to $\mathcal{B}(1)$, which shows in particular that $K_{V,m}$ is simply connected and compact, and, moreover, $\Omega = \partial K_{V,m}$ is C^σ -diffeomorphic to the unit sphere, which shows that $K_{V,m}$ has a C^σ boundary. ■

Remark 2.9. A simple choice for the function $q(\mathbf{x})$ in Theorem 2.8 is $q(\mathbf{x}) = 1 + \|\mathbf{f}(\mathbf{x})\|_2^2$ and we will use it in the examples.

Considering the PDE $\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = -p(\mathbf{x})q(\mathbf{x})$ as in (2.4) instead of $\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = -p(\mathbf{x})$ as in [7, Theorem 2.46] has the advantage that the sublevel sets $K_{V,m}$ of V are necessarily compact.

The following example shows that the solution of $\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = -p(\mathbf{x})$ has not necessarily compact sublevel sets. Consider the one-dimensional system $\dot{x} = f(x)$, where

$$f(x) = \begin{cases} -\frac{1}{2}x & \text{for } x < 0, \\ -\frac{1}{2}x(1+x^2)^2 & \text{for } x \geq 0. \end{cases}$$

Note that $f \in C^1(\mathbb{R}, \mathbb{R})$. 0 is the only equilibrium; it is exponentially stable with basin of attraction $\mathcal{A} = \mathbb{R}$. The Lyapunov function

$$V(x) = \begin{cases} x^2 & \text{for } x < 0, \\ \frac{x^2}{1+x^2} & \text{for } x \geq 0, \end{cases}$$

is a $C^1(\mathbb{R}, \mathbb{R})$ function, satisfying $V'(x) = -x^2$ for all $x \in \mathcal{A}$. However, sublevel sets $K_{V,m}$ with $m \geq 1$ are not compact.

3. Mesh-free collocation. In this section we discuss the RBF method, a mesh-free collocation method to solve linear PDE's, in our case the PDE $V'(\mathbf{x}) = -p(\mathbf{x})q(\mathbf{x})$. Mesh-free collocation provides a powerful tool for solving generalized interpolation problems efficiently, in particular linear PDE's [36, 6, 35]. In this paper we use kernels based on Radial Basis Functions, in particular Wendland's compactly supported Radial Basis Functions [34]. The advantages of mesh-free collocation over other methods to solve PDE's include that collocation points can be scattered and the approximation is a smooth function.

3.1. Mesh-free collocation and RBFs. In this section we follow [13, 7]. Let $H \subset C^0(\mathcal{B})$, where $\mathcal{B} \subset \mathbb{R}^n$ is a domain, be a Hilbert space of functions $G : \mathcal{B} \rightarrow \mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle_H$ and corresponding norm $\| \cdot \|_H$. Let H^* be its dual with inner product $\langle \cdot, \cdot \rangle_{H^*}$ and corresponding norm $\| \cdot \|_{H^*}$. We consider a generalized interpolation problem of the following form.

Definition 3.1. *Given N linearly independent functionals $\lambda_1, \lambda_2, \dots, \lambda_N \in H^*$ and N values $G_1, G_2, \dots, G_N \in \mathbb{R}$, a generalized interpolant is a function $g \in H$ satisfying $\lambda_j(g) = G_j$, $j = 1, 2, \dots, N$. The norm-minimal interpolant s_G is the interpolant that, in addition, minimizes the norm of the Hilbert space; i.e., s_G is the solution to*

$$(3.1) \quad \min\{\|g\|_H : \lambda_j(g) = G_j, j = 1, 2, \dots, N\}.$$

It is well known that the norm-minimal generalized interpolant is a linear combination of the Riesz representers of the functionals and that the coefficients can be computed by solving a linear system. We will assume that H is a reproducing kernel Hilbert space (RKHS).

Definition 3.2. *A reproducing kernel Hilbert space (RKHS) is a Hilbert space H , such that there exists a unique kernel $\Psi : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{R}$, satisfying*

- (i) $\Psi(\cdot, \mathbf{x}) \in H$ for all $\mathbf{x} \in \mathcal{B}$,
- (ii) $g(\mathbf{x}) = \langle g, \Psi(\cdot, \mathbf{x}) \rangle_H$ for all $\mathbf{x} \in \mathcal{B}$ and all $g \in H$.

We denote the RKHS with kernel Ψ by $H = \mathcal{N}_\Psi$.

For a RKHS, the point evaluation functionals $\delta_{\mathbf{x}}$ are continuous, i.e., $\delta_{\mathbf{x}} \in H^*$ [13, Theorem 10.2] and the Riesz representer of a functional $\lambda \in H^*$ is given by $\lambda^{\mathbf{y}}\Psi(\cdot, \mathbf{y})$ [13, Theorem 16.7]; here, $\lambda^{\mathbf{y}}$ denotes the application of λ with respect to the variable \mathbf{y} .

Lemma 3.3 (see [13, Theorem 16.1]). *If H is a reproducing kernel Hilbert space, then the solution s_G of (3.1) is given by*

$$s_G(\mathbf{x}) = \sum_{j=1}^N \alpha_j \lambda_j^{\mathbf{y}} \Psi(\mathbf{x}, \mathbf{y}),$$

where $\alpha \in \mathbb{R}^N$ is the solution of the linear system $A\alpha = G$ with $A = (a_{ij})_{i,j=1,\dots,N} = (\lambda_i^{\mathbf{x}} \lambda_j^{\mathbf{y}} \Psi(\mathbf{x}, \mathbf{y}))_{i,j=1,\dots,N}$ and $G = (G_j)_{j=1,\dots,N}$.

Note that

$$(3.2) \quad a_{ij} = \lambda_i^{\mathbf{x}} \lambda_j^{\mathbf{y}} \Psi(\mathbf{x}, \mathbf{y}) = \langle \lambda_i^{\mathbf{x}} \Psi(\cdot, \mathbf{x}), \lambda_j^{\mathbf{y}} \Psi(\cdot, \mathbf{y}) \rangle_H = \langle \lambda_i, \lambda_j \rangle_{H^*}$$

(see [13, Theorem 16.7]), and hence A is positive semidefinite. Since the functionals are assumed to be linearly independent, the matrix is even positive definite and, in particular, the linear system $A\boldsymbol{\alpha} = G$ has a unique solution.

The Wendland functions are positive definite functions with compact support, on their support they are polynomials. They are defined by the following recursion with respect to the parameter k .

Definition 3.4 (Wendland functions, [34]). *Let $l \in \mathbb{N}$, $k \in \mathbb{N}_0$. We define by recursion*

$$(3.3) \quad \psi_{l,0}(r) = (1-r)_+^l,$$

$$(3.4) \quad \text{and } \psi_{l,k+1}(r) = \int_r^1 t \psi_{l,k}(t) dt$$

for $r \in \mathbb{R}_0^+$. Here we set $x_+ = x$ for $x \geq 0$ and $x_+ = 0$ for $x < 0$. Setting $l := \lfloor \frac{n}{2} \rfloor + k + 1$, $\Phi(\mathbf{x}) := \psi_{l,k}(c\|\mathbf{x}\|_2)$ belongs to $C^{2k}(\mathbb{R}^n)$ for any $c > 0$.

For the rest of this paper, we consider kernels of the form $\Psi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y})$, where Φ is given by a Wendland function, and we make the following assumptions, also including the discussion at the end of section 2.

Assumptions 3.5. Let V be the Lyapunov function from Theorem 2.8. Denote by k the smoothness index of the compactly supported Wendland function $\psi_{l,k}$ where $l = \lfloor \frac{n}{2} \rfloor + k + 1$; let $k \geq 2$ if n is odd and $k \geq 3$ if n is even. Let $c > 0$ and define the kernel $\Psi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y})$ by the RBF $\Phi(\mathbf{x}) = \psi(\|\mathbf{x}\|_2)$, where $\psi(r) = \psi_{l,k}(cr)$. Set $\tau = k + \frac{n+1}{2}$ and $\sigma = \lceil \tau \rceil$.

Since the Fourier transform of Φ given by the Wendland function as in Assumptions 3.5 satisfies

$$(3.5) \quad c_1(1 + \|\omega\|_2^2)^{-\tau} \leq \widehat{\Phi}(\omega) \leq c_2(1 + \|\omega\|_2^2)^{-\tau}, \quad \omega \in \mathbb{R}^n,$$

with $\tau = k + (n+1)/2$, we have the following result.

Lemma 3.6. *For Φ defined by a Wendland function with index $k \in \mathbb{N}_0$ as in Assumptions 3.5, the RKHS $\mathcal{N}_\Phi(\mathbb{R}^n)$ consists of the same functions as the Sobolev space $W_2^{k+(n+1)/2}(\mathbb{R}^n)$ and their norms are equivalent [13, Theorem 10.35]. If the domain $\mathcal{B} \subset \mathbb{R}^n$ has a Lipschitz boundary, then $\mathcal{N}_\Phi(\mathcal{B})$ consists of the same functions as $W_2^{k+(n+1)/2}(\mathcal{B})$ and their norms are equivalent. This can be shown as in [13, Corollary 10.48].*

Note that in Lemma 3.6 the Sobolev spaces $W_2^{k+(n+1)/2}(\mathbb{R}^n)$ and $W_2^{k+(n+1)/2}(\mathcal{B})$ are of fractional order if n is an even number.

3.2. RBF Lyapunov function. In this section we apply the general theory to the generalized interpolation problem given by the linear PDE (2.4) $V'(\mathbf{x}) = -p(\mathbf{x})q(\mathbf{x})$ in a compact set $\mathcal{D} \in \mathbb{R}^n$. For more details and the derivation of the formulas, cf. [7].

We choose pairwise distinct points $\mathbf{x}_j \in \mathcal{D}$, $\mathbf{f}(\mathbf{x}_j) \neq \mathbf{0}$, for $j = 1, 2, \dots, N$ and denote this set of collocation points by $X := \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$.

Given ψ , we define ψ_1 and ψ_2 by

$$(3.6) \quad \psi_1(r) = \frac{\frac{d}{dr}\psi(r)}{r} \text{ for } r > 0,$$

$$(3.7) \quad \psi_2(r) = \frac{\frac{d}{dr}\psi_1(r)}{r} \text{ for } r > 0.$$

Note that by the definition of Wendland functions and since $k \geq 2$ (see Assumptions 3.5), ψ_1 and ψ_2 can be extended continuously to 0; for formulas of ψ_1 and ψ_2 , see [7, Appendix B.1].

The linearly independent functionals are now given by $\lambda_j = (\delta_{\mathbf{x}_j} \circ L)$, where $LV(\mathbf{x}) := V'(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$ denotes the linear operator of the orbital derivative and $\delta_{\boldsymbol{\xi}}$ denotes Dirac's delta functional, i.e., $\delta_{\boldsymbol{\xi}}g = g(\boldsymbol{\xi})$. They are linearly independent, since the points are pairwise distinct and no equilibria, which are the only singular points of the linear operator [13, Definition 3.2, Proposition 3.3]. By Lemma 3.3, the solution s_V of the minimization problem (3.1) is given by

$$(3.8) \quad s_V(\mathbf{x}) = \sum_{j=1}^N \alpha_j (\delta_{\mathbf{x}_j} \circ L)^{\mathbf{y}} \Phi(\mathbf{x} - \mathbf{y}),$$

where the superscript \mathbf{y} denotes the application of the operator with respect to \mathbf{y} . Note that, since L is a first-order differential operator, we have $s_V \in C^{2k-1}(\mathbb{R}^n, \mathbb{R})$.

The coefficient vector $\boldsymbol{\alpha} \in \mathbb{R}^N$ is determined by the linear system

$$(3.9) \quad A\boldsymbol{\alpha} = G,$$

where the right-hand side of (3.9) is given by $G_j = -p(\mathbf{x}_j)q(\mathbf{x}_j)$ for $j = 1, 2, \dots, N$ and the matrix A has the elements

$$(3.10) \quad \begin{aligned} a_{ij} &= (\delta_{\mathbf{x}_i} \circ L)^{\mathbf{x}} (\delta_{\mathbf{x}_j} \circ L)^{\mathbf{y}} \Phi(\mathbf{x} - \mathbf{y}) \\ &= \psi_2(\|\mathbf{x}_i - \mathbf{x}_j\|_2) \langle \mathbf{x}_i - \mathbf{x}_j, \mathbf{f}(\mathbf{x}_i) \rangle_2 \langle \mathbf{x}_j - \mathbf{x}_i, \mathbf{f}(\mathbf{x}_j) \rangle_2 \\ &\quad - \psi_1(\|\mathbf{x}_i - \mathbf{x}_j\|_2) \langle \mathbf{f}(\mathbf{x}_i), \mathbf{f}(\mathbf{x}_j) \rangle_2, \end{aligned}$$

where ψ_1 and ψ_2 were defined in (3.6) and (3.7) and $\langle \cdot, \cdot \rangle_2$ denotes the Euclidean scalar product. The approximate solution s_V and its orbital derivative $(s_V)'$ are given by (see [7])

$$(3.11) \quad s_V(\mathbf{x}) = \sum_{j=1}^N \alpha_j \langle \mathbf{x}_j - \mathbf{x}, \mathbf{f}(\mathbf{x}_j) \rangle_2 \psi_1(\|\mathbf{x} - \mathbf{x}_j\|_2),$$

$$(3.12) \quad \begin{aligned} (s_V)'(\mathbf{x}) &= \sum_{j=1}^N \alpha_j \left[\psi_2(\|\mathbf{x} - \mathbf{x}_j\|_2) \langle \mathbf{x} - \mathbf{x}_j, \mathbf{f}(\mathbf{x}) \rangle_2 \langle \mathbf{x}_j - \mathbf{x}, \mathbf{f}(\mathbf{x}_j) \rangle_2 \right. \\ &\quad \left. - \psi_1(\|\mathbf{x} - \mathbf{x}_j\|_2) \langle \mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}_j) \rangle_2 \right]. \end{aligned}$$

We will show in Lemma 3.8 that the orbital derivative $(s_V)'$ of s_V approximates the orbital derivative V' of V . Note that the solution of $V'(\mathbf{x}) = -p(\mathbf{x})q(\mathbf{x})$ is unique up to a constant; cf. the proof of Theorem 2.8. As we later estimate the error of the RBF approximation to V , we define V_R which is s_V shifted by a constant so that $V_R(\mathbf{0}) = 0$. We summarize the definition of the RBF approximation s_V and V_R .

Definition 3.7 (RBF approximation of a Lyapunov function). *Let V be the Lyapunov function from Theorem 2.8. We now have the following:*

- By the RBF problem with respect to the $X = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$ and using the basis function ψ , we mean the RBF problem (3.9), where $G_j = -p(\mathbf{x}_j)q(\mathbf{x}_j)$ and A is given by (3.10).
- By the solution s_V to the RBF problem, we mean the function (3.11) where $\boldsymbol{\alpha}$ is the solution of (3.9).
- By the RBF approximation V_R of V with respect to X and using the basis function ψ , we mean the function $V_R(\mathbf{x}) := s_V(\mathbf{x}) - s_V(\mathbf{0})$.

3.3. Error estimates. In this section we first prove in Lemma 3.8 error estimates between V and the RBF approximation V_R in terms of their orbital derivative (3.13). This enables us in Theorem 3.9 to show that the error between V_R and V , both between the functions and their orbital derivatives, is small if the fill distance of the collocation points is small.

Moreover, we prove uniform bounds on all approximations s_V , which only depend on V and not on the set of collocation points X ; see (3.14) to (3.16).

Lemma 3.8. *Let Assumptions 3.5 hold. Let $\mathcal{B} \subset \mathcal{A}$ be a bounded open set with C^1 boundary and let $X \subset \mathcal{B}$ be a set of points with sufficiently small fill distance² $h_X := \sup_{\mathbf{x} \in \mathcal{B}} \min_{\mathbf{x}_j \in X} \|\mathbf{x} - \mathbf{x}_j\|_2$. Let s_V and V_R be the RBF approximations of V with respect to X and the RBF ψ in the sense of Definition 3.7.*

Then there exists a constant C , independent of X and the approximations s_V and V_R , such that

$$(3.13) \quad \|V'_R - V'\|_{L^\infty(\mathcal{B})} \leq Ch_X^{k-\frac{1}{2}} \|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})},$$

$$(3.14) \quad \|s_V\|_{W_2^{k+(n+1)/2}(\mathcal{B})} \leq C \|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})},$$

$$(3.15) \quad \|s_V - V\|_{W_2^{k+(n+1)/2}(\mathcal{B})} \leq C \|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})},$$

$$(3.16) \quad \|s_V\|_{C^2(\bar{\mathcal{B}})} \leq C \|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})}.$$

Proof. The first estimate follows from [13, Corollary 4.11], which holds similarly for the set \mathcal{B} .

For the second and third estimate note that by Lemma 3.6 there are constants $c_1, c_2 > 0$ such that

$$c_1 \|W\|_{\mathcal{N}_\Phi(\mathcal{B})} \leq \|W\|_{W_2^{k+(n+1)/2}(\mathcal{B})} \leq c_2 \|W\|_{\mathcal{N}_\Phi(\mathcal{B})}$$

for all $W \in W_2^{k+(n+1)/2}(\mathcal{B})$. Using that the approximation is norm-minimal in $\mathcal{N}_\Phi(\mathcal{B})$ (see section 3.1), i.e., in particular $\|s_V\|_{\mathcal{N}_\Phi(\mathcal{B})} \leq \|V\|_{\mathcal{N}_\Phi(\mathcal{B})}$, we thus get

$$\begin{aligned} \|s_V\|_{W_2^{k+(n+1)/2}(\mathcal{B})} &\leq c_2 \|s_V\|_{\mathcal{N}_\Phi(\mathcal{B})} \\ &\leq c_2 \|V\|_{\mathcal{N}_\Phi(\mathcal{B})} \\ &\leq \frac{c_2}{c_1} \|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})}, \end{aligned}$$

which shows the second inequality. The third inequality follows directly from the second one.

²That is, h_X needs to be smaller than a constant h^* , only depending on the set \mathcal{B} and the Sobolev space $W_2^{k+(n+1)/2}(\mathcal{B})$.

To obtain the last inequality, we use general Sobolev inequalities, e.g., [5, Chapter 5.7, Theorem 6]. Note that \mathcal{B} is a bounded open subset of \mathbb{R}^n with C^1 boundary.

Case 1. If $\frac{n}{2}$ is not an integer (i.e., n is odd), then $k + (n + 1)/2$ is an integer larger than $\frac{n}{2}$ and we have the estimate

$$\|s_V\|_{C^{k+(n+1)/2-\lfloor\frac{n}{2}\rfloor-1,\gamma}(\overline{\mathcal{B}})} \leq C\|s_V\|_{W_2^{k+(n+1)/2}(\mathcal{B})},$$

where the constant C does not depend on s_V . We have $\gamma = \lfloor\frac{n}{2}\rfloor + 1 - \frac{n}{2} = \frac{1}{2}$ and the Hölder space is $C^{k+(n+1)/2-\lfloor\frac{n}{2}\rfloor-1,\gamma}(\overline{\mathcal{B}}) = C^{k,1/2}(\overline{\mathcal{B}})$.

Case 2. If $\frac{n}{2}$ is an integer (i.e., n is even), then $k + n/2$ is an integer larger than $\frac{n}{2}$ and we have the estimate

$$\|s_V\|_{C^{k+n/2-\lfloor\frac{n}{2}\rfloor-1,\gamma}(\overline{\mathcal{B}})} \leq C\|s_V\|_{W_2^{k+n/2}(\mathcal{B})} \leq C\|s_V\|_{W_2^{k+(n+1)/2}(\mathcal{B})},$$

where the constant C does not depend on s_V . The Hölder space is $C^{k+n/2-\lfloor\frac{n}{2}\rfloor-1,\gamma}(\overline{\mathcal{B}}) = C^{k-1,\gamma}(\overline{\mathcal{B}})$.

By the definition of the Hölder norm and under the assumptions on k , we have thus in both cases that $s_V \in C^{2,\gamma}(\overline{\mathcal{B}}) \subset C^2(\overline{\mathcal{B}})$ and thus

$$\|s_V\|_{C^2(\overline{\mathcal{B}})} \leq \|s_V\|_{C^{2,\gamma}(\overline{\mathcal{B}})} \leq C\|s_V\|_{W_2^{k+(n+1)/2}(\mathcal{B})} \leq C\|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})},$$

taking (3.14) into account. \blacksquare

In the following theorem we derive error estimates between the RBF approximation V_R and V , both for the orbital derivatives and, by estimates near the equilibrium and integration along solutions, also for the functions.

Theorem 3.9. *Let Assumptions 3.5 hold. Let $\mathcal{K} \in \mathfrak{N}^n$ be a subset of \mathcal{A} and let \mathcal{B} be a bounded, open, and forward invariant set with C^1 boundary such that $\mathcal{K} \subset \mathcal{B} \subset \mathcal{A}$.*

Then, given any $\delta > 0$ there is an $h_R^ > 0$ such that for any set of collocation points $X \subset \mathcal{B}$ with fill distance $h_X := \sup_{\mathbf{x} \in \mathcal{B}} \min_{\mathbf{x}_j \in X} \|\mathbf{x} - \mathbf{x}_j\|_2 \leq h_R^*$ and not including the origin, the RBF approximation V_R to V with respect to X (cf. Definition 3.7) fulfills*

$$(3.17) \quad |V(\mathbf{x}) - V_R(\mathbf{x})| \leq \delta \text{ and}$$

$$(3.18) \quad |V'(\mathbf{x}) - V'_R(\mathbf{x})| \leq \delta$$

for all $\mathbf{x} \in \mathcal{K}$.

Proof. We denote the RKHS for the kernel defined by the RBF Φ by $\mathcal{N}_\Phi(\mathcal{B})$. By Lemma 3.6 there is a constant $C^* > 0$ such that

$$(3.19) \quad \|W\|_{\mathcal{N}_\Phi(\mathcal{B})} \leq C^*\|W\|_{W_2^{k+(n+1)/2}(\mathcal{B})} \quad \text{for all } W \in W_2^{k+(n+1)/2}(\mathcal{B}).$$

Choose $m_0 > 0$ so small that $\overline{\mathcal{B}(m_0)} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 \leq m_0\} \subset \mathcal{K}^\circ$, and

$$(3.20) \quad 2m_0 \max_{\tilde{r} \in [0, m_0]} \left| \frac{d}{d\tilde{r}} \psi(\tilde{r}) \right| \leq \left(\frac{\delta}{2C^*C\|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})}} \right)^2$$

hold where $\psi(r) := \psi_{l,k}(cr)$, C^* was defined in (3.19) and C comes from Lemma 3.8. This is possible since $\frac{d}{dr}\psi(r)$ is bounded.

Choose $r_0 > 0$ so small that

$$\Omega := \{\mathbf{x} \in \mathcal{A} : V(\mathbf{x}) \leq r_0\} \subset \overline{\mathcal{B}(m_0)}$$

holds. Define

$$\Gamma = \partial\Omega = \{\mathbf{x} \in \mathcal{A} : V(\mathbf{x}) = r_0\} \subset \overline{\mathcal{B}(m_0)}.$$

Γ is a noncharacteristic hypersurface by [7, Lemma 2.37] and, hence, by [7, Theorem 2.38] there exists a function $\theta \in C^\sigma(\mathcal{A} \setminus \{\mathbf{0}\}, \mathbb{R})$ defined implicitly by $S_{\theta(\mathbf{x})}\mathbf{x} \in \Gamma$; $\theta(\mathbf{x})$ denotes the unique time when the solution through \mathbf{x} intersects with Γ . Set

$$\theta_0 := \max_{\mathbf{x} \in \mathcal{K}} \theta(\mathbf{x}) > 0.$$

Define $h_R^* > 0$ such that both the fill distance h_R^* is small enough in the sense of Lemma 3.8 and

$$(3.21) \quad C(h_R^*)^{k-\frac{1}{2}} \|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})} \leq \min\left(\frac{\delta}{2\theta_0}, \delta\right),$$

where C is from Lemma 3.8. Now fix $X \subset \mathcal{B}$ with fill distance $h_X \leq h_R^*$ and denote by s_V and V_R the RBF approximations to V with respect to X .

We have $V_R(\mathbf{x}) := s_V(\mathbf{x}) - s_V(\mathbf{0})$ and thus $V_R(\mathbf{0}) = 0 = V(\mathbf{0})$. For $\mathbf{x}^* \in \overline{\mathcal{B}(m_0)}$ we have with $\delta_{\mathbf{x}^*}, \delta_{\mathbf{0}} \in \mathcal{N}_\Phi(\mathcal{B})^*$,

$$(3.22) \quad \begin{aligned} |V(\mathbf{x}^*) - V_R(\mathbf{x}^*)| &= |(\delta_{\mathbf{x}^*} - \delta_{\mathbf{0}})(V - s_V + s_V(\mathbf{0}))| \\ &= |(\delta_{\mathbf{x}^*} - \delta_{\mathbf{0}})(V - s_V)| \\ &\leq \|\delta_{\mathbf{x}^*} - \delta_{\mathbf{0}}\|_{\mathcal{N}_\Phi(\mathcal{B})^*} \cdot \|V - s_V\|_{\mathcal{N}_\Phi(\mathcal{B})} \\ &\leq C^* \|\delta_{\mathbf{x}^*} - \delta_{\mathbf{0}}\|_{\mathcal{N}_\Phi(\mathcal{B})^*} \cdot \|V - s_V\|_{W_2^{k+(n+1)/2}(\mathcal{B})} \\ &\leq C^* C \|\delta_{\mathbf{x}^*} - \delta_{\mathbf{0}}\|_{\mathcal{N}_\Phi(\mathcal{B})^*} \cdot \|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})} \end{aligned}$$

using (3.19) and (3.15).

We use (3.2) for the norm in $H^* = \mathcal{N}_\Phi(\mathcal{B})^*$. Moreover, Taylor expansion yields the existence of an $\tilde{r} \in [0, \rho]$, where $\rho := \|\mathbf{x}^*\|_2 \leq m_0$ such that

$$\begin{aligned} \|\delta_{\mathbf{x}^*} - \delta_{\mathbf{0}}\|_{\mathcal{N}_\Phi(\mathcal{B})^*}^2 &= (\delta_{\mathbf{x}^*} - \delta_{\mathbf{0}})^{\mathbf{x}} (\delta_{\mathbf{x}^*} - \delta_{\mathbf{0}})^{\mathbf{y}} \Phi(\mathbf{x} - \mathbf{y}) \\ &= (\delta_{\mathbf{x}^*} - \delta_{\mathbf{0}})^{\mathbf{x}} [\psi(\|\mathbf{x} - \mathbf{x}^*\|_2) - \psi(\|\mathbf{x}\|_2)] \\ &= 2[\psi(0) - \psi(\|\mathbf{x}^*\|_2)] \\ &= -2 \frac{d}{dr} \psi(\tilde{r}) \rho \\ &\leq \left(\frac{\delta}{2C^* C \|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})}} \right)^2; \text{ cf. (3.20)}. \end{aligned}$$

Hence, we have with (3.22)

$$(3.23) \quad |V(\mathbf{x}^*) - V_R(\mathbf{x}^*)| \leq \frac{\delta}{2} \quad \text{for all } \mathbf{x}^* \in \overline{\mathcal{B}(m_0)}.$$

For $\mathbf{x} \in \Gamma$, i.e., $V(\mathbf{x}) = r_0$, we have $\mathbf{x} \in \overline{\mathcal{B}(m_0)}$ and hence

$$(3.24) \quad V_R(\mathbf{x}) \in \left[r_0 - \frac{\delta}{2}, r_0 + \frac{\delta}{2} \right] \quad \text{for all } \mathbf{x} \in \Gamma.$$

For the orbital derivatives we have, using (3.13) of Lemma 3.8 and (3.21),

$$(3.25) \quad |V'_R(\mathbf{x}) - V'(\mathbf{x})| \leq C(h_R^*)^{k-\frac{1}{2}} \|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})} \leq \min\left(\frac{\delta}{2\theta_0}, \delta\right)$$

for all $\mathbf{x} \in \mathcal{B}$, which shows (3.18) as $\mathcal{K} \subset \mathcal{B}$.

We will now show (3.17), namely $|V(\mathbf{x}) - V_R(\mathbf{x})| \leq \delta$ for all $\mathbf{x} \in \mathcal{K}$. If $\mathbf{x} \in \Omega \subset \overline{\mathcal{B}(m_0)}$, then the statement follows from (3.23).

Now let $\mathbf{x} \in \mathcal{K} \setminus \Omega$, i.e., in particular that $\theta(\mathbf{x}) \in [0, \theta_0]$. Then we have, using $S_{\theta(\mathbf{x})}\mathbf{x} \in \Gamma$, (3.24) and (3.25)

$$\begin{aligned} V_R(\mathbf{x}) &= V_R(S_{\theta(\mathbf{x})}\mathbf{x}) - \int_0^{\theta(\mathbf{x})} V'_R(S_\tau\mathbf{x}) d\tau \\ &\leq r_0 + \frac{\delta}{2} + \int_0^{\theta(\mathbf{x})} \left(-V'(S_\tau\mathbf{x}) + \frac{\delta}{2\theta_0} \right) d\tau \\ &\leq \underbrace{V(S_{\theta(\mathbf{x})}\mathbf{x}) - \int_0^{\theta(\mathbf{x})} V'(S_\tau\mathbf{x}) d\tau}_{=V(\mathbf{x})} + \frac{\delta}{2} + \frac{\delta}{2} \\ &= V(\mathbf{x}) + \delta. \end{aligned}$$

Similarly, we can show $V_R(\mathbf{x}) \geq V(\mathbf{x}) - \delta$, which proves the theorem. \blacksquare

The error estimates in Theorem 3.9 show that the RBF method will construct a Lyapunov function V_R such that both the function and its orbital derivative are close to V and V' , respectively. In particular, sublevel sets of V_R and V are close if the fill distance of the collocation points is small.

The drawback of this theorem, however, is that the relation between δ and h_R^* , i.e., error and fill distance, involves unknown quantities such as $\|V\|_{W_2^{k+(n+1)/2}(\mathcal{B})}$. Hence, the error estimate provides no direct tool to determine how dense the collocation points must be chosen, and not even to verify that an approximation V_R has negative orbital derivative, i.e., is a Lyapunov function for the system.

In this paper, we will derive a verification method to ensure that the constructed function is a Lyapunov function; this is a major improvement compared to the original RBF method. Note, however, that the verified function is not the RBF approximation itself, but a CPA interpolation if it.

4. CPA verification. In the remainder of this paper, we will present an algorithm involving a verification procedure, which will guarantee that we have constructed a Lyapunov function. This Lyapunov function will not be V_R itself, but an interpolation of V_R by a CPA function, denoted V_C . The function V_C interpolates V_R on a triangulation, using the values of V_R at the vertices. If finitely many inequalities are satisfied, then the function V_C is a Lyapunov function. Moreover, we will prove that the algorithm will construct a function V_C which can be verified to be a Lyapunov function by finitely many inequalities, if both the RBF collocation points and the triangulation are fine enough.

The inequalities are taken from the CPA method to construct Lyapunov functions [18], where the values of the Lyapunov function at the vertices are determined by linear programming. Here, however, the method is used to verify rather than to construct a Lyapunov function, so no computationally demanding linear programming problem has to be solved. This is a major improvement with respect to the computation time compared to the original CPA method.

We will first discuss the triangulation before we present the CPA interpolation of a function and the inequalities to verify that a function has negative orbital derivative. In the following section 5, we will then combine the RBF method to construct and the CPA method to verify Lyapunov functions, and prove that it always succeeds.

4.1. Triangulation. A triangulation \mathcal{T} of a subset of \mathbb{R}^n consists of countably many n -simplices \mathfrak{S}_ν . To simplify notation we often write $\mathcal{T} = (\mathfrak{S}_\nu)$, where it is to be understood that $\nu = 1, 2, \dots, N$, if \mathcal{T} has a finite number N of (different) simplices, or $\nu \in \mathbb{N}$, if \mathcal{T} is infinite.

Let us first recall the definition of a simplex. Let $(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m)$ be an ordered $(m + 1)$ -tuple of vectors in \mathbb{R}^n . The set of all *convex combinations* of these vectors is denoted by $\text{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m) := \{\sum_{i=0}^m \lambda_i \mathbf{x}_i : 0 \leq \lambda_i \leq 1, \sum_{i=0}^m \lambda_i = 1\}$. The vectors $(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m)$ are called *affinely independent* if $\sum_{i=1}^m \lambda_i (\mathbf{x}_i - \mathbf{x}_0) = \mathbf{0}$ implies $\lambda_i = 0$ for all $i = 1, 2, \dots, m$. If $(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m)$ are affinely independent, then the set $\mathfrak{S} := \text{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m)$ is called an m -simplex and the vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m$ are said to be its vertices. We consider two simplices \mathfrak{S}_1 and \mathfrak{S}_2 to be equal if they are equal as sets, although we represent them as the convex combination of an ordered tuple $(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m)$ of vertices.

We will briefly describe triangulations suited for our needs; for more details, cf. [9, 10]. We start by defining general triangulations and CPA functions, then we define the triangulations which we use in this paper and derive their basic properties.

Definition 4.1 (triangulation). *Let \mathcal{T} be a collection of n -simplices \mathfrak{S}_ν in \mathbb{R}^n . \mathcal{T} is called a triangulation if for every $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$, $\nu \neq \mu$, either $\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \emptyset$ or \mathfrak{S}_ν and \mathfrak{S}_μ intersect in a common face. Recall that a face of an n -simplex $\text{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$ is a k -simplex $\text{co}(\mathbf{x}_{i_0}, \mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_k})$, where $0 \leq k < n$ and the $0 \leq i_0, i_1, \dots, i_k \leq n$ are pairwise different integers.*

For a triangulation \mathcal{T} we define

$$\mathcal{V}_{\mathcal{T}} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \text{ is a vertex of a simplex in } \mathcal{T}\}$$

and

$$\mathcal{D}_{\mathcal{T}} := \bigcup_{\mathfrak{S}_\nu \in \mathcal{T}} \mathfrak{S}_\nu.$$

We call $\mathcal{V}_{\mathcal{T}}$ the vertex set of the triangulation \mathcal{T} and we say that \mathcal{T} is a triangulation of the set $\mathcal{D}_{\mathcal{T}}$.

We will introduce and use some standard triangulations in this paper; see Definition 4.8. For some figures of these triangulations together with a discussion on how they can be efficiently generated, see [19, 10].

Definition 4.2 (CPA function). Let $\mathcal{T} = (\mathfrak{S}_{\nu})$ be a triangulation of a set $\mathcal{D}_{\mathcal{T}} \subset \mathbb{R}^n$. A continuous piecewise affine function $P : \mathcal{D}_{\mathcal{T}} \rightarrow \mathbb{R}$ can be defined by fixing its value at every vertex of the vertex set $\mathcal{V}_{\mathcal{T}}$.

More exactly, assume that for every $\mathbf{x} \in \mathcal{V}_{\mathcal{T}}$ we are given a number $P_{\mathbf{x}} \in \mathbb{R}$. Then we can uniquely define a continuous function $P : \mathcal{D}_{\mathcal{T}} \rightarrow \mathbb{R}$ through the following:

- (i) $P(\mathbf{x}) := P_{\mathbf{x}}$ for every $\mathbf{x} \in \mathcal{V}_{\mathcal{T}}$,
- (ii) P is affine on every simplex $\mathfrak{S}_{\nu} \in \mathcal{T}$, i.e., there is a vector $\mathbf{a}_{\nu} \in \mathbb{R}^n$ and a number $b_{\nu} \in \mathbb{R}$, such that $P(\mathbf{x}) = \mathbf{a}_{\nu}^T \mathbf{x} + b_{\nu}$ for all $\mathbf{x} \in \mathfrak{S}_{\nu}$.

The set of all such continuous piecewise affine functions $\mathcal{D} \rightarrow \mathbb{R}$ fulfilling (i) and (ii) is denoted by $\text{CPA}[\mathcal{T}]$ or $\text{CPA}[(\mathfrak{S}_{\nu})]$.

For every $\mathfrak{S}_{\nu} \in \mathcal{T}$ we define $\nabla P_{\nu} = \nabla P|_{\mathfrak{S}_{\nu}} := \mathbf{a}_{\nu}$, where $\mathbf{a}_{\nu} \in \mathbb{R}^n$ is as in (ii). Note that $\nabla P_{\nu} = \mathbf{a}_{\nu}$ is unique for every simplex \mathfrak{S}_{ν} .

Remark 4.3. If $\mathbf{x} \in \mathfrak{S}_{\nu} = \text{co}(\mathbf{x}_0^{\nu}, \mathbf{x}_1^{\nu}, \dots, \mathbf{x}_n^{\nu}) \in \mathcal{T}$, then \mathbf{x} can be written uniquely as a convex combination $\mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}_i^{\nu}$, $0 \leq \lambda_i \leq 1$ for all $i = 0, 1, \dots, n$, and $\sum_{i=0}^n \lambda_i = 1$ of the vertices of \mathfrak{S}_{ν} and

$$P(\mathbf{x}) = P\left(\sum_{i=0}^n \lambda_i \mathbf{x}_i^{\nu}\right) = \sum_{i=0}^n \lambda_i P(\mathbf{x}_i^{\nu}) = \sum_{i=0}^n \lambda_i P_{\mathbf{x}_i^{\nu}}.$$

For a triangulation we can define the set of shape matrices, built from the vectors that span each simplex. They will be used to define a family of regular triangulations below.

Definition 4.4. Let $\mathcal{T} = (\mathfrak{S}_{\nu})$ be a triangulation.

- (i) For a simplex $\mathfrak{S}_{\nu} = \text{co}(\mathbf{x}_0^{\nu}, \mathbf{x}_1^{\nu}, \dots, \mathbf{x}_n^{\nu}) \in \mathcal{T}$, we define its diameter h_{ν} through

$$h_{\nu} := \max_{\mathbf{x}, \mathbf{y} \in \mathfrak{S}_{\nu}} \|\mathbf{x} - \mathbf{y}\|_2.$$

- (ii) For a simplex $\mathfrak{S}_{\nu} = \text{co}(\mathbf{x}_0^{\nu}, \mathbf{x}_1^{\nu}, \dots, \mathbf{x}_n^{\nu}) \in \mathcal{T}$, we define its shape matrix $X_{\nu} \in \mathbb{R}^{n \times n}$ through

$$X_{\nu} := (\mathbf{x}_1^{\nu} - \mathbf{x}_0^{\nu}, \mathbf{x}_2^{\nu} - \mathbf{x}_0^{\nu}, \dots, \mathbf{x}_n^{\nu} - \mathbf{x}_0^{\nu})^T.$$

Thus, the matrix X_{ν} is defined by writing the entities of the vector $\mathbf{x}_i^{\nu} - \mathbf{x}_0^{\nu}$ in the i th row of X_{ν} for $i = 1, 2, \dots, n$.

- (iii) We refer to the set $\{X_{\nu} : \mathfrak{S}_{\nu} \in \mathcal{T}\}$ as the shape matrices of the triangulation \mathcal{T} .

It is not difficult to derive a formula for ∇P_{ν} in terms of the shape matrix X_{ν} of \mathfrak{S}_{ν} and the values of P at the vertices of \mathfrak{S}_{ν} ; cf. [11, Remark 9].

Lemma 4.5. Let \mathcal{T} be a triangulation, let $P \in \text{CPA}[\mathcal{T}]$, and let $\mathfrak{S}_{\nu} = \text{co}(\mathbf{x}_0^{\nu}, \mathbf{x}_1^{\nu}, \dots, \mathbf{x}_n^{\nu}) \in \mathcal{T}$. Then $\nabla P_{\nu} = X_{\nu}^{-1} \mathbf{p}$, where $\mathbf{p} = (p_1, p_2, \dots, p_n)^T$ is a column vector with $p_i := P_{\mathbf{x}_i^{\nu}} - P_{\mathbf{x}_0^{\nu}}$ for $i = 1, 2, \dots, n$.

We will need triangulations which become finer and finer with a parameter ρ , but still stay regular, in the sense that the opening angles of the corners of the simplices remain bounded

from below. The precise conditions for a set of triangulation to be regular are given below.

Definition 4.6. Let $\mathcal{T} = (\mathfrak{S}_\nu)$ be a triangulation. Let $h, d > 0$ be constants. The triangulation \mathcal{T} is said to be (h, d) -bounded if it fulfills the following:

- (i) The diameter of every simplex $\mathfrak{S}_\nu \in \mathcal{T}$ is bounded by h , i.e.,

$$h_\nu = \text{diam}(\mathfrak{S}_\nu) := \max_{\mathbf{x}, \mathbf{y} \in \mathfrak{S}_\nu} \|\mathbf{x} - \mathbf{y}\|_2 < h.$$

- (ii) The degeneracy of every simplex $\mathfrak{S}_\nu \in \mathcal{T}$ is bounded by d in the sense that

$$h_\nu \|X_\nu^{-1}\|_1 \leq d,$$

where X_ν is the shape matrix of the simplex \mathfrak{S}_ν .

We will focus on a particular type of parameterized triangulations $\mathcal{T}_\rho^{\text{std}}$, $\rho > 0$, that can be computed easily. We will later show that $\mathcal{T}_\rho^{\text{std}}$ is an (h, d) -bounded triangulation, where $h > \sqrt{n}\rho$ and d only depends on the dimension n and not on h or ρ .

Remark 4.7. For the construction of our triangulations we use the set S_n of all permutations of the numbers $1, 2, \dots, n$, the characteristic function $\chi_{\mathcal{J}}(i)$ equal to one if $i \in \mathcal{J}$ and equal to zero if $i \notin \mathcal{J}$, and the standard orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ of \mathbb{R}^n . Further, we use the functions $\mathbf{R}^{\mathcal{J}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined for every $\mathcal{J} \subset \{1, 2, \dots, n\}$ by

$$\mathbf{R}^{\mathcal{J}}(\mathbf{x}) := \sum_{i=1}^n (-1)^{\chi_{\mathcal{J}}(i)} x_i \mathbf{e}_i.$$

$\mathbf{R}^{\mathcal{J}}(\mathbf{x})$ puts a minus in front of the coordinate x_i of \mathbf{x} whenever $i \in \mathcal{J}$.

Definition 4.8 (standard triangulations). We are interested in two general triangulations \mathcal{T}^{std} and $\mathcal{T}_\rho^{\text{std}}$ of \mathbb{R}^n .

- (i) The triangulation \mathcal{T}^{std} consists of the simplices

$$\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \text{co}(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \dots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma})$$

for all $\mathbf{z} \in \mathbb{N}_0^n$, all $\mathcal{J} \subset \{1, 2, \dots, n\}$, and all $\sigma \in S_n$ (cf. Remark 4.7 for notation), where

$$(4.1) \quad \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} := \mathbf{R}^{\mathcal{J}} \left(\mathbf{z} + \sum_{j=1}^i \mathbf{e}_{\sigma(j)} \right) \text{ for } i = 0, 1, 2, \dots, n.$$

- (ii) Now choose a constant $\rho > 0$ and scale the simplices in the triangulation \mathcal{T}^{std} by the mapping $\mathbf{x} \mapsto \rho \mathbf{x}$. We denote by $\mathcal{T}_\rho^{\text{std}}$ the resulting set of n -simplices, i.e., $\text{co}(\rho \mathbf{x}_0, \rho \mathbf{x}_1, \dots, \rho \mathbf{x}_n) \in \mathcal{T}_\rho^{\text{std}}$ whenever $\text{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathcal{T}^{\text{std}}$.

The following lemma follows directly by step 3 in the proof of [11, Theorem 5].

Lemma 4.9. Consider the triangulation $\mathcal{T}_\rho^{\text{std}}$ of \mathbb{R}^n as in Definition 4.8. It is an (h, d) -bounded triangulation (cf. Definition 4.6), where $h > \sqrt{n}\rho$ and d is a constant only dependent on n .

One can show for \mathcal{T}^{std} that indeed $\|X_\nu^{-1}\|_2 \leq 4$ for all $n \in \mathbb{N}$. Thus $\mathcal{T}_\rho^{\text{std}}$ is an $(h, 4n)$ -bounded triangulation of \mathbb{R}^n with $h > \sqrt{n}\rho$.

For a given set $\mathcal{M} \in \mathfrak{N}^n$ and a given triangulation \mathcal{T} , we consider the smallest triangulation covering \mathcal{M} .

Definition 4.10. Let $\mathcal{M} \subset \mathbb{R}^n$, $\mathcal{M} \in \mathfrak{N}^n$, and \mathcal{T} be an (h, d) -bounded triangulation such that $\mathcal{D}_{\mathcal{T}} \supset \mathcal{M}$. We define the triangulation, restricted to \mathcal{M} by

$$(4.2) \quad \mathcal{T}^{\mathcal{M}} := \{\mathfrak{S} \in \mathcal{T} : \mathfrak{S} \cap \mathcal{M}^\circ \neq \emptyset\}.$$

Choosing the standard triangulation $\mathcal{T}_\rho^{\text{std}}$ as \mathcal{T} , we denote its restriction to \mathcal{M} by $\mathcal{T}_\rho^{\mathcal{M}}$.

Corollary 4.11. Let $\mathcal{M} \subset \mathbb{R}^n$, $\mathcal{M} \in \mathfrak{N}^n$, and \mathcal{T} be an (h, d) -bounded triangulation such that $\mathcal{D}_{\mathcal{T}} \supset \mathcal{M}$. Then we have the following:

- (i) $\mathcal{T}^{\mathcal{M}}$ is an (h, d) -bounded triangulation,
- (ii) $\mathcal{D}_{\mathcal{T}^{\mathcal{M}}} \in \mathfrak{N}^n$,
- (iii) $\mathcal{M} \subset \mathcal{D}_{\mathcal{T}^{\mathcal{M}}} \subset \mathcal{M}_h$.

Note that $\mathcal{T}_\rho^{\text{std}}$ as in Definition 4.10 with $h > \sqrt{n} \rho$ and d the constant from Lemma 4.9, is an example for \mathcal{T} .

Proof. Statements (i) and (iii) are obvious: for the inclusion $\mathcal{M} \subset \mathcal{D}_{\mathcal{T}^{\mathcal{M}}}$ let $\mathbf{x} \in \mathcal{M}^\circ$, then there exists an $\mathfrak{S} \in \mathcal{T}$ such that $\mathbf{x} \in \mathfrak{S}$. By definition, $\mathfrak{S} \in \mathcal{T}^{\mathcal{M}}$. As $\overline{\mathcal{M}^\circ} = \mathcal{M}$ and $\mathcal{D}_{\mathcal{T}^{\mathcal{M}}}$ are compact, the inclusion follows. Statement (ii) follows by [11, Lemma 2]. $\mathcal{T}_\rho^{\mathcal{M}}$ is an (h, d) -bounded triangulation by Lemma 4.9. ■

In the next corollary (for a schematic illustration, see Figure 2), we want to construct a triangulation \mathcal{T} that is between two sets \mathcal{D}_I and \mathcal{D}_O in the sense that $\mathcal{D}_I \subset \mathcal{D}_{\mathcal{T}} \subset \mathcal{D}_O$. Moreover, we want to restrict a given triangulation $\mathcal{T}^{\mathcal{M}}$ further to $\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}$ by removing some simplices near the origin. More precisely, we keep the ones that have nonempty intersection with the complement of \mathcal{F} , where \mathcal{F} is a small neighborhood of the origin. In particular, we want to ensure that the remaining simplices $\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}$ lie between the two sets \mathcal{F}_I and $\mathcal{F}_O = \mathcal{F}$ as well, i.e., $\mathcal{D}_I \setminus \mathcal{F}_O \subset \mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}} \subset \mathcal{D}_O \setminus \mathcal{F}_I$. An example of such a triangulation is given by starting with the standard triangulation $\mathcal{T} = \mathcal{T}_\rho^{\text{std}}$.

Corollary 4.12. Let $\mathcal{F}_I, \mathcal{F}_O =: \mathcal{F}, \mathcal{M} := \mathcal{D}_I, \mathcal{D}_O \in \mathfrak{N}^n$, where $\mathcal{F}_I \subset \mathcal{F}_O^\circ$, $\mathcal{F}_O \subset \mathcal{D}_I^\circ$, and $\mathcal{D}_I \subset \mathcal{D}_O^\circ$. Define

$$(4.3) \quad h^* := \min(d(\mathcal{F}_I, (\mathcal{F}_O)^C), d(\mathcal{F}_O, (\mathcal{D}_I)^C), d(\mathcal{D}_I, (\mathcal{D}_O)^C)),$$

where $d(A, B) = \inf_{\mathbf{x} \in A, \mathbf{y} \in B} \|\mathbf{x} - \mathbf{y}\|_2$. Let $0 < h \leq h^*$.

Let \mathcal{T} be an (h, d) -bounded triangulation such that $\mathcal{D}_{\mathcal{T}} \supset \mathcal{M}$. Define

$$\begin{aligned} \mathcal{T}_{\mathcal{F}}^{\mathcal{M}} &:= \bigcup \{\mathfrak{S} \in \mathcal{T}^{\mathcal{M}} : \mathfrak{S} \cap \mathcal{F}^C \neq \emptyset\} \\ &= \bigcup \{\mathfrak{S} \in \mathcal{T} : \mathfrak{S} \cap \mathcal{M}^\circ \neq \emptyset \text{ and } \mathfrak{S} \cap \mathcal{F}^C \neq \emptyset\}. \end{aligned}$$

Then

- $\mathcal{T}^{\mathcal{M}}$ is an (h, d) -bounded triangulation of $\mathcal{D}_{\mathcal{T}^{\mathcal{M}}}$ and $\mathcal{D}_I \subset \mathcal{D}_{\mathcal{T}^{\mathcal{M}}} \subset \mathcal{D}_O$ holds.
- $\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}$ is an (h, d) -bounded triangulation of $\mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}}$ and we have

$$\mathcal{D}_I \setminus \mathcal{F}_O \subset \mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}} \subset \mathcal{D}_O \setminus \mathcal{F}_I.$$

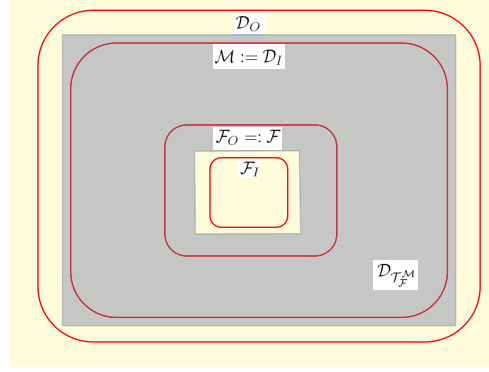


Figure 2. Schematic figures of the sets $\mathcal{F}_I \subset \mathcal{F}_O \subset \mathcal{D}_I \subset \mathcal{D}_O$ in Corollary 4.12 (red outer boundaries). The shaded area is the set $\mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}}$ that is triangulated by $\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}$. Note that because $\mathcal{D}_I \setminus \mathcal{F}_O \subset \mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}} \subset \mathcal{D}_O \setminus \mathcal{F}_I$, the triangulated area covers the set $\mathcal{M} \setminus \mathcal{F}$.

Moreover, for any $0 < \rho < h/\sqrt{n}$ with d the constant from Lemma 4.9 we have that $\mathcal{T} := \mathcal{T}_{\rho}^{\text{std}}$ is a triangulation satisfying the assumptions on \mathcal{T} .

For a schematic figure of the sets $\mathcal{F}_I, \mathcal{F}_O, \mathcal{D}_I, \mathcal{D}_O$ and the area $\mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}}$ triangulated by $\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}$, see Figure 2.

Proof. Most statements follow directly from Corollary 4.11. The inclusion $\mathcal{D}_I \subset \mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}} \subset \mathcal{D}_O$ and thus $\mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}} \subset \mathcal{D}_O$ follows from (iii) of that corollary. To show that $\mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}} \subset \mathcal{D}_O \setminus \mathcal{F}_I$, let $\mathcal{S} \in \mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}}$, i.e., $\mathcal{S} \cap \mathcal{F}^C \neq \emptyset$. Hence, there exists an $\mathbf{x} \in \mathcal{S} \setminus \mathcal{F}_O$ and for all $\mathbf{y} \in \mathcal{S}$ we have $\|\mathbf{x} - \mathbf{y}\|_2 < h$, i.e., $\mathbf{y} \notin \mathcal{F}_I$. This shows that $\mathcal{S} \cap \mathcal{F}_I = \emptyset$.

To show the inclusion $\mathcal{D}_I \setminus \mathcal{F}_O \subset \mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}}$, let $\mathbf{x} \in \mathcal{D}_I \setminus \mathcal{F}$. Since $\mathcal{D}_{\mathcal{T}_{\mathcal{F}}^{\mathcal{M}}} \supset \mathcal{D}_I$, there is an $\mathcal{S} \in \mathcal{T}^{\mathcal{M}}$ with $\mathbf{x} \in \mathcal{S}$. Since $\mathbf{x} \notin \mathcal{F}$, $\mathcal{S} \in \mathcal{T}_{\mathcal{F}}^{\mathcal{M}}$. ■

4.2. CPA interpolation. In this section we define the CPA interpolation of a function P by choosing the values of P at the vertices of a given triangulation, and thus defining a CPA function.

Definition 4.13. Let $P : \mathcal{U} \rightarrow \mathbb{R}$, $\mathcal{U} \subset \mathbb{R}^n$, be a function and let \mathcal{T} be a triangulation with $\mathcal{D}_{\mathcal{T}} \subset \mathcal{U}$. We call the function $Q \in \text{CPA}[\mathcal{T}]$, where $Q_{\mathbf{x}} := P(\mathbf{x})$ for every $\mathbf{x} \in \mathcal{V}_{\mathcal{T}}$, the CPA interpolation of the function P on \mathcal{T} . We denote the CPA interpolation of P by P_C .

The next lemma shows that if finitely many inequalities (4.5) at the vertices are satisfied, then the orbital derivative is negative for all points in a simplex. This implies later that the Dini orbital derivative is negative, i.e., we can show that V is a Lyapunov function.

Lemma 4.14. Consider the system from Definition 2.1 and assume that $\sigma \geq 2$. Let \mathcal{T} be a triangulation and let $V \in \text{CPA}[\mathcal{T}]$.

Fix $\mathcal{S}_{\nu} \in \mathcal{T}$ and define

$$(4.4) \quad B_{\nu} \geq \max_{\substack{m,r,s=1,2,\dots,n \\ \mathbf{z} \in \mathcal{S}_{\nu}}} \left| \frac{\partial^2 f_m}{\partial x_r \partial x_s}(\mathbf{z}) \right|.$$

Assume there exists a constant $\alpha > 0$ such that for every vertex $\mathbf{x}_i \in \mathcal{S}_{\nu}$

$$(4.5) \quad \nabla V_{\nu} \cdot \mathbf{f}(\mathbf{x}_i) + n B_{\nu} h_{\nu}^2 \|\nabla V_{\nu}\|_1 \leq -\alpha$$

holds true. Then we have

$$\nabla V_\nu \cdot \mathbf{f}(\mathbf{x}) \leq -\alpha \quad \text{for all } \mathbf{x} \in \mathfrak{S}_\nu.$$

Proof. The proof follows by [2, Theorem 4.12]. ■

In the next lemma we give an estimate of the error between a C^2 function W and its CPA interpolation W_C . This will be used in Theorem 4.16 to show that the CPA interpolation W_C of a Lyapunov function W is also a Lyapunov function and, moreover, this can be verified as W_C satisfies inequalities of the form (4.5).

Lemma 4.15. *Let $\mathcal{T} = (\mathfrak{S}_\nu)$ be an (h, d) -bounded triangulation, let $\mathcal{D}_O \in \mathfrak{N}^n$, and let $\mathcal{D}_O \supset \mathcal{D}_\mathcal{T}$. Let $W \in C^2(\mathcal{D}_O)$ and define*

$$(4.6) \quad C := 1 + \frac{dn^{3/2}}{2}.$$

Denote by W_C the CPA interpolation of W on \mathcal{T} ; cf. Definition 4.13. Then the following estimates hold true:

$$(4.7) \quad |W_C(\mathbf{x}) - W(\mathbf{x})| \leq \sqrt{nh^2} \|W\|_{C^2(\mathcal{D}_O)} \quad \text{for all } \mathbf{x} \in \mathcal{D}_\mathcal{T},$$

$$(4.8) \quad \|\nabla(W_C)_\nu - \nabla W(\mathbf{x})\|_1 \leq hC \|W\|_{C^2(\mathcal{D}_O)} \quad \text{for all } \mathfrak{S}_\nu \in \mathcal{T} \text{ and all } \mathbf{x} \in \mathfrak{S}_\nu,$$

$$(4.9) \quad \|\nabla(W_C)_\nu\|_1 \leq (1 + hC) \|W\|_{C^2(\mathcal{D}_O)} \quad \text{for all } \mathfrak{S}_\nu \in \mathcal{T}.$$

Proof. Let $\mathbf{x} \in \mathcal{D}_\mathcal{T}$. We use Proposition 4.1, 4.2, and Lemma 4.2 from [1] to obtain

$$|W_C(\mathbf{x}) - W(\mathbf{x})| \leq h^2 \max_{\mathbf{z} \in \mathcal{D}_\mathcal{T}} \|H_W(\mathbf{z})\|_2 \leq \sqrt{nh^2} \|W\|_{C^2(\mathcal{D}_O)},$$

where $H_W(\mathbf{z})$ denotes the Hessian of W at \mathbf{z} ; this shows (4.7).

Inequality (4.8) can be shown as in step 4 of the proof of [11, Theorem 5]. The inequality (4.9) follows immediately from (4.8) and $\|\nabla W(\mathbf{x})\|_1 \leq \|W\|_{C^2(\mathcal{D}_O)}$. ■

The following theorem, Theorem 4.16, shows that given a C^2 Lyapunov function W on $\mathcal{M} \setminus \mathcal{F}^\circ$, its CPA interpolation W_C with respect to a triangulation which is fine enough, is not only a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$, too (see (c)), but we can verify this through finitely many inequalities (4.11) at the vertices of the triangulation; see (b). Moreover, by (a), W and W_C are close together.

Theorem 4.16. *Consider the system from Definition 2.1 and assume that $\sigma \geq 2$. Let the sets $\mathcal{F}_I, \mathcal{F}_O =: \mathcal{F}, \mathcal{M} := \mathcal{D}_I, \mathcal{D}_O \in \mathfrak{N}^n$ be as in Corollary 4.12.*

Let $W \in C^2(\mathcal{D}_O)$ and assume that

$$\nabla W(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq -\beta \quad \text{for all } \mathbf{x} \in \mathcal{D}_O \setminus \mathcal{F}_I^\circ$$

holds for a constant $\beta > 0$. Let

$$B \geq \max_{\substack{m, r, s=1, 2, \dots, n \\ \mathbf{z} \in \mathcal{D}_O}} \left| \frac{\partial^2 f_m}{\partial x_r \partial x_s}(\mathbf{z}) \right|,$$

and let $c, d > 0$ and $0 < \alpha < \beta$ be arbitrary constants.

Then there exists a constant $h > 0$ such that for every (h, d) -bounded triangulation \mathcal{T} with $\mathcal{D}_\mathcal{T} \supset \mathcal{M}$ we have for the CPA interpolation $W_C \in \text{CPA}[\mathcal{T}]$ of W :

(a)

$$(4.10) \quad |W_C(\mathbf{x}) - W(\mathbf{x})| \leq c \quad \text{for all } \mathbf{x} \in \mathcal{D}_{\mathcal{T}^M}.$$

(b) For every $\mathfrak{S}_\nu \in \mathcal{T}_{\mathcal{F}}^M$ and every vertex \mathbf{x}_i of \mathfrak{S}_ν we have

$$(4.11) \quad \nabla(W_C)_\nu \cdot \mathbf{f}(\mathbf{x}_i) + nBh_\nu^2 \|\nabla(W_C)_\nu\|_1 \leq -\alpha.$$

(c) Inequality (4.11) implies

$$D^+(W_C)(\mathbf{x}) \leq -\alpha \quad \text{for all } \mathbf{x} \in \mathcal{D}_{\mathcal{T}_{\mathcal{F}}^M}^\circ.$$

Moreover, for any $0 < \rho < h/\sqrt{n}$ we have that $\mathcal{T} := \mathcal{T}_\rho^{std}$ is a triangulation satisfying the assumptions on \mathcal{T} with d the constant from Lemma 4.9.

Proof. Define $F := \max_{\mathbf{x} \in \mathcal{D}_O} \|\mathbf{f}(\mathbf{x})\|_\infty$ and let the constant C be defined as in Lemma 4.15. Let $h > 0$ be so small that both

$$(4.12) \quad h\|W\|_{C^2(\mathcal{D}_O)}CF + nBh^2(1+hC)\|W\|_{C^2(\mathcal{D}_O)} \leq \beta - \alpha$$

$$(4.13) \quad \text{and } \sqrt{n}h^2\|W\|_{C^2(\mathcal{D}_O)} \leq c \text{ hold}$$

and that $h \leq h^*$, where $h^* > 0$ is defined in (4.3) of Corollary 4.12, so that every (h, d) -bounded triangulation \mathcal{T} with $\mathcal{D}_{\mathcal{T}} \supset \mathcal{M}$ satisfies $\mathcal{D}_{\mathcal{T}^M} \subset \mathcal{D}_O$ and $\mathcal{D}_I \setminus \mathcal{F}_O \subset \mathcal{D}_{\mathcal{T}_{\mathcal{F}}^M} \subset \mathcal{D}_O \setminus \mathcal{F}_I$; the same corollary also shows that \mathcal{T}_ρ^{std} satisfies these assumption.

Now (4.10) follows from Lemma 4.15, in particular (4.7), applied to \mathcal{T}^M , which shows (a) using (4.13).

Further, to show (b), let \mathbf{x}_i be an arbitrary vertex of an arbitrary simplex $\mathfrak{S}_\nu \subset \mathcal{D}_{\mathcal{T}_{\mathcal{F}}^M} \subset \mathcal{D}_O \setminus \mathcal{F}_I$. Then by Lemma 4.15, in particular (4.8) and (4.9), applied to $\mathcal{T}_{\mathcal{F}}^M$, we have

$$\begin{aligned} \nabla(W_C)_\nu \cdot \mathbf{f}(\mathbf{x}_i) &\leq \nabla W(\mathbf{x}_i) \cdot \mathbf{f}(\mathbf{x}_i) + \|\nabla(W_C)_\nu - \nabla W(\mathbf{x}_i)\|_1 \|\mathbf{f}(\mathbf{x}_i)\|_\infty \\ &\leq -\beta + h\|W\|_{C^2(\mathcal{D}_O)}CF \end{aligned}$$

and

$$nBh_\nu^2 \|\nabla(W_C)_\nu\|_1 \leq nBh^2(1+hC)\|W\|_{C^2(\mathcal{D}_O)}.$$

Together, the last two inequalities deliver with (4.12)

$$\begin{aligned} \nabla(W_C)_\nu \cdot \mathbf{f}(\mathbf{x}_i) + nBh_\nu^2 \|\nabla(W_C)_\nu\|_1 &\leq -\beta + h\|W\|_{C^2(\mathcal{D}_O)}CF \\ &\quad + nBh^2(1+hC)\|W\|_{C^2(\mathcal{D}_O)} \\ &\leq -\alpha, \end{aligned}$$

i.e., (4.11).

To show (c), note that the inequality (4.11) implies (4.5), so that, using Lemma 4.14, this shows $\nabla(W_C)_\nu \cdot \mathbf{f}(\mathbf{x}) \leq -\alpha$ for all $\mathbf{x} \in \mathfrak{S}_\nu$ and all $\mathfrak{S}_\nu \in \mathcal{T}_{\mathcal{F}}^M$. The statement (c) can now be shown following the argumentation of [8, Proof of Theorem 2.6]. \blacksquare

5. RBF-CPA Lyapunov function. In this section we combine the RBF approximation with the CPA interpolation. We first approximate the function V by the RBF approximation V_R . Then we interpolate V_R to obtain a CPA interpolation V_C of V_R . The first step involves a set of RBF collocation points with fill distance $h_R \leq h_R^*$, and the second step an (h_C^*, d) -bounded triangulation. If h_R^* and h_C^* are small enough, then V_C fulfills finitely many inequalities (5.1) which can be verified and imply that V_C is a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$, in particular $\mathcal{L}_{V_C}^{\text{inf}} \neq \emptyset$. Given any small set \mathcal{S} and large set $\mathcal{C} \subset \mathcal{A}$, we have $\mathcal{L}_{V_C}^{\text{inf}} \subset \mathcal{S}$ and $\mathcal{C} \subset \mathcal{L}_{V_C}^{\text{sup}}$.

Theorem 5.1. *Let Assumptions 3.5 hold. Let $\mathcal{S}, \mathcal{C} \in \mathfrak{N}^n$, $\mathcal{S} \subset \mathcal{C}^\circ \subset \mathcal{C} \subset \mathcal{A}$.*

Then there are sets $\mathcal{F}, \mathcal{M} \in \mathfrak{N}^n$ and an open and bounded set \mathcal{B} with $\mathcal{F} \subset \mathcal{S}$ and $\mathcal{C} \subset \mathcal{M} \subset \mathcal{B} \subset \mathcal{A}$ and constants $h_C^, h_R^* > 0$, such that*

- *for all RBF approximations $V_R \in C^{2k-1}(\mathbb{R}^n, \mathbb{R})$ of the Lyapunov function V of Theorem 2.8 with respect to the RBF ψ and the collocation points $X \subset \mathcal{B} \setminus \{\mathbf{0}\}$ with fill distance $h_R \leq h_R^*$; cf. Definition 3.7*
- *and all CPA interpolations V_C of the above V_R with respect to an (h_C^*, d) -bounded triangulation \mathcal{T} such that $\mathcal{D}_{\mathcal{T}} \supset \mathcal{M}$.*

We also have the following:

- (a) *for every $\mathfrak{S}_\nu \in \mathcal{T}_{\mathcal{F}}^{\mathcal{M}}$ and every vertex \mathbf{x}_i of \mathfrak{S}_ν :*

$$(5.1) \quad \nabla(V_C)_\nu \cdot \mathbf{f}(\mathbf{x}_i) + nBh_\nu^2 \|\nabla(V_C)_\nu\|_1 \leq -\alpha$$

holds, where

$$B \geq \sup_{\substack{m,r,s=1,2,\dots,n \\ \mathbf{z} \in \mathcal{B}}} \left| \frac{\partial^2 f_m}{\partial x_r \partial x_s}(\mathbf{z}) \right|,$$

- (b) $\emptyset \neq \mathcal{L}_{V_C}^{\text{inf}} \subset \mathcal{S}$,

- (c) *note that (a) and (b) imply that V_C is a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$,*

- (d) $\mathcal{L}_{V_C}^{\text{sup}} \supset \mathcal{C}$.

Moreover, for any $0 < \rho < h_C^/\sqrt{n}$ we have that $\mathcal{T} := \mathcal{T}_\rho^{\text{std}}$ is a triangulation satisfying the assumptions on \mathcal{T} with d the constant from Lemma 4.9.*

Proof. Step 1: *Definition of sets.* Denote sublevel sets of the Lyapunov function V by

$$K_{V,m} = \{\mathbf{x} \in \mathcal{A} : V(\mathbf{x}) \leq m\};$$

by Theorem 2.8 they are all in \mathfrak{N}^n and with a C^1 boundary for any $m > 0$.

Since $V(\mathbf{0}) = 0$, $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$, and V is continuous, there is $l := \min_{\mathbf{x} \in \partial \mathcal{S}} V(\mathbf{x}) > 0$ such that $K_{V,l} \subset \mathcal{S}$. We show that $K_{V,l}^\circ \subset \mathcal{S}^\circ$, from which $K_{V,l} \subset \mathcal{S}$ follows. Indeed, assume in contradiction that $\mathbf{x} \in K_{V,l}^\circ \setminus \mathcal{S}^\circ$, then there exists a continuous map $\gamma: [0, 1] \rightarrow K_{V,l}^\circ$ such that $\gamma(0) = \mathbf{0}$ and $\gamma(1) = \mathbf{x}$. Since $\gamma(0) \in \mathcal{S}^\circ$ and $\gamma(1) \notin \mathcal{S}^\circ$, there exists a $\theta \in [0, 1]$ such that $\gamma(\theta) \in \partial \mathcal{S}$. We have $V(\gamma(\theta)) < l$, since $\gamma(\theta) \in K_{V,l}^\circ$ and $V(\gamma(\theta)) \geq \min_{\mathbf{x} \in \partial \mathcal{S}} V(\mathbf{x}) = l$, which is a contradiction.

There is also $L := \max_{\mathbf{x} \in \mathcal{C}} V(\mathbf{x}) > l$ such that $K_{V,L} \supset \mathcal{C}$. Choose $c > 0$ so small that $c \leq \frac{l}{7}$. Set $m := l - 2c$ and $M := L + 3c$. We have $m - 4c = l - 6c \geq \frac{l}{7} > 0$. Moreover, set $\mathcal{F} := K_{V,m-3c}$ and $\mathcal{M} := K_{V,M+3c}$. Then $K_{V,k} = \overline{\mathcal{L}_{V,k}} \neq \emptyset$ for all $m - 3c < k < M + 3c$,

noting that all of these sets are in \mathfrak{N}^n and homeomorphic to the unit ball; for the definition of $\mathcal{L}_{V,k}$, see Definition 2.3. We define the following sets, in decreasing order:

$$\begin{aligned}\mathcal{B} &:= K_{V,M+5c}^\circ, \\ \mathcal{D}_O &:= K_{V,M+4c}, \\ \mathcal{M} &:= \mathcal{D}_I := K_{V,M+3c}, \\ \text{note that } \mathcal{C} &\subset \overline{\mathcal{L}_{V,M-3c}} \subset \mathcal{L}_{V,M-2c}, \\ \mathcal{S} &\supset \overline{\mathcal{L}_{V,m+2c}}, \\ \mathcal{F} &:= \mathcal{F}_O := K_{V,m-3c}, \\ \mathcal{F}_I &:= K_{V,m-4c}.\end{aligned}$$

Furthermore, set

$$(5.2) \quad \varepsilon := \frac{1}{3} \min_{\mathbf{x} \in \mathcal{D}_O \setminus \mathcal{F}_I^\circ} p(\mathbf{x})q(\mathbf{x}) > 0.$$

Step 2: RBF approximation. Choose $h_R^* > 0$ so small that Theorem 3.9 holds with \mathcal{B} , which is bounded, open, forward invariant, and has a C^1 boundary, $\mathcal{K} = \mathcal{D}_O$ and $\delta = \min(\varepsilon, c)$. Then

$$(5.3) \quad |V(\mathbf{x}) - V_R(\mathbf{x})| \leq c \quad \text{and} \quad |V'(\mathbf{x}) - V'_R(\mathbf{x})| \leq \varepsilon$$

for all $\mathbf{x} \in \mathcal{D}_O$. By the first inequality of (5.3) and Lemma 2.4 we get

$$(5.4) \quad \mathcal{F}_O \subset \mathcal{L}_{V,m-2c} \subset \mathcal{L}_{V_R,m-c} \subset \mathcal{L}_{V,m} \subset \mathcal{L}_{V_R,m+c} \subset \mathcal{L}_{V,m+2c} \subset \mathcal{S}$$

since the sets $\mathcal{L}_{V,\cdot}$ in (5.4) are not empty and

$$(5.5) \quad \mathcal{C} \subset \mathcal{L}_{V,M-2c} \subset \mathcal{L}_{V_R,M-c} \subset \mathcal{L}_{V,M} \subset \mathcal{L}_{V_R,M+c} \subset \mathcal{L}_{V,M+2c} \subset \mathcal{D}_I$$

since the sets $\mathcal{L}_{V,\cdot}$ in (5.5) are not empty. By the second inequality of (5.3) we obtain

$$\begin{aligned}V'_R(\mathbf{x}) &\leq V'(\mathbf{x}) + \varepsilon \leq -2\varepsilon \text{ by (5.2) and (2.4)} \\ &=: -\beta \quad \text{for all } \mathbf{x} \in \mathcal{D}_O \setminus \mathcal{F}_I^\circ.\end{aligned}$$

Step 3: Triangulation. We choose h_C^* as the minimum of h^* of Corollary 4.12 and h of Theorem 4.16 with $\mathcal{F}_I, \mathcal{F}_O, \mathcal{D}_I, \mathcal{D}_O \in \mathfrak{N}^n$ and $c > 0$ as above, $\alpha = \varepsilon$, $\beta = 2\varepsilon$, and $W = V_R$. Then by Corollary 4.12 for every (h_C^*, d) -bounded triangulation \mathcal{T} such that $\mathcal{D}_{\mathcal{T}} \supset \mathcal{M}$ we have $\mathcal{D}_I \subset \mathcal{D}_{\mathcal{T}\mathcal{M}}$ and $\mathcal{D}_I \setminus \mathcal{F}_O \subset \mathcal{D}_{\mathcal{T}\mathcal{M}} \subset \mathcal{D}_O \setminus \mathcal{F}_I$. The standard triangulation \mathcal{T}_ρ^{std} with $0 < \rho < h_C^*/\sqrt{n}$ is an example for such a triangulation with d the constant from Lemma 4.9.

Step 4: CPA interpolation. We now apply Theorem 4.16 with $\mathcal{F}_I, \mathcal{F}_O, \mathcal{D}_I, \mathcal{D}_O \in \mathfrak{N}^n$ and $c > 0$ as above, $\alpha = \varepsilon$, $\beta = 2\varepsilon$, and $W = V_R$. Then (4.10) holds in $\mathcal{D}_I \subset \mathcal{D}_{\mathcal{T}\mathcal{M}}$ and thus with Lemma 2.4 and (5.4) we get

$$(5.6) \quad \mathcal{F}_O \subset \mathcal{L}_{V_R,m-c} \subset \mathcal{L}_{V_C,m} \subset \mathcal{L}_{V_R,m+c} \subset \mathcal{S}$$

since the sets $\mathcal{L}_{V_R, \cdot}$ are not empty, which implies that $\emptyset \neq \mathcal{L}_{V_C}^{\text{inf}} \subset \mathcal{L}_{V_C, m} \subset \mathcal{S}$, proving (b). Moreover, using Lemma 2.4 and (5.5) we obtain

$$(5.7) \quad \mathcal{C} \subset \mathcal{L}_{V_R, M-c} \subset \mathcal{L}_{V_C, M} \subset \mathcal{L}_{V_R, M+c} \subset \mathcal{D}_I$$

since the sets $\mathcal{L}_{V_R, \cdot}$ are not empty, which implies that $\mathcal{L}_{V_C}^{\text{sup}} \supset \mathcal{L}_{V_C, M} \supset \mathcal{C}$, proving (d). The inequality (4.11) shows for every $\mathfrak{S}_\nu \in \mathcal{T}_{\mathcal{F}}^M$ and every vertex \mathbf{x}_i of \mathfrak{S}_ν

$$(5.8) \quad \nabla(V_C)_\nu \cdot \mathbf{f}(\mathbf{x}_i) + nBh_\nu^2 \|\nabla(V_C)_\nu\|_1 \leq -\varepsilon,$$

proving (a).

In particular, by Theorem 4.16 (c), $D^+(V_C)(\mathbf{x}) \leq -\varepsilon$ holds for all $\mathbf{x} \in \mathcal{D}_{\mathcal{T}_{\mathcal{F}}}^\circ$. Since $\mathcal{M} \setminus \mathcal{F}^\circ \subset \mathcal{D}_{\mathcal{T}_{\mathcal{F}}}^\circ$, V_C is a Lyapunov function on $\mathcal{M} \setminus \mathcal{F}^\circ$, proving (c). \blacksquare

Theorem 5.1 requires the collocation points for the RBF approximation as well as the triangulation for the CPA interpolation to be *uniformly* dense in the sense that the fill distance of X is smaller than h_R^* and the triangulation is an (h_C^*, d) triangulation. In practice, we start with a coarse set of collocation points and a coarse triangulation and refine them until the estimate (5.1) holds. In contrast to *uniform* refinement, *local* refinements, only where the estimate (5.1) is violated, could result in the same estimate with fewer collocation points and less simplices. While Example 2 in the next section shows that this indeed can work well in examples, the general result in Theorem 5.1 involves global quantities. The following remark explains that local refinements are desirable, but not straightforward.

Remark 5.2 (local refinements). From Lemma 4.15 one easily deduces that the local error in the $\|\cdot\|_{C^1}$ norm of the CPA interpolation of a function is only dependent on the diameter and degeneracy of the local simplices. Thus, it is possible to refine the simplicial complex only in the area where the CPA interpolation fails to have a negative orbital derivative. Note, however, that one must take care that the resulting triangulation is regular in the sense of Definition 4.1 and that its degeneracy in the sense of Definition 4.6 does not increase. Such a refinement is not trivial and is a matter of current research.

For the RBF approximation the estimate (3.13) depends on the fill distance of the whole set of collocation points, which is not a local property. Thus, one cannot deduce by Lemma 3.8 that the local error decreases when the fill distance is smaller locally. However, as the third run in Example 2 suggests, this might be the case and we are currently examining this. For further studies on refinement for the RBF method, see also [27].

In this context, note also that a Lyapunov function is a global object on the basin of attraction of an attractor. One can, in general, not change it locally without influencing the object as a whole. In particular, estimates on the basin of attraction are obtained from the sublevel sets of a Lyapunov function; they are globally influenced by a local change.

6. Examples. The examples were computed on a PC (i4790K@4.6GHz, 32GB RAM). We programmed in Visual C++ Express 2013 and included the matrix library Armadillo [32], that uses LAPACK and BLAS for the actual computations. To generate the pictures we used Scilab [33] for Example 1 and MATLAB for Example 2. All the software used, except MATLAB, is available for free on the internet.

As discussed earlier, our method is guaranteed to construct a Lyapunov function V_C on $\mathcal{M} \setminus \mathcal{F}^\circ$, but, in general, there are simplices near the equilibrium, where V_C has nonnegative

orbital derivative. However, if we can guarantee that \mathcal{F} is in the basin of attraction, then sublevel sets of V_C lie in the basin of attraction; cf. Theorem 2.6 (d).

As the equilibrium is assumed to be exponentially stable, we can show $\mathcal{F} \subset \mathcal{A}$ using a quadratic Lyapunov function for the linearized system near the equilibrium, which is locally a Lyapunov function for the nonlinear system. Such a quadratic Lyapunov function can be computed by standard means by solving the continuous-time Lyapunov equation.

The complexity of our method is the same as the RBF method, i.e., solving a linear system of equations where the number of equations is the same as the collocation points used in the RBF method. To get an idea of how much faster our method is compared with the rigorous CPA method solving a linear optimization problem, cf., e.g., [25], where it takes more than two hours to solve a two-dimensional problem with 3,641 vertices. The second run of Example 1 in this paper with 3,120 points for the RBF method and 80,000 triangles for the CPA verification altogether took 3.11 seconds. A direct comparison of computation times for Example 1 is not possible because the LP problem in the CPA method is not feasible. This is because there is no CPA Lyapunov function on this triangulation.

A further advantage of the combined method is that it delivers information on where the orbital derivative of V_C fails to be negative, whereas the CPA method either constructs V_C such that the orbital derivative is negative everywhere or delivers the message that such a function does not exist.

6.1. Example 1. We consider the following system from [17, Example 2]:

$$(6.1) \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \text{with } \mathbf{f}(x, y) = \begin{pmatrix} y \\ -x + \frac{1}{3}x^3 - y \end{pmatrix}.$$

It is simple to verify that this system has equilibria at $(0, 0)$ and $(\pm\sqrt{3}, 0)$ and that the equilibrium at the origin is exponentially stable.

By linearizing and solving the continuous-time Lyapunov function, the quadratic Lyapunov function $V_q(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$ with

$$P = \frac{1}{2} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

is obtained for the linearized system. In [17, p. 673] a generally applicable method is used to prove that V_q has negative orbital derivative in the cube $[-0.52, 0.52]^2$ for the nonlinear system (6.1). Sublevel sets of V_q in this cube are subsets of the basin of attraction.

We use the Wendland function for $k = 3$,

$$\psi_{5,3}(cr) = (1 - cr)_+^8 [32(cr)^3 + 25(cr)^2 + 8cr + 1]$$

with $c = 1$; for the formulas of ψ_1 and ψ_2 , see [7, Appendix B.1]. We solve the RBF problem

$$V'(\mathbf{x}) = -p(\mathbf{x})q(\mathbf{x}), \quad \text{with } p(\mathbf{x}) = \|\mathbf{x}\|_2^2 \quad \text{and } q(\mathbf{x}) = 1 + \|\mathbf{f}(\mathbf{x})\|_2^2.$$

The collocation points used to solve the RBF problem are the vertices of a hexagonal grid as described in [7, p. 134]. Such a grid is optimal for the RBF method (cf. [21]) in the sense that it has a small fill distance, but at the same time a large separation distance; note that

if two points are close to each other, i.e., the separation distance is small, then the condition number of the interpolation matrix is large.

More exactly, we define the vectors

$$\mathbf{g}_1 = (1, 0)^T, \quad \mathbf{g}_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)^T, \quad \text{and} \quad \mathbf{g}_0 = (\mathbf{g}_1 + \mathbf{g}_2)/2.$$

For a parameter $c_R > 0$ the set of collocation points X_{c_R} consists of all points

$$X_{c_R} := \{\mathbf{x} = c_R(\mathbf{g}_0 + z_1\mathbf{g}_1 + z_2\mathbf{g}_2), z_1, z_2 \in \mathbb{Z}\} \cap]-2.6, 2.6[^2.$$

The fill distance h_R of X_{c_R} is proportional to c_R , and X_{c_R} does not include any equilibrium point due to the shift by \mathbf{g}_0 .

For the CPA method we used triangulations $\mathcal{T}_\rho^\mathcal{M}$ with $\mathcal{M} = [-2.5, 2.5]^2$ and different parameters $\rho > 0$; cf. Definition 4.10. The constants B_ν in the CPA problem were assigned the values

$$B_\nu = 2 \max_{(x,y) \in \mathfrak{S}_\nu} |x| \quad \text{for all simplices } \mathfrak{S}_\nu.$$

We did two different runs with different parameters for both the RBF and the CPA method.

In the first run we set $c_R = 0.2$ and $\rho = 0.05$. The set $X_{0.2}$ contains 780 points and the triangulation $\mathcal{T}_{0.05}^\mathcal{M}$ 20,000 triangles. The RBF problem was created and solved in 0.033 sec., delivering V_R . The triangulation and the inequalities were created in 0.25 sec. and the values of V_R were written in the inequalities of the CPA problem in 0.11 seconds. The negativity of the orbital derivative of V_C was verified in 0.003 sec. with the result that it was negative in all but 428 or 2.14% of the simplices. In Figure 3 some level sets of V_C are drawn in blue and the simplices where the orbital derivative V_C' is nonnegative are marked with red dots. For comparison the largest sublevel set of V_q , $\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^2 : V_q(\mathbf{x}) \leq 0.225\}$, which is guaranteed to be a subset of the basin of attraction, is drawn in black. With these parameters for the RBF and CPA method, we can conclude the positive invariance of the innermost sublevel set of V_C in the figure, but not that it is a subset of the basin of attraction.

In the second run we set $c_R = 0.1$ and $\rho = 0.025$. The set $X_{0.1}$ contains 3,120 points and the triangulation $\mathcal{T}_{0.025}^\mathcal{M}$ 80,000 triangles. The RBF problem was created and solved in 1.19 sec., delivering V_R . The triangulation and the inequalities were created in 1.04 sec. and the values of V_R were written in the inequalities of the CPA problem in 0.87 seconds. The negativity of the orbital derivative of V_C was verified in 0.01 sec. with the result that it was negative in all but 146 or 0.18% of the simplices. In Figure 4 some level sets of V_C are drawn in blue and the simplices where the orbital derivative V_C' is nonnegative are marked with red dots. For comparison the largest sublevel set of V_q , $\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^2 : V_q(\mathbf{x}) \leq 0.225\}$, which is guaranteed to be a subset of the basin of attraction, is drawn in black. With these parameters for the RBF and CPA method, we can conclude that the innermost sublevel set of V_C in the figure is in the basin of attraction. Indeed, all simplices in this sublevel set where the orbital derivative of V_C is nonnegative fit in the ball $\overline{\mathcal{B}(0.075)}$ which lies inside \mathcal{E} . By choosing the set $\mathcal{F} \subset \mathcal{E}$ such that the simplices marked in red lie in \mathcal{F} , and choosing \mathcal{M} larger than the innermost sublevel set, but not including any of the simplices marked red outside \mathcal{E} , we can

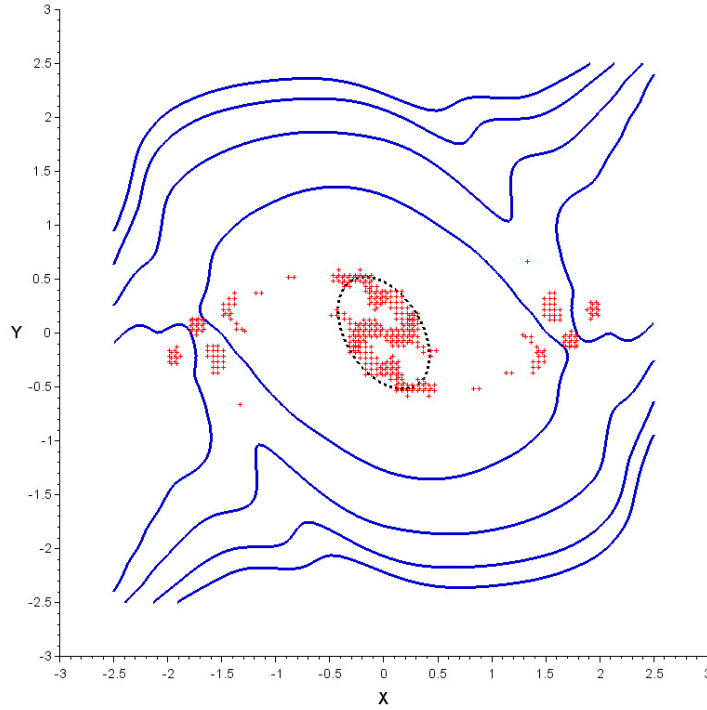


Figure 3. First set of parameters. Some level sets of V_C (blue), the triangles where the orbital derivative of V_C is nonnegative (red), and the largest level set of V_q (dashed black) which lies in the basin of attraction. We can conclude that the innermost sublevel set of V_C is forward invariant, but not that it is in the basin of attraction.

apply Theorem 2.6 (d). In Figure 5, the graph of the computed Lyapunov function V_C is drawn.

6.2. Example 2. We consider the following three-dimensional system (cf. [7, Example 6.4] and [27, Example 4]):

$$\begin{cases} \dot{x} = x(x^2 + y^2 - 1) - y(z^2 + 1), \\ \dot{y} = y(x^2 + y^2 - 1) + x(z^2 + 1), \\ \dot{z} = 10z(z^2 - 1). \end{cases}$$

The system has an exponentially stable equilibrium at $(0, 0, 0)$ and its basin of attraction is given by the cylinder

$$\mathcal{A} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1, |z| < 1\}.$$

In [7, Example 6.4], an RBF approximation with 137 points in a hexagonal grid resulted in a large area near the equilibrium with positive orbital derivative.

As generally the basin of attraction is not known, [27, Example 4] chose to place the collocation points in the set $K = [-0.9, 0.9]^3$, which is not a subset of the basin of attraction. The authors of [27, Example 4] started with a coarse set and applied a refinement algorithm

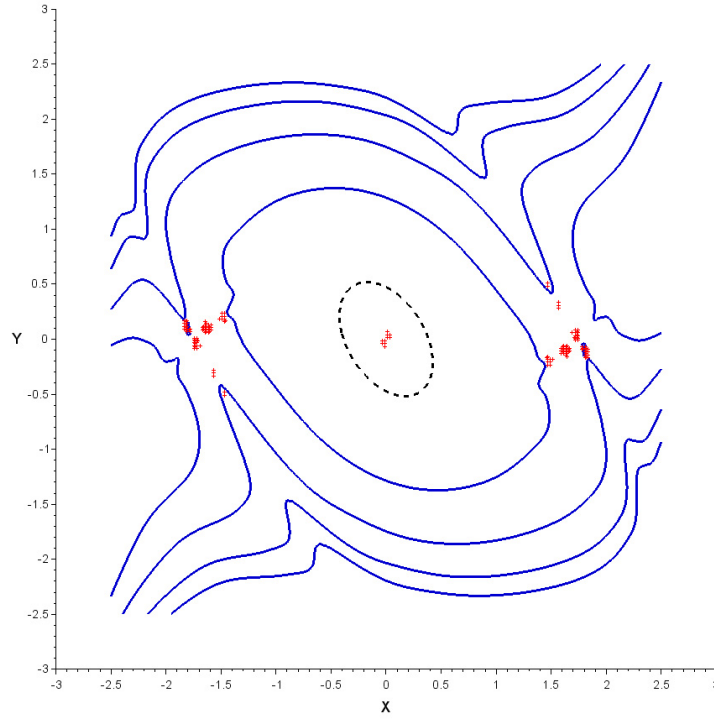


Figure 4. Second set of parameters. We display some level sets of V_C (blue), the triangles where the orbital derivative of V_C is nonnegative (red), and the largest level set of V_q (dashed black) which lies in the basin of attraction. We can conclude that the innermost sublevel set of V_C is in the basin of attraction.

to refine the set for the RBF method. The resulting RBF approximation was checked to be negative on a fine grid, but no verification was given.

In this paper we first determine a subset of the basin of attraction using the quadratic Lyapunov function $V_q(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$ with

$$P = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/20 \end{pmatrix}.$$

A direct calculation shows that $V_q'(\mathbf{x}) = r^2(r^2 - 1) + z^2(z^2 - 1)$, where $r = \sqrt{x^2 + y^2}$, so that $V_q'(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathcal{A}$. Hence

$$\mathcal{E} = \left\{ (x, y, z) \in \mathbb{R}^3 : V(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1}{20}z^2 < \frac{1}{20} \right\}$$

is the largest sublevel set of V_q that is a subset of the basin of attraction; see Figure 6.

We use the Wendland function for $k = 3$

$$\psi_{5,3}(cr) = (1 - cr)_+^8 [32(cr)^3 + 25(cr)^2 + 8cr + 1]$$

with $c = 0.6$; for the formulas of ψ_1 and ψ_2 , see [7, Appendix B.1]. We solve the RBF problem

$$V'(\mathbf{x}) = -p(\mathbf{x})q(\mathbf{x}), \quad \text{with } p(\mathbf{x}) = \|\mathbf{x}\|_2^2 \text{ and } q(\mathbf{x}) = 1 + \|\mathbf{f}(\mathbf{x})\|_2^2.$$

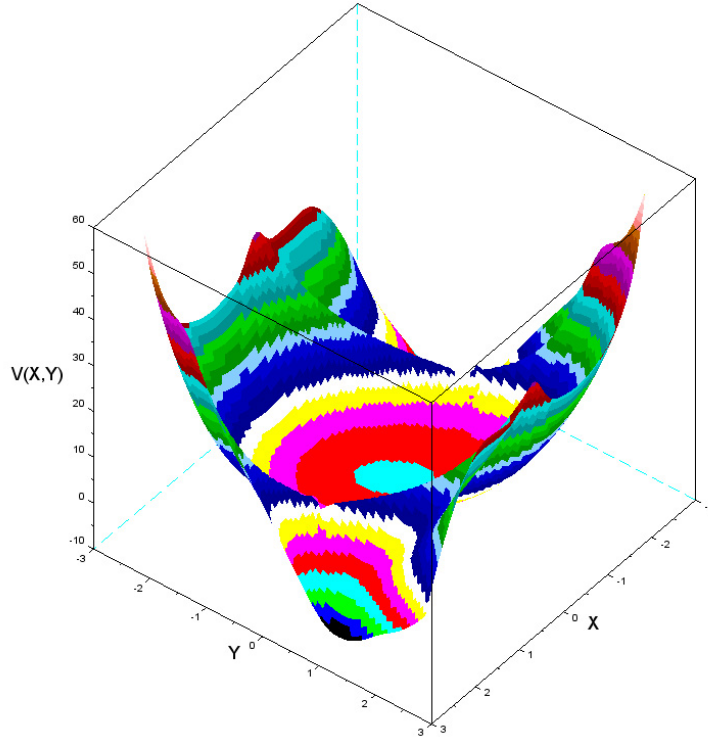


Figure 5. The computed Lyapunov function V_C for the system (6.1) with the second set of parameters.

The collocation points of the RBF problem were from the hexagonal grid as described in [21]; see also [7, p. 134]. More exactly, we define the vectors

$$\mathbf{g}_1 = (1, 0, 0)^T, \quad \mathbf{g}_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)^T, \quad \mathbf{g}_3 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}}\right)^T,$$

and $\mathbf{g}_0 = (\mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3)/2$. For a parameter $c_R > 0$ the set X_{c_R} consists of all points

$$X_{c_R} := \{\mathbf{x} = c_R(\mathbf{g}_0 + z_1\mathbf{g}_1 + z_2\mathbf{g}_2 + z_3\mathbf{g}_3), z_1, z_2, z_3 \in \mathbb{Z}\} \cap \mathcal{A}.$$

The fill distance h_R of X_{c_R} is proportional to c_R .

For the CPA method we used triangulations $T_\rho^\mathcal{M}$ with $\mathcal{M} = [-0.99, 0.99]^3$ and different parameters $\rho > 0$; cf. Definition 4.10. Note that \mathcal{M} has areas which do not lie in \mathcal{A} . The constants B_ν in the CPA problem were assigned the values

$$B_\nu = \max_{(x,y,z) \in \mathfrak{S}_\nu} \max\{6|x|, 6|y|, 60|z|\} \quad \text{for all simplices } \mathfrak{S}_\nu.$$

We did three different runs with different parameters for the RBF method and the CPA method. For creating the CPA problem we used a code that actually creates an LP problem, but instead of solving the LP problem we assign values to the variables of the problem and

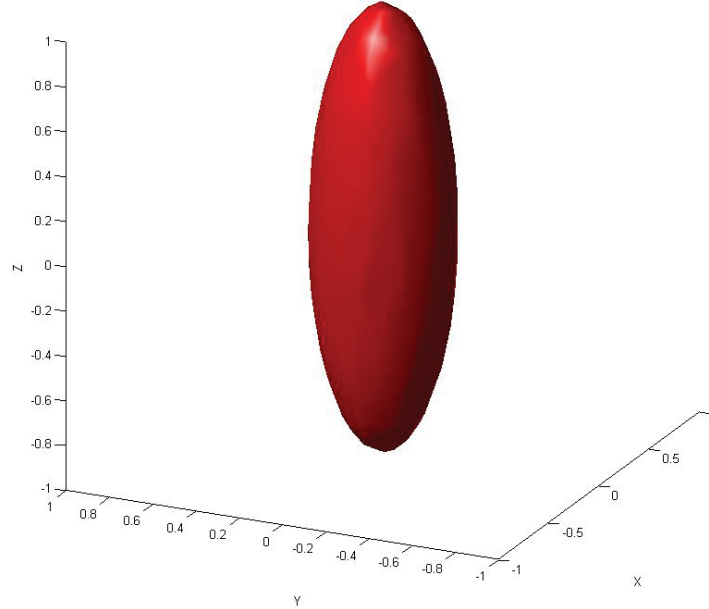


Figure 6. The largest sublevel set \mathcal{E} of the quadratic Lyapunov function V_q which lies in the origin's basin of attraction.

then verify them. We have not optimized this code for our application in this paper. In the runs below we used $\rho = 0.045$, resulting in the triangulation $\mathcal{T}_{0.045}^{\mathcal{M}}$ consisting of 511,104 tetrahedra, and $\rho = 0.03$ resulting in 1,724,976 tetrahedra. The creation times of the resulting LP problems are 13sec. and 46sec., respectively. Note that after the LP problem has been created, different values can be written in its variables.

In the first run we set $c_R = 0.24$ and $\rho = 0.045$. The set $X_{0.24}$ contains 658 points and the RBF problem was created and solved in 0.06sec., delivering V_R . The values of V_R were written in the inequalities of the CPA problem in 2.1seconds. The negativity of the orbital derivative of V_C was verified in 0.11sec. with the result that it was negative in 85.4% of the simplices. In Figure 7 the simplices where the orbital derivative V_C' is nonnegative are marked with blue points. The simplices where V_C' is nonnegative are spread around the planes $z = -1$, $z = 0$, and $z = 1$. These results are not useful for getting a better estimate for the basin of attraction than delivered by the sublevel sets of the quadratic Lyapunov function V_q . Setting $\rho = 0.03$ in the CPA interpolation and using the same parameter value $c_R = 0.24$ for the RBF method delivered qualitatively similar results.

In the second run we set $c_R = 0.15$ and $\rho = 0.045$. The set $X_{0.15}$ contains 2,658 points and the RBF problem was created and solved in 2.2sec., delivering V_R . The values of V_R were written in the inequalities of the CPA problem in 4.8seconds. The negativity of the

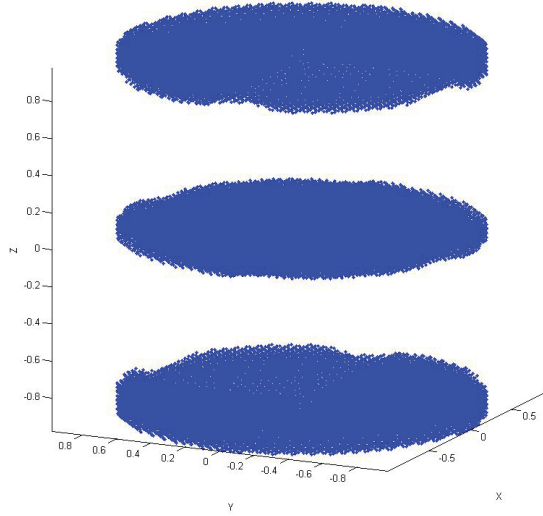


Figure 7. First set of parameters: $c_R = 0.24$ and $\rho = 0.045$. The simplices where the orbital derivative V'_C is nonnegative are marked with blue points.

orbital derivative of V_C was verified in 0.1 sec. with the result that it was negative in 87% of the simplices. The orbital derivative is nonnegative predominantly in the planes $z = -1$ and $z = 1$. In Figure 8 the level set $V_C^{-1}(0.88)$ is depicted together with the simplices where the orbital derivative V'_C fails to be negative. Inside the sublevel set all simplices where V'_C is nonnegative are contained in the ball $\overline{\mathcal{B}_{0.077}}$. This ball fits with ease in the sublevel set \mathcal{E} of the quadratic Lyapunov function V_q and the sublevel set depicted on Figure 8 is thus a much better lower bound on the basin of attraction than delivered by the quadratic Lyapunov function V_q ; cf. Theorem 2.6 (d). Setting $\rho = 0.03$ in the CPA interpolation and using the same parameter value $c_R = 0.15$ for the RBF method delivered qualitatively similar results, but the radius of the ball at the origin containing the simplices where V'_C fails to be negative can be reduced to 0.028.

For the third run we used a set of collocation points for the RBF method, which was not a uniform refinement from the first to the second run, but a local one. We started with the same collocation points for the RBF method as in the first run, and added a denser layer around the plane $z = 0$, the area inside of \mathcal{M} where the calculations failed to deliver V_C with a negative orbital derivative. More exactly, we added the points of $X_{0.12}$, corresponding to $c_R = 0.12$, directly above and below the plane $z = 0$, i.e., the points $(x, y, z) \in X_{0.12}$ and $|z| < 0.1$. The RBF problem then has 1,162 collocation points. For the CPA verification we used the same triangulation as in the first and second run with 511,104 tetrahedra. The RBF problem was created and solved in 0.24 sec., delivering V_R . The values of V_R were written in the inequalities of the CPA problem in 2.7 seconds. The negativity of the orbital derivative of V_C was verified in 0.1 sec. with the result that it was negative in 84.3% of the simplices, with exceptions predominantly at the boundary of \mathcal{M} . In Figure 9 the level set $V_C^{-1}(0.88)$ is depicted together with the simplices where the orbital derivative V'_C fails to be negative.

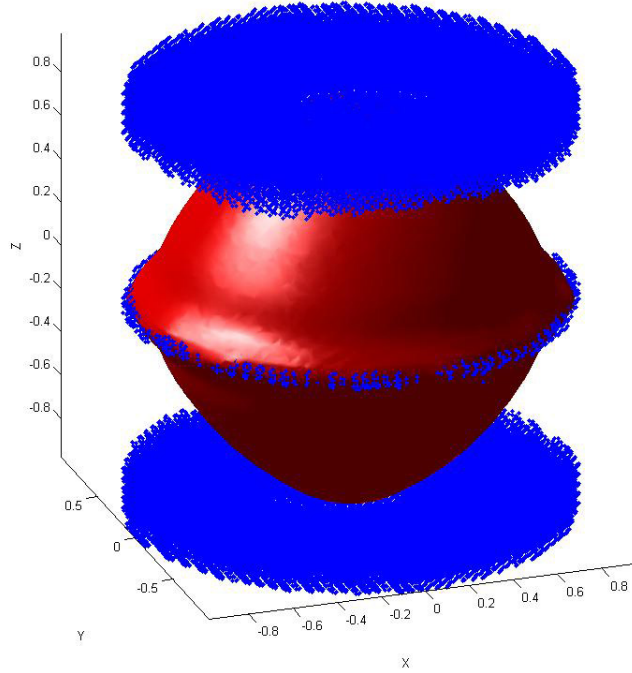


Figure 8. *Second set of parameters. The level set $V_C^{-1}(0.88)$ for V_C computed in the second run (red) and simplices where the orbital derivative of V_C fails to be negative (blue). Inside the sublevel set all simplices such that the orbital derivative V'_C is nonnegative are contained in a ball centered at the origin with radius 0.077.*

Inside the sublevel set all simplices where V'_C is nonnegative are contained in the ball $\overline{\mathcal{B}_{0.1}}$. The results are thus only marginally worse than in the second run, although we used much fewer collocation points. Especially, we can conclude that the sublevel set in Figure 9 is in the origin's basin of attraction. Note that after the LP problem from the CPA method has been created in 13 sec. the computation time for creating and solving the RBF problem and the verifying the negativity of V'_C is reduced from 7.1 sec. in the second run to 3.04 sec. in the third run.

Setting $\rho = 0.03$ in the CPA interpolation we got, again, qualitatively similar results, but the radius of the ball at the origin containing the simplices where V'_C fails to be negative can be reduced to 0.028; the same improvement as in the second run. Note that the theory of this paper is only applicable to simplices of the triangulation which lie in the basin of attraction and indeed, in this area a finer triangulation improves the results as shown in the second and third run.

7. Conclusions. In this paper we have presented a method to both compute and verify a Lyapunov function for a general nonlinear ODE in \mathbb{R}^n with an exponentially stable equilibrium. The computation of a Lyapunov function candidate is achieved using mesh-free collocation and Radial Basis Functions, solving a linear PDE. Then the function is interpolated by a

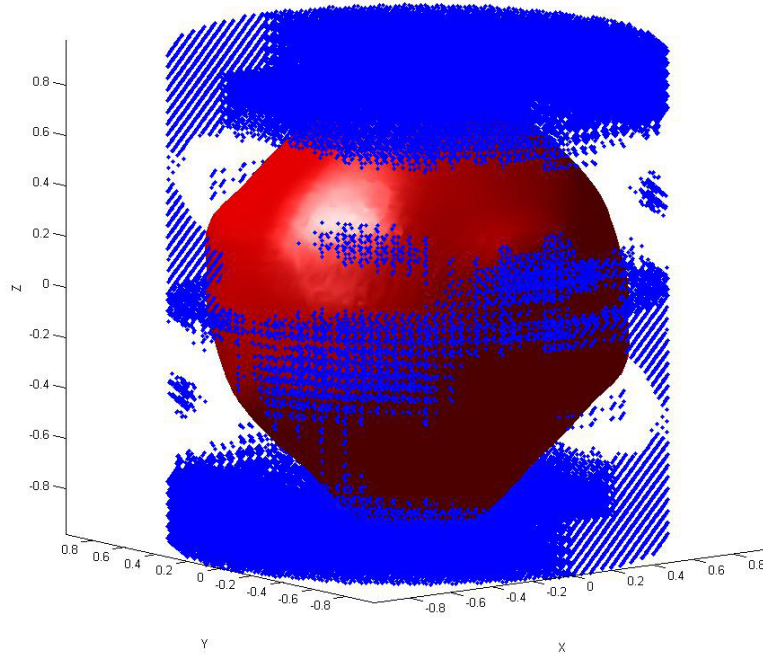


Figure 9. *Third set of parameters. The level set $V_C^{-1}(0.88)$ for V_C computed in the third run (red) and simplices where the orbital derivative of V_C fails to be negative (blue). Inside the sublevel set all simplices such that the orbital derivative V'_C is nonnegative are contained in a ball centered at the origin with radius 0.1.*

continuous piecewise affine (CPA) function on a triangulation and, checking finitely many inequalities at the vertices of the triangulation, the properties of a Lyapunov function can be rigorously verified. Hence, sublevel sets of the Lyapunov function are subsets of the basin of attraction of the equilibrium.

We have proved that this method succeeds in the computation and verification of a Lyapunov function if both the collocation points for the mesh-free collocation are dense enough and the triangulation for the CPA interpolation is fine enough. Moreover, sublevel sets of the computed Lyapunov function cover any given compact subset of the basin of attraction, hence, the method can determine compact subsets arbitrarily close to the (open) basin of attraction.

The proposed method improves the existing RBF method as it adds a verification which proves that the constructed function is indeed a Lyapunov function. It improves the original CPA method by reducing the computation time considerably. Further research will include suitable local refinements of both the set of collocation points and the triangulation to reduce the number of points and thus the computation time further.

The combination of the RBF and CPA methods is a powerful method which was demonstrated on a two- and a three-dimensional example. Combining the strengths of both existing methods, the new method is both as computationally efficient as the RBF method and includes rigorous estimates as the CPA method, and thus will be useful for applications, particularly

in higher dimensions.

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