Lyapunov Functions by Interpolating Numerical Quadrature: Proof of Convergence

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Abstract. Lyapunov functions for nonlinear systems, whose dynamics are defined by ordinary differential equations, are computed by solving linear programming feasibility problems in the CPA method. Further, the CPA method is constructive and can generate a Lyapunov function on any compact subset of the basin of attraction of an asymptotically stable equilibrium. Instead of solving the linear programming feasibility problem, one can use converse theorems to determine a candidate solution and then verify the constraints of the feasibility problem. This procedure has the advantage of being usually much faster. Further, a partial solution to the feasibility problem that violates the constraints in some areas can be analyzed, whereas a solver either generates a feasible solution or assures that a feasible solution does not exist. In this paper we prove that the numerical quadrature of numerically integrated solutions will deliver a feasible solution to the linear programming problem, given that the time horizon is large enough and the time steps are small enough in the numerical integration and quadrature. Further, the relevant theorems are general enough to allow for considerable flexibility in the particular implementation as they cover a wider range of numerical methods both for integration and quadrature.

Keywords: Lyapunov function, CPA algorithm, Numerical Integration, Numerical Quadrature

1 Introduction

The Lyapunov stability theory is a mathematical abstraction of the concept of dissipative energy in physics. It is the centerpiece of practical and theoretical stability analysis and is treated in various detail in virtually all textbooks and monographs on linear and nonlinear systems, cf. e.g. [28, 45, 46] or [32, 39, 43] for a more recent discussion. In a physical system the (free) energy is an obvious choice for a Lyapunov function and a dissipative system approaches a local minimum of the available energy.

For a general dynamical system modelling nonphysical phenomena there is usually no obvious candidate for a Lyapunov function and there are no analytical methods to obtain a Lyapunov function. Therefore numerous numerical methods for the numerical computation of Lyapunov functions have been developed. To name a few, in [41, 42] the computation of rational Lyapunov functions was studied, in [2, 37] sum-of-squared

(SOS) polynomial Lyapunov functions were computed using semi-definite optimization (SOS method), see also [38, 30] for other approaches using polynomials, and in [11] a Zubov type PDE was numerically solved using radial basis functions (RBF method). For an overview of more methods see, e.g., the review paper [15].

In [29, 34] linear programming was used to compute continuous and piecewise affine (CPA) Lyapunov functions; this approach is referred to as the CPA method. In the CPA method the domain, where the Lyapunov function is to be computed, is triangulated, i.e. subdivided into simplices, and a feasibility problem is derived, such that its feasible solution can be used to define a CPA Lyapunov function for the system. In [13, 19, 20] it was established that the CPA method can generate a Lyapunov function for a general nonlinear system with an asymptotically stable equilibrium, if the simplices in the triangulation are sufficiently small and non-degenerate in a certain sense.

In [14], the CPA and the RBF method were combined to deliver a method that is as fast as the RBF method and delivers a verified Lyapunov function as the CPA method. This was achieved by solving a system of linear equations in the RBF method rather than a linear optimization problem as in the CPA method. At the same time, the obtained function is verified to be a Lyapunov function by checking that it satisfies the constraints of the feasibility problem. Further, the authors proved that this approach is constructive and one is always able to compute a verified Lyapunov function in any compact subset of an exponentially stable equilibrium's basin of attraction. In numerous papers a similar approach has been followed, where one uses numerical solutions of the system under consideration to generate values for the variables of the feasibility problem of the CPA method and then verifies the constraints, see [4–6, 9, 10, 23–27, 33] and also [18, 21] for more implementation oriented papers. This technique works well in practice, but there was no proof available that this approach always works. In this paper we deliver the proof that this method will always work for sufficiently fine triangulations and sufficiently accurate numerical approximations.

We consider an ordinary differential equation (ODE) of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{f} \in C^s(\mathbb{R}^n, \mathbb{R}^n), \quad s \geq 1, \tag{1}$$

with an exponentially stable equilibrium. Without loss of generality we assume that the equilibrium is at the origin. Recall that *exponentially stable* means that there are constants $\delta, \alpha, M > 0$ such that any solution $t \mapsto \phi(t, \xi)$ to system (1) starting at $\xi \in B_\delta := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < \delta\}$ at time $t = 0$ fulfills the inequality

$$\|\phi(t, \xi)\|_2 \leq M \|\xi\|_2 \mathrm{e}^{-\alpha t} \quad \text{for all } t \geq 0.$$

This can be easily checked by investigating the eigenvalues of the Jacobian $D\mathbf{f}(\mathbf{0})$ of $\mathbf{f}$ at the origin. In particular, it is easy to compute a quadratic Lyapunov function for the linearization, which is locally also a Lyapunov function for the nonlinear system and thus determines a (usually small) subset of the basin of attraction. We are interested in the equilibrium's *basin of attraction*

$$\mathcal{A} := \{\xi \in \mathbb{R}^n : \lim_{t \to \infty} \phi(t, \xi) = \mathbf{0}\},$$

and seek to determine subsets of $\mathcal{A}$, which are as large as possible, by sublevel sets of suitable Lyapunov functions. We will see that the Lyapunov functions that we construct will not be valid in a small local neighborhood of the equilibrium, where a local Lyapunov function as described above can be used.

The paper is organized as follows. After introducing some notations, we prove general error estimates for the type of Lyapunov function we are approximating in Section 2, and more specific error estimates for the numerical integration and quadrature in Section 3 to compute approximations and bound the approximation error of the values of the Lyapunov function. In Section 4, we define triangulations and CPA functions, while in Section 5 we outline the CPA feasibility problem and prove our main result, showing that using numerical integration and quadrature always succeeds in computing and verifying a Lyapunov function if the triangulation is sufficiently fine and the numerical approximations sufficiently accurate. Finally, we conclude the paper in Section 6.

1.1 Prerequisites and Notation

$\mathbb{N}$ denotes the set $\{1,2,\ldots,\}$ and $\mathbb{N}_0 := \{0\}\cup\mathbb{N}$, by $\mathbb{R}_+ = \{x\in\mathbb{R} \mid x\geq 0\}$ we denote the non-negative reals. We utilize a bold-face font for (column) vectors, e.g. $\mathbf{x}\in\mathbb{R}^{n\times 1} = \mathbb{R}^n$. For a vector $\mathbf{x}$ we write x_i or $[\mathbf{x}]_i$ for its ith component and we define the norm $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for $p\geq 1$. We also define $\|\mathbf{x}\|_\infty = \max_{i\in\{1,\ldots,n\}} |x_i|$. We will repeatedly use the norm equivalence relation

$$\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq n^{q^{-1}-p^{-1}}\|\mathbf{x}\|_p \quad \text{for } p>q.$$

The *induced matrix norm* $\|\cdot\|_p$ is defined by $\|A\|_p = \max_{\|\mathbf{x}\|_p=1}\|A\mathbf{x}\|_p$. Clearly $\|A\mathbf{x}\|_p \leq \|A\|_p\|\mathbf{x}\|_p$. For a matrix A we write A^{T} for its transpose.

We denote by $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n$ the standard orthonormal basis of $\mathbb{R}^n$ and by I the identity matrix. We denote the interior of a set $\mathcal{S}\subset\mathbb{R}^n$ by $\mathcal{S}^\circ$ and its closure by $\overline{\mathcal{S}}$. An open ball in the Euclidian norm $\|\cdot\|_2$ on $\mathbb{R}^n$, centered at the origin and with radius $R>0$, is denoted B_R. A continuous function $\alpha\colon\mathbb{R}_+\to\mathbb{R}_+$, $\mathbb{R}_+ := [0,\infty)$, is said to be of class $\mathcal{K}$ if $\alpha(0)=0$ and it is strictly monotonically increasing. If additionally $\lim_{x\to\infty}\alpha(x)=\infty$ we say α is of class $\mathcal{K}_\infty$. Furthermore, we write $C\subset\subset A$ if C is a compact subset of A.

A continuous function $\beta\colon\mathbb{R}^n\to\mathbb{R}$ is called positive definite, if $\beta(\mathbf{0})=0$ and $\beta(\mathbf{x})>0$ for all $\mathbf{x}\neq\mathbf{0}$.

2 Error estimates

In the next two lemmas we state and prove some results that will be used later. Recall that the basin of attraction $\mathcal{A}$ is an open set.

Lemma 1. *Consider system* (1) *and let $\mathcal{C}\subset\subset\mathcal{A}$ be a compact subset of its equilibrium's (at the origin) basin of attraction. Then:*

i) $\overline{\phi([0,\infty),\mathcal{C})}\subset\subset\mathcal{A}$.
ii) With $L>0$ a Lipschitz constant for $\mathbf{f}$ on $\overline{\phi([0,\infty),\mathcal{C})}$ and for any $\delta,T\geq 0$, we have

$$\phi([0,T],\mathcal{C}\setminus B_\delta)\subset\overline{\phi([0,\infty),\mathcal{C})}\setminus B_{\delta^*},\quad \delta^* := \delta\,\mathrm{e}^{-LT}.$$

Proof. Proof of i): Let $R \geq r > 0$ be such that $\mathcal{C} \subset\subset B_R$ and $\overline{B_r} \subset\subset \mathcal{A}$. Since the origin is exponentially stable and $\mathcal{C} \cup \overline{B_r}$ compact, there exist constants $\alpha > 0$ and $M \geq 1$ such that $\|\phi(t,\xi)\|_2 \leq M\|\xi\|_2 e^{-\alpha t}$ for all $\xi \in \mathcal{C} \cup \overline{B_r}$; see e.g. [6, Lem. 1]. Set $S = \alpha^{-1}\ln(M^2R/r)$. Then for every $\xi \in \mathcal{C}$ we have $\|\phi(S,\xi)\|_2 \leq MRe^{-\alpha S} \leq r/M$, i.e. $\phi(S,\mathcal{C}) \subset \overline{B_{r/M}}$. Since clearly $\phi([0,\infty),\overline{B_{r/M}}) \subset \overline{B_r}$, it follows that

$$\phi([0,\infty),\mathcal{C}) = \phi([0,S],\mathcal{C}) \cup \phi([0,\infty),\phi(S,\mathcal{C})) \subset \phi([0,S],\mathcal{C}) \cup \overline{B_r}$$

and because $\phi([0,S],\mathcal{C})$ is compact, statement i) is proved.

Proof of ii): Note that for $\xi \in \overline{\phi([0,\infty),\mathcal{C})}$ and $t \geq 0$ we have

$$\begin{aligned}\left|\frac{d}{dt}\|\phi(t,\xi)\|_2^2\right| &= \left|2\dot{\phi}(t,\xi)^{\mathrm{T}}\phi(t,\xi)\right| = 2\left|\mathbf{f}(\phi(t,\xi))^{\mathrm{T}}\phi(t,\xi)\right| \\ &\leq 2\|\mathbf{f}(\phi(t,\xi))\|_2\|\phi(t,\xi)\|_2 \leq 2L\|\phi(t,\xi)\|_2^2,\end{aligned}$$

i.e.

$$-2L\|\phi(t,\xi)\|_2^2 \leq \frac{d}{dt}\|\phi(t,\xi)\|_2^2.$$

Hence,

$$-2L \leq \frac{1}{\|\phi(t,\xi)\|_2^2}\frac{d}{dt}\|\phi(t,\xi)\|_2^2 = \frac{d}{dt}\ln(\|\phi(t,\xi)\|_2^2)$$

and by integrating both sides from 0 to t we obtain

$$-2Lt \leq \ln(\|\phi(t,\xi)\|_2^2) - \ln(\|\xi\|_2^2) = 2\ln\left(\frac{\|\phi(t,\xi)\|_2}{\|\xi\|_2}\right),$$

i.e.

$$e^{-Lt}\|\xi\|_2 \leq \|\phi(t,\xi)\|_2.$$

In particular, we have $\|\phi(t,\xi)\|_2 \geq \delta^*$, $\delta^* = \delta e^{-LT}$, for $t \in [0,T]$ and $\|\xi\|_2 \geq \delta$, which proves statement ii).

Lemma 2. *Consider system* (1) *and let $\mathcal{C} \subset\subset \mathcal{A}$ be a compact subset of its equilibrium's (at the origin) basin of attraction. Let $\beta\colon \mathbb{R}^n \to \mathbb{R}_+$ be a continuous and positive definite function that is C^s on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Fix $\delta > 0$ such that $\mathcal{C} \setminus B_\delta \neq \emptyset$.*

Then there exists a $T^* \geq 0$ and an $m > 0$ such that for all $T \geq T^*$ the function

$$V(\mathbf{x}) = \int_0^T \beta(\phi(t,\mathbf{x}))dt \tag{2}$$

is a C^s function on $\mathcal{C} \setminus B_\delta$, which satisfies

$$\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq -m < 0 \ \textit{ for all } \mathbf{x} \in \mathcal{C} \setminus B_\delta. \tag{3}$$

Proof. Let $m := \frac{1}{2}\min_{\mathbf{x}\in\mathcal{C}\setminus B_\delta}\beta(\mathbf{x}) > 0$ and let $r > 0$ be so small that

$$\max_{\mathbf{x}\in\overline{B_r}}\beta(\mathbf{x}) \leq m;$$

this is possible by the assumptions on β.

Choose $T^* > 0$ so large such that $\phi(t+T^*,\mathcal{C}) \subset B_r$ for all $t \geq 0$; this is possible since 0 is asympotically stable and $\mathcal{C}$ is compact. Indeed, by the stability there exists $\tilde{r} > 0$ such that $\mathbf{x} \in B_{\tilde{r}}$ implies $\phi(t,\mathbf{x}) \in B_r$ for all $t \geq 0$ and by the attractivity there exists T^* such that $\phi(T^*,\mathcal{C}) \subset B_{\tilde{r}}$.

Then, for $T \geq T^*$ and $\mathbf{x} \in \mathcal{C} \setminus B_\delta$, we have

$$\begin{aligned}
\nabla V(\mathbf{x}) \bullet \mathbf{f}(\mathbf{x}) &= \frac{d}{dt} V(\phi(t,\mathbf{x}))\Big|_{t=0} \\
&= \frac{d}{dt} \int_0^T \beta(\phi(t+s,\mathbf{x}))ds\Big|_{t=0} \\
&= \frac{d}{dt} \int_t^{T+t} \beta(\phi(s,\mathbf{x}))ds\Big|_{t=0} \\
&= \beta(\phi(T,\mathbf{x})) - \beta(\mathbf{x}) \\
&\leq m - 2m.
\end{aligned}$$

Note that due to Lemma 1 ii) $\phi([0,t],\mathcal{C} \setminus B_\delta)$ lies outside B_{δ^*} and thus V is C^s on $\mathcal{C} \setminus B_\delta$, which proves the lemma.

3 Computing values for V_ξ directly

Instead of solving the linear optimization problem generated to obtain a Lyapunov function, one can use a different method to make *educated guesses* of their values and then verify if the linear constraints are fulfilled for these values. For the system (1), i.e. $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with an exponentially stable equilibrium at the origin, a Lyapunov function is given by

$$V(\xi) = \int_0^T \beta(\phi(s,\xi))ds, \tag{4}$$

where $\beta : \mathbb{R}^n \to \mathbb{R}_+$ is a continuous and positive definite function, for sufficiently large T by Lemma 2. The idea for constructing such Lyapunov functions goes back to Massera [36], see also [31], and is discussed in many textbooks on nonlinear systems, e.g. [32, 43]. Typically $\beta(\mathbf{x}) = \beta^*(\|\mathbf{x}\|_2)$, where $\beta^* : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}_\infty$, but there are other choices that may be more advantageous, for example

$$\beta(\mathbf{x}) = \|\mathbf{f}(\mathbf{x})\|_2 \quad \text{or} \quad \beta(\mathbf{x}) = \frac{\|\mathbf{x}\|_2}{\delta + \|\mathbf{x}\|_2^p}, \;\; \delta > 0, \;\; p \in \mathbb{R}, \tag{5}$$

are used in [5, 6] and [26, 27] respectively.

Note that although (4) gives an explicit formula for a Lyapunov function, this formula includes the solution $\phi(s,\xi)$ to the differential equation and the solution is usually not known. It can, however, be approximated at the vertices $\mathcal{V}_\mathcal{T}$ of a simplicial complex $\mathcal{T}$ with a subsequent verification of the constraints of the linear programming problem.

The idea is now to approximate (4) numerically. There are two approximations necessary: first, we need to approximate the solution $\phi(s,\xi)$ (numerical integration) and finally one needs a quadrature rule to approximate the integral (numerical quadrature).

For example, we can use the Adam-Bashforth four-step method for obtaining numerically solutions $t \mapsto \phi(t,\xi)$ to the initial-value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \xi,$$

on $[0,T]$ and the composite Simpson's Rule to integrate $\beta(\phi(s,\xi))$ over the same interval. Both are standard methods that are described in most textbooks on numerical analysis, cf. e.g. [40].

This, or a similar approach, is followed in, e.g. [4–6, 9, 10, 14, 23, 24, 33], and it generates the values much faster than solving the linear programming problem. An additional advantage is that one can localize the area where the constraints are not fulfilled, whereas a solver will simply state that the linear programming problem does not possess a feasible solution when that is the case.

In the following subsections we will quantify the error due to the numerical integration and the numerical quadrature, before estimating the overall error between the function value $V(\xi)$ and its numerical approximation V_ξ.

3.1 Error in the Numerical Integration

We say tha a method to approximate solutions to system (1) is a *one step integrator with order of consistency* $p \in \mathbb{N}$, if for every compact set $\mathcal{K} \subset \mathbb{R}^n$ there exist constants $C, \tau' > 0$ such that the local truncation error satisfies

$$\|\phi(\tau,\xi) - \widetilde{\phi_1}\|_2 \leq C\tau^{p+1}$$

independent of $\xi \in \mathcal{K}$ and $0 < \tau \leq \tau'$. Here $\widetilde{\phi_1}$ is the approximation by the method to the true solution $\phi(\tau,\xi)$ at time $t = \tau$ when starting at ξ at time $t = 0$.

Most one step methods are one step integrators with order of consistency p, e.g. Euler's method, where $p = 1$, and the Runge-Kutta family, where p is the order of the method.

Lemma 3. *Consider the system* (1) *and a a one step integrator of the system with order of consistency $p \leq s$ (recall $\mathbf{f} \in C^s(\mathbb{R}^n,\mathbb{R}^n)$). Let $\mathcal{C} \subset\subset \mathcal{A}$ be a neighbourhood of the (exponentially stable) origin. Then there exists a $\mathcal{C} \subset \mathcal{K} \subset\subset \mathcal{A}$, that is forward invariant, both for the system* (1) *and the numerical method with time step size τ, $0 < \tau \leq \tau'$, where $\tau' > 0$ depends on $\mathcal{K}$. Let $L > 0$ be a Lipschitz constant for $\mathbf{f}$ on $\mathcal{K}$. By the assumptions there exists a constant $C_\phi > 0$ such that the local truncation error satisfies*

$$\|\phi_1 - \widetilde{\phi_1}\|_2 \leq C_\phi \tau^{p+1}$$

independent of $\xi \in \mathcal{K}$ and $0 < \tau \leq \tau'$ (same $\tau' > 0$ for which $\mathcal{K}$ is forward invariant). Denote by $\phi_i := \phi(i\cdot\tau,\xi)$ the true solution at time $t = i\cdot\tau$ to (1)*, starting at ξ at time $t = 0$, and by $\widetilde{\phi}_i$ the approximation at time $t = i\cdot\tau$ computed with the integrator.*

Then, for a given time interval $T > 0$, the error at time $t = i\cdot\tau \leq T$ is bounded by

$$\|\phi_i - \widetilde{\phi}_i\|_2 \leq C_\phi \frac{e^{LT}-1}{L}\tau^p \quad \textit{for} \quad 0 \leq i\tau \leq T. \tag{6}$$

For a proof of the lemma see the Appendix in [17]. Note that in general the constant C_ϕ depends on the $(p+1)$-st derivatives of ϕ. If $\mathbf{f}$ in (1) is C^p then so are ϕ and $\dot\phi$, see e.g. [44, III.§13.XI], and we need to assume that $\mathbf{f}$ is C^p on $\mathcal{K}$ for the estimate to hold true. Recall that $\dot\phi = \mathbf{f}(\phi)$ and therefore we save one derivative, i.e. we do not have to assume that $\mathbf{f}$ is C^{p+1}. Further note, that the numerical method is forward invariant on $\mathcal{K}$ means that for any fixed step size $0 < \tau \le \tau'$, we have for $\xi \in \mathcal{K}$ that $\widetilde{\phi}_i \in \mathcal{K}$ for all $i \in \mathbb{N}_0$.

3.2 Error in the Numerical Quadrature

Assumption 1 *Let $T > 0$ and let $\alpha \in C^k([0,T])$. We use a quadrature rule for the integral*

$$I_\alpha := \int_0^T \alpha(t)dt$$

that, for a given $N_T \in \mathbb{N}$, subdivides the interval $[0,T]$ into intervals of length $\tau := T/N_T$ and approximates I_α by a sum of the form

$$\widetilde{I_\alpha} := T\sum_{i=0}^{N_T} c_i\alpha_i, \quad \sum_{i=0}^{N_T} c_i = 1, \quad c_i > 0, \quad \textit{and} \quad \alpha_i := \alpha(i\cdot\tau).$$

Moreover, we assume that the difference between I_α and $\widetilde{I_\alpha}$ is bounded by

$$|I_\alpha - \widetilde{I_\alpha}| \le C_I\tau^k, \tag{7}$$

where C_I is a constant depending on the derivatives of α up to and including order k.

There are very good quadrature methods, e.g. Gaussian quadrature, that do not use uniformly distributed evaluations of α, but since we will generate approximations to α at equidistant points these methods are less appropriate for our purpose. Numerous classical rules, however, fit into this framework. For example, the composite Trapezoidal rule, the composite Simpson's rule, and the composite Boole's rule give orders $k = 2$, $k = 4$, $k = 6$ with constants

$$C_I = \frac{T}{12}\cdot\max_{t\in[0,T]}|\alpha^{(2)}(t)|, \;\; C_I = \frac{T}{180}\cdot\max_{t\in[0,T]}|\alpha^{(4)}(t)|, \;\; C_I = \frac{2T}{945}\cdot\max_{t\in[0,T]}|\alpha^{(6)}(t)|,$$

respectively. All these rules are Newton–Cotes quadrature rules and the latter two are also obtained using Richardson extrapolation on the Trapezoidal rule. For $N_T = 2^p m$, $p, m \in \mathbb{N}$, one can even apply Richardson extrapolation p times on the composite Trapezoidal rule and thus obtains in general a better estimate on the integral; e.g. in Romberg's method this is used to its full extent for $N_T = 2^p$. As shown in [22], Richardson extrapolation is very cheap computationally in our setting. Further, it does not suffer from Runge's phenomenon like some higher-order Newton–Cotes quadrature rules. For a general error formula for such an extrapolation we refer to formula (3.5.13) in [8] and for a detailed discussion to [7, Ch. 9.4].

3.3 Error in Evaluating V_ξ

Now we use the previous estimates to obtain a bound on the difference between the function value $V(\xi)$ and its numerical approximation V_ξ.

Theorem 2. *Consider system* (1) *and let* $\delta > 0$ *and* $\mathcal{C} \subset\subset \mathcal{A}$ *be given such that* $\mathcal{C} \setminus B_\delta \neq \emptyset$.

Further, let $\beta\colon \mathbb{R}^n \to \mathbb{R}_+$ *be a locally Lipschitz continuous and positive definite function, that is* C^s *on* $\mathbb{R}^n \setminus \{\mathbf{0}\}$. *Consider a numerical integration method with order of consistency* $p \le s$ *for system* (1) *and let* $\mathcal{K} \supset \overline{\phi([0,\infty),\mathcal{C})}$ *be forward invariant both for system* (1) *and for the numerical method with step size* $0 < \tau \le \tau'$, *as in Lemma 3. Further, consider a quadrature method of order k,* $p \le k \le s$ *as in Assumption 1.*

Then, there exists $T^* > 0$ *such that for all fixed* $T \ge T^*$ *and* $\varepsilon > 0$ *there exists* $N_T^* \in \mathbb{N}$ *such that for all* $N_T \in \mathbb{N}$ *with* $N_T \ge N_T^*$ *and*

$$V_\xi := T \sum_{i=0}^{N_T} c_i \beta(\widetilde{\phi}_i), \tag{8}$$

where the c_i *are from the quadrature rule (see Assumption 1) and the* $\tilde{\phi}_i$ *are the approximations of* $\phi_i := \phi(i \cdot \tau, \xi)$ *with* $\tau := T/N_T$ *(see Lemma 3), we have for all* $\xi \in \mathcal{C} \setminus B_\delta$

$$\left|V(\xi) - V_\xi\right| < \varepsilon.$$

Remark 1. Note that we do not assume that β is s-times differentiable at the origin and the derivatives are even allowed to diverge at the origin. We can do this because we established in Lemma 1 ii) that the solution trajectories of (1) starting in $\mathcal{A} \setminus B_\delta$ will stay away from the origin for $t \in [0,T]$, i.e. $\|\phi(t,\xi)\|_2 \ge \delta^* > 0$. This results in more freedom in choosing the function β which is advantageous in applications; e.g. as in (5).

Proof. We choose $T^* > 0$ according to Lemma 2 and choose $T \ge T^*$. We have

$$V(\xi) = I_{\beta(\phi(\cdot,\xi))} := \int_0^T \beta(\phi(t,\xi))dt. \tag{9}$$

Note that by Lemma 1 i) and Lemma 3 we only need to consider $\mathbf{f}$ on the compact set $\mathcal{K}$, $\overline{\phi([0,\infty),\mathcal{C})} \subset \mathcal{K} \subset\subset \mathcal{A}$, and β on a ball $B_R \supset \mathcal{K}$. By assumption $\mathbf{f}$ has a Lipschitz constant $L_\mathbf{f} > 0$ on $\mathcal{K}$ and β has a Lipschitz constant $L_\beta > 0$ on $\overline{B_R^*} \supset \mathcal{K}$.

Note that an error bound constant C_I for the functions $t \mapsto \beta(\phi(t,\xi))$, $t \in [0,T]$, uniformly for $\xi \in \mathcal{K} \setminus B_{\delta^*}$ exists, because C_I is bounded above by derivatives of $t \mapsto \beta(\phi(t,\xi))$, which are continuous on the compact set $\mathcal{K} \setminus B_{\delta^*}$. However, C_I does depend on the set $\mathcal{C}$ and the parameters $\delta > 0$ and $T > 0$, because $\delta^* = \delta e^{-LT}$.

Choose $N_T^* \in \mathbb{N}$ so large that $\tau^* := T/N_T^*$ fulfills $\tau^* \le \tau'$ and

$$C_I(\tau^*)^k + L_\beta T C_\phi \frac{e^{L_\mathbf{f} T} - 1}{L_\mathbf{f}} (\tau^*)^p < \varepsilon. \tag{10}$$

Fix $N_T \ge N_T^*$ and $\tau = T/N_T \le \tau^*$. Let $\xi \in \mathcal{C} \setminus B_\delta$. By the assumptions on the numerical quadrature method we have

$$\left|I_{\beta(\phi(\cdot,\xi))} - \widetilde{I}_{\beta(\phi(\cdot,\xi))}\right| \le C_I \tau^k. \tag{11}$$

Further, we have

$$\|\phi_i - \widetilde{\phi}_i\|_2 \le C_\phi \frac{e^{L_{\mathbf{f}}T} - 1}{L_{\mathbf{f}}} \tau^p \le C_\phi \frac{e^{L_{\mathbf{f}}T} - 1}{L_{\mathbf{f}}} T^p$$

by Lemma 3. Then, because the numbers T and c_i and the function β are nonnegative, we have

$$\begin{aligned}
\left|\widetilde{I}_{\beta(\phi(\cdot,\xi))} - V_\xi\right| &= \left|T\sum_{i=0}^{N_T} c_i\beta(\phi_i) - T\sum_{i=0}^{N_T} c_i\beta(\widetilde{\phi}_i)\right| \\
&\le T\sum_{i=0}^{N_T} c_i\left|\beta(\phi_i) - \beta(\widetilde{\phi}_i)\right| \\
&\le T\sum_{i=0}^{N_T} c_i L_\beta \|\phi_i - \widetilde{\phi}_i\|_2 \\
&\le L_\beta T C_\phi \frac{e^{L_{\mathbf{f}}T} - 1}{L_{\mathbf{f}}} \tau^p.
\end{aligned}$$

Using (9) and (10) we obtain

$$\begin{aligned}
\left|V(\xi) - V_\xi\right| &\le \left|I_{\beta(\phi(\cdot,\xi))} - \widetilde{I}_{\beta(\phi(\cdot,\xi))}\right| + \left|\widetilde{I}_{\beta(\phi(\cdot,\xi))} - V_\xi\right| \\
&\le C_I\tau^k + L_\beta T C_\phi \frac{e^{L_{\mathbf{f}}T} - 1}{L_{\mathbf{f}}} \tau^p < \varepsilon.
\end{aligned}$$

This concludes the proof of the theorem.

Hence, we have established that we can approximate $V(\xi)$ arbitrarily well numerically for every $\xi \in \mathcal{C} \setminus B_\delta$. In the next sections we first introduce triangulations and CPA functions and then show that the feasibility problem with these values V_ξ fulfills certain linear constraints that verify a CPA Lyapunov function.

4 Triangulations and CPA functions

In this section we will introduce triangulations and CPA functions as well as the definition of (h,d)-bounded triangulations.

Definition 1. *We define the following*:

i) The **convex-combination** of vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n$, denoted

$$\mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\},$$

is the set of all sums

$$\sum_{i=0}^{m} \lambda_i \mathbf{x}_i, \quad \text{where} \quad \sum_{i=0}^{m} \lambda_i = 1$$

and $\forall i: 0 \le \lambda_i \le 1$.

ii) The vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n$ are said to be **affinely-independent** if

$$\sum_{i=0}^{m} \lambda_i \mathbf{x}_i = \mathbf{0} \quad \text{and} \quad \sum_{i=0}^{m} \lambda_i = 0$$

implies $\lambda_0 = \lambda_1 = \cdots = \lambda_m = 0$.

iii) If $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n$ are affinely-independent, then the set $S = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}$ is called an m**-simplex**. The vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m$ are called the **vertices** of S. The set of vertices for an m-simplex is sometimes denoted by $\mathrm{ve}\, S = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}$. In $\mathbb{R}^n$ an n-simplex is often referred to as just a simplex.

iv) For an m-simplex S, define its **diameter** as:

$$\mathrm{diam}(S) := \max_{\mathbf{x},\mathbf{y} \in S} \|\mathbf{x} - \mathbf{y}\|_2.$$

We now define a *triangulation*. It simplifies the discussion to have the order of the vertices of every simplex in the triangulation fixed, as in e.g. [14]. Note however, that this is not necessary and the results are essentially the same for any ordering, see [16]. For an n-tuple of vertices $C = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ we define $\mathrm{co}\, C = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and for the simplex $S = \mathrm{co}\, C$ we define $\mathrm{ve}\, C = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ as the set of the vertices.

Definition 2 (Triangulation). *Let I be a set of indices. A* triangulation $\mathcal{T} = \{S_\nu\}_{\nu \in I}$ *in $\mathbb{R}^n$ is a set of n-simplices S_ν with ordered vertices $C_\nu = \left(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\right)$ for all $\nu \in I$, such that*

$$S_\mu \cap S_\nu = \mathrm{co}\, \mathrm{ve}\, S_\mu \cap \mathrm{co}\, \mathrm{ve}\, S_\nu = \mathrm{co}(\mathrm{ve}\, S_\mu \cap \mathrm{ve}\, S_\nu) \tag{12}$$

for all $\mu, \nu \in I$. We call $\mathcal{T}$ locally finite, if for every compact $\mathcal{C} \subset \mathbb{R}^n$ the set $\{S_\nu \in \mathcal{T} \mid S_\nu \cap \mathcal{C} \neq \emptyset\}$ is finite.

The domain of $\mathcal{T}$ is defined as

$$\mathcal{D}_\mathcal{T} := \bigcup_{\nu \in I} S_\nu$$

and its complete set of vertices is denoted by

$$\mathcal{V}_\mathcal{T} := \bigcup_{\nu \in I} \mathrm{ve}\, S_\nu.$$

Further, we define the diameter of $\mathcal{T}$ as

$$\mathrm{diam}(\mathcal{T}) := \sup\nolimits_{S \in \mathcal{T}} \mathrm{diam}(S).$$

Given a triangulation $\mathcal{T}$, a continuous and piecewise affine function, i.e. CPA function, can be defined by fixing its values at $\mathcal{V}_\mathcal{T}$.

Definition 3 (CPA function). *Let $\mathcal{T}$ be a triangulation in $\mathbb{R}^n$. We denote by* $\mathrm{CPA}[\mathcal{T}]$ *the set of all continuous functions*

$$W : \mathcal{D}_\mathcal{T} \to \mathbb{R}$$

that are affine on each simplex $S_\nu \in \mathcal{T}$, i.e. for each $S_\nu \in \mathcal{T}$ there exists a vector $\mathbf{g}_\nu \in \mathbb{R}^n$ and a number $a_\nu \in \mathbb{R}$ such that

$$W(\mathbf{x}) = \mathbf{g}_\nu \bullet \mathbf{x} + a_\nu \quad \text{for all } \mathbf{x} \in S_\nu.$$

We define the (column) vector $\nabla W_\nu := \mathbf{g}_\nu$ for every $S_\nu \in \mathcal{T}$.

Let $W \in \mathrm{CPA}[\mathcal{T}]$ and $\mathbf{x} \in \mathcal{D}_{\mathcal{T}}$. Then there is a simplex $S_\nu = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{T}$ such that $\mathbf{x} \in S$. Further, $\mathbf{x}$ has a unique representation as the convex combination of the vertices of S_ν, i.e. there are unique numbers $\lambda_i^{\mathbf{x}} \in [0,1]$, $i = 0, 1, \ldots, n$, such that

$$\mathbf{x} = \sum_{i=0}^{n} \lambda_i^{\mathbf{x}} \mathbf{x}_i \quad \text{and} \quad \sum_{i=0}^{n} \lambda_i^{\mathbf{x}} = 1.$$

Since for every $\mathbf{x} \in S_\nu$ we have

$$W(\mathbf{x}) = \mathbf{g}_\nu \bullet \sum_{i=0}^{n} \lambda_i^{\mathbf{x}} \mathbf{x}_i + a_\nu = \sum_{i=0}^{n} \lambda_i^{\mathbf{x}} (\mathbf{g}_\nu \bullet \mathbf{x}_i + a_\nu) = \sum_{i=0}^{n} \lambda_i^{\mathbf{x}} W(\mathbf{x}_i),$$

each $W \in \mathrm{CPA}[\mathcal{T}]$ is completely determined by its values in the vertex set $\mathcal{V}_{\mathcal{T}}$.

To have concrete examples of triangulations useful for the CPA algorithm we recall the definition of the standard triangulation $\mathcal{T}_{\mathrm{std}}$ as given in [1]; for a graphical representation see Figure 1.

Definition 4 (The Standard Triangulation of $\mathbb{R}^n$). *The* Standard Triangulation *is a triangulation $\mathcal{T}_{\mathrm{std}} = \{S_\nu\}_{\nu \in I}$ with indices $\nu = (\mathbf{z}, \sigma, \mathbf{J}) \in \mathbb{N}_0^n \times Sym(n) \times \{-1,+1\}^n =: I$ and vertices $C_\nu = (\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)$ given by:*

$$\mathbf{x}_k^\nu = R_{\mathbf{J}} \left(\mathbf{z} + \sum_{l=1}^{k} \mathbf{e}_{\sigma(l)} \right) = R_{\mathbf{J}} \mathbf{z} + R_{\mathbf{J}} \mathbf{u}_k^\sigma. \tag{13}$$

Here, $\mathbf{J} = (J_1, J_2, \ldots, J_n)^{\mathrm{T}} \in \{-1,+1\}^n$ and $R_{\mathbf{J}} = \mathrm{diag}(\mathbf{J}) \in \mathbb{R}^{n \times n}$ is a matrix corresponding to the reflection specified by $\mathbf{J} \in \{-1,+1\}^n$. Further, $Sym(n)$ denotes the set of permutations $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ and

$$\mathbf{u}_k^\sigma = \sum_{l=1}^{k} \mathbf{e}_{\sigma(l)}.$$

We now define the shape-matrix of a simplex, of which the vertices are in a particular order. This is needed to define (h,d)-bounded triangulations. We will explain the importance of shape-matrices in computing CPA Lyapunov functions below.

Definition 5. *For an n-simplex S_ν of a triangulation with vertices $C_\nu = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ its* **shape-matrix** *X_ν is defined by*

$$X_\nu := \begin{pmatrix} (\mathbf{x}_1 - \mathbf{x}_0)^{\mathrm{T}} \\ (\mathbf{x}_2 - \mathbf{x}_0)^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_n - \mathbf{x}_0)^{\mathrm{T}} \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Remark 2. The vectors $\nabla W_\nu = \mathbf{g}_\nu$ in Definition 3 are given by the formula

$$\nabla W_\nu = \mathbf{g}_\nu = X_\nu^{-1} \mathbf{v}_\nu \quad \text{where} \quad \mathbf{v}_\nu := \begin{pmatrix} W(\mathbf{x}_1) - W(\mathbf{x}_0) \\ W(\mathbf{x}_2) - W(\mathbf{x}_0) \\ \vdots \\ W(\mathbf{x}_n) - W(\mathbf{x}_0) \end{pmatrix}, \tag{14}$$

see e.g. [13, Rem. 9].

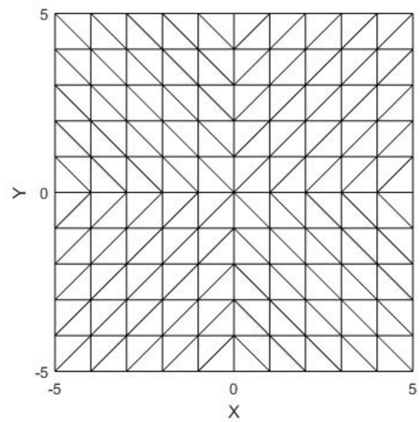

Fig. 1. The standard triangulation $\mathcal{T}_{\text{std}}$ in $\mathbb{R}^2$ on $[-5,5]^2$

We define the degeneracy of a triangulation and (h,d)-bounded triangulations as in [14]. Note that the degeneracy depends on the order of the vertices of the simplex, but it was shown in [16] that if a triangulation is (h,d)-bounded then any reordering of the vertices will results in a (h,d^*)-bounded triangulation with $d^* = d(1+d\sqrt{n-1})$. The motivation for the definition comes from the fact that the CPA method always succeeds in computing a Lyapunov function if one exists. The proof of this fact in [13] uses a sequence of finite triangulations $\mathcal{T}_k$ with the following properties:

- the simplices become smaller, i.e. $\text{diam}(\mathcal{T}_k) \to 0$ as $k \to \infty$, and
- $\sup_{S_\nu \in \mathcal{T}_k} \text{diam}(S_\nu)^2 \cdot \|X_\nu^{-1}\|_2 \to 0$ as $k \to \infty$, or, as a sufficient condition, that

$$\sup_{S_\nu \in \mathcal{T}_k} \text{diam}(S_\nu) \cdot \|X_\nu^{-1}\|_2 \le d$$

is bounded.

Definition 6. *We define the* **degeneracy** *of the triangulation $\mathcal{T}$ to be the quantity*

$$\sup_{S_\nu \in \mathcal{T}} \text{diam}(S_\nu) \|X_\nu^{-1}\|_2,$$

*where X_ν is the shape-matrix of S_ν. We say that the triangulation $\mathcal{T}$ is (h,d)-***bounded** *for constants $h,d > 0$, if $\text{diam}(\mathcal{T}) < h$ and the degeneracy of $\mathcal{T}$ is bounded by d, i.e. $\sup_{S_\nu \in \mathcal{T}} \text{diam}(S_\nu)\|X_\nu^{-1}\|_2 \le d$.*

Practically, to obtain a (h,d)-bounded triangulation, one can scale down the standard triangulation in the following way: to scale down the simplex S we multiply the vertices of S with a number $0 < \rho < 1$, then $\text{diam}(\rho S) = \rho\, \text{diam}(S)$ and $\|X_{\rho S}^{-1}\|_2 = \rho^{-1}\|X_S^{-1}\|_2$. Here and in the following $X_{\rho^k S}$, $k \in \mathbb{N}_0$, denotes the shape-matrix of the simplex $\rho^k S$ with the order of the vertices determined by the standard triangulation for the simplex S.

We have $\mathrm{diam}(\mathcal{T}_{\mathrm{std}}) = \sqrt{n}$ and from [26, Remark 2] we know that $\sup_{S_\nu \in \mathcal{T}_{\mathrm{std}}} \|X_\nu^{-1}\|_2 \leq 2$. Fix a constant ρ fulfilling $0 < \rho < 1$ and a compact set $\mathcal{C}$, and define

$$\mathcal{T}_k := \{\rho^k S_\nu \,:\, (\rho^k S_\nu) \cap \mathcal{C}^\circ \neq \emptyset\}$$

for $k \in \mathbb{N}_0$. Then for each $k \in \mathbb{N}_0$, $\mathcal{T}_k$ consists of a finite number of simplices, and we have

$$\mathrm{diam}(\mathcal{T}_k) = \rho^k \sqrt{n}$$
$$\text{and } \sup_{\rho^k S \in \mathcal{T}_k} \mathrm{diam}(\rho^k S)\|X_{\rho^k S}^{-1}\|_2 \leq \rho^k \sqrt{n} \cdot \rho^{-k} 2 = 2\sqrt{n}.$$

Thus

$$\mathrm{diam}(\mathcal{T}_k) \to 0 \text{ as } k \to \infty$$
$$\text{and } \sup_{\rho^k S \in \mathcal{T}_k} \mathrm{diam}(\rho^k S)\|X_{\rho^k S}^{-1}\|_2 \leq 2\sqrt{n} =: d$$

is bounded.

5 The CPA feasibility problem

The linear constraints of the CPA algorithm to compute a Lyapunov function W for system (1) consist of two sets of constraints: the first set forces W to have a minimum at the equilibrium at the origin and the second set forces W to be decreasing along all solution trajectories. By assigning $W(\xi) := V_\xi$ with, the formula (20) the first set is automatically taken care of if the second set is fulfilled. This remains essentially true, even if one removes a small neighbourhood of the equilibrium from the domain of the Lyapunov function, see [14] for the details. Therefore we are only interested in the second set of constraints. Further, we assume that $\mathbf{f}$ in system (1) is C^2, i.e. $s \geq 2$. It is possible to derive constraints for the CPA method with $s = 1$, see [3], but in this case one needs much smaller simplices for the key estimates to hold true.

Definition 7. *Let $\mathcal{T}$ be a triangulation and let $W \in \mathrm{CPA}[\mathcal{T}]$. We denote by $\mathcal{T}_{dec} \subset \mathcal{T}$ the set of simplices $S_\nu \in \mathcal{T}$ with ordered vertices $C_\nu = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ such that*

$$\nabla W_\nu \bullet \mathbf{f}(\mathbf{x}_i) + \|\nabla W_\nu\|_1 E_i^\nu < 0 \quad \text{for } i = 0, 1, \ldots, n, \tag{15}$$

holds, where

$$\nabla W_\nu \;=\; X_\nu^{-1}\mathbf{w}_\nu \;\; \text{with} \;\; \mathbf{w}_\nu := \begin{pmatrix} W(\mathbf{x}_1) - W(\mathbf{x}_0) \\ W(\mathbf{x}_2) - W(\mathbf{x}_0) \\ \vdots \\ W(\mathbf{x}_n) - W(\mathbf{x}_0) \end{pmatrix}, \tag{16}$$

$$E_i^\nu := \max_{j=0,1,\ldots,n} \sum_{r,s=1}^{n} \frac{B_{r,s}}{2} |\mathbf{e}_r \bullet (\mathbf{x}_i - \mathbf{x}_0)| \left(|\mathbf{e}_s \bullet (\mathbf{x}_j - \mathbf{x}_0)| + |\mathbf{e}_s \bullet (\mathbf{x}_i - \mathbf{x}_0)|\right) \tag{17}$$

$$\text{and } B_{r,s}^\nu \;\geq\; \max_{k=1,2,\ldots,n} \max_{\mathbf{x} \in S} \left| \frac{\partial^2 f_k}{\partial x_r \partial x_s}(\mathbf{x}) \right|, \quad r, s = 1, 2, \ldots, n, \tag{18}$$

are given constants for S_ν.

Note that the constants E_i^ν are chosen such that if the constraints (15) hold true for S_ν, i.e. $S_\nu \in \mathcal{T}_{\text{dec}}$, then $\nabla W_\nu \bullet \mathbf{f}(\mathbf{x}) < 0$ for all $\mathbf{x} \in S_\nu$. For this fact see [34, 35].

It it shown in, e.g., [12], that

$$\limsup_{h\to 0+} \frac{W(\mathbf{x}+h\mathbf{f}(\mathbf{x}))-W(\mathbf{x})}{h} \le \max_{\nu:\ \mathbf{x}\in S_\nu} \nabla W_\nu \bullet \mathbf{f}(\mathbf{x}) < 0 \tag{19}$$

for all $\mathbf{x} \in \mathcal{D}^\circ_{\mathcal{T}_{\text{dec}}}$, i.e. the mapping $t \mapsto W(\phi(t,\xi))$ is strictly decreasing for all $\phi(t,\xi)$ in the interior of the domain of the triangulation $\mathcal{T}_{\text{dec}}$.

Remark 3. The only input to the verification problem are the constants $B^\nu_{r,s}$, that are upper bounds on the second order derivatives of the components of the vector field $\mathbf{f}$ defining the dynamics of system (1). The constants E_i^ν are computed algorithmically from the $B^\nu_{r,s}$ and the geometry of the simplex $S_\nu = \mathrm{co}\,C_\nu$. Note that any upper bounds $B^\nu_{r,s}$ will suffice, but obviously it is easier to fulfill the constraints for smaller values of the $B^\nu_{r,s}$.

Let us recap the situation. Assume the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^s(\mathbb{R}^n,\mathbb{R}^n)$, $s \ge 2$, has an exponentially stable equilibrium at the origin and let $\beta\colon \mathbb{R}^n \to \mathbb{R}_+$ be a locally Lipschitz and positive definite function that is C^s on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. For sufficiently large T the function

$$V(\xi) = \int_0^T \beta(\phi(t,\xi))dt$$

is a Lyapunov function for the system, which we approximate using numerical integration and numerical quadrature.

In the next theorem we prove that we can verify the condition (15) in Definition 7 for all vertices in $\mathcal{C}\setminus B_\delta$ for sufficiently fine triangulations and sufficiently accurate numerical approximations.

Theorem 3. *Consider system* (1) *with* $s \ge 2$. *Let* $\delta > 0$, $d > 0$ *and* $\mathcal{C} \subset\subset \mathcal{A}$ *be given such that* $\mathcal{C}\setminus B_\delta \neq \emptyset$. *Further, let* $\beta\colon \mathbb{R}^n \to \mathbb{R}_+$ *be a locally Lipschitz continuous and positive definite function, that is* C^s *on* $\mathbb{R}^n \setminus \{\mathbf{0}\}$. *Consider a numerical integration method for system* (1), *with order of consistency* $1 \le p \le s$ *as in Lemma 3 on* $\mathcal{K}$, $\mathcal{A} \supset\supset \mathcal{K} \supset \overline{\phi([0,\infty),\mathcal{C})}$, *that is forward invariant both for* (1) *and the numerical method with step size* $0 < \tau \le \tau'$ *on* $\mathcal{K}$, *and a quadrature method of order k,* $p \le k \le s$ *as in Assumption 1. Fix a constant*

$$B \ge \max_{\mathbf{z}\in\mathcal{C}} \max_{k,r,s=1,2,\dots,n} \left| \frac{\partial^2 f_k}{\partial x_r \partial x_s}(\mathbf{z}) \right|.$$

Then there exists $T^* > 0$, *such that for any fixed* $T \ge T^*$ *and any small enough* $h > 0$ *we have: for any fixed* (h,d)*-bounded and locally finite triangulation* $\mathcal{T}$, *there exists* $N_T^* \in \mathbb{N}$ *such that for all* $N_T \in \mathbb{N}$ *with* $N_T \ge N_T^*$ *and*

$$W(\xi) = V_\xi := T\sum_{i=0}^{N_T} c_i \beta(\widetilde{\phi}_i) \quad \textit{for all } \xi \in \mathcal{V}_\mathcal{T}, \tag{20}$$

where the c_i are from the quadrature rule (see Assumption 1) and the $\tilde{\phi}_i$ are the approximations of $\phi_i := \phi(i \cdot \tau, \xi)$ with $\tau := T/N_T$ (see Lemma 3), the following inclusion holds

$$\{S \in \mathcal{T} \mid S \subset \mathcal{C} \setminus B_\delta\} \subset \mathcal{T}_{dec}.$$

Here $\mathcal{T}_{dec}$ denotes the simplices satisfying the condition (15) *with $B^\nu_{r,s} \le B$, see Definition 7.*

Proof. Define $T^* > 0$ according to Lemma 2 and fix $T \ge T^*$. Define m by

$$\max_{\mathbf{x} \in \mathcal{C} \setminus B_\delta} \nabla V(\mathbf{x}) \bullet \mathbf{f}(\mathbf{x}) =: -3m < 0,$$

see (3). Denote by H_V the Hessian matrix of V, which is C^s on $\mathbb{R}^n \setminus B_\delta$, and set

$$H := \max_{\mathbf{z} \in \mathcal{C} \setminus B_\delta} \|H_V(\mathbf{z})\|_2 \tag{21}$$

$$G := \max_{\mathbf{z} \in \mathcal{C} \setminus B_\delta} \|\nabla V(\mathbf{z})\|_2 \tag{22}$$

$$F := \max_{\mathbf{z} \in \mathcal{C}} \|\mathbf{f}(\mathbf{z})\|_2. \tag{23}$$

Fix $h > 0$ so small that

$$\frac{d\sqrt{n}+2}{2} Hh \left(F + n^{\frac{5}{2}} Bh^2\right) + n^{\frac{5}{2}} BGh^2 \le m. \tag{24}$$

Choose an (h,d)-bounded, locally finite triangulation $\mathcal{T}$ and consider the (finite) set of simplices

$$\mathcal{T}^* = \{S \in \mathcal{T} \mid S \subset \mathcal{C} \setminus B_\delta\}.$$

Define

$$h_* = \min_{S \in \mathcal{T}^*} \operatorname{diam}(S) > 0,$$

set

$$\varepsilon := \frac{m}{d\sqrt{n}} \left(\frac{F}{h_*} + n^{\frac{5}{2}} Bh\right)^{-1}, \tag{25}$$

and choose N_T^* as in Theorem 2 for $\varepsilon/2$, and let $N_T \ge N_T^*$. Thus $|V(\xi) - V_\xi| < \varepsilon/2$ for all $\xi \in \mathcal{C} \setminus B_\delta$.

Let us consider an arbitrary simplex $S_\nu \subset \mathcal{T}^*$ with vertices $C_\nu = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ and shape-matrix X_ν. Set $h_\nu := \operatorname{diam}(S_\nu)$. For $\mathbf{x}, \mathbf{y} \in S_\nu$ we have

$$\|\nabla V(\mathbf{x}) - \nabla V(\mathbf{y})\|_2 \le Hh_\nu \le Hh \tag{26}$$

and for some $\mathbf{z}^* \in S_\nu$ on the line segment between $\mathbf{x}$ and $\mathbf{y}$ that

$$V(\mathbf{x}) = V(\mathbf{y}) + \nabla V(\mathbf{y}) \bullet (\mathbf{x} - \mathbf{y}) + \frac{1}{2}(\mathbf{x} - \mathbf{y})^{\mathrm{T}} H_V(\mathbf{z}^*)(\mathbf{x} - \mathbf{y}),$$

i.e.

$$|V(\mathbf{x})-V(\mathbf{y})-\nabla V(\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})| = \frac{1}{2}\left|(\mathbf{x}-\mathbf{y})^{\mathrm{T}}H_V(\mathbf{z}^*)(\mathbf{x}-\mathbf{y})\right| \le \frac{1}{2}Hh_\nu^2. \tag{27}$$

Further,

$$E_i^\nu \le n^2Bh_\nu^2 \quad \text{for } i=0,1,\ldots,n.$$

With $W(\mathbf{x}_i)=V_{\mathbf{x}_i}$ for $i=0,1,\ldots,n$ we have

$$\nabla W_\nu = X_\nu^{-1}\mathbf{w}_\nu, \quad \text{where} \quad \mathbf{w}_\nu = \begin{pmatrix} V_{\mathbf{x}_1}-V_{\mathbf{x}_0} \\ V_{\mathbf{x}_2}-V_{\mathbf{x}_0} \\ \vdots \\ V_{\mathbf{x}_n}-V_{\mathbf{x}_0} \end{pmatrix}; \quad \text{define} \quad \mathbf{v}_\nu := \begin{pmatrix} V(\mathbf{x}_1)-V(\mathbf{x}_0) \\ V(\mathbf{x}_2)-V(\mathbf{x}_0) \\ \vdots \\ V(\mathbf{x}_n)-V(\mathbf{x}_0) \end{pmatrix}.$$

Note that

$$\|\mathbf{v}_\nu-\mathbf{w}_\nu\|_\infty \le \varepsilon. \tag{28}$$

Now

$$\begin{aligned}\|\nabla W_\nu-\nabla V(\mathbf{x}_0)\|_2 &= \|X_\nu^{-1}\mathbf{w}_\nu-\nabla V(\mathbf{x}_0)\|_2 = \|X_\nu^{-1}(\mathbf{w}_\nu-X_\nu\nabla V(\mathbf{x}_0))\|_2 \\ &\le \|X_\nu^{-1}\|_2\|\mathbf{w}_\nu-X_\nu\nabla V(\mathbf{x}_0)\|_2. \end{aligned} \tag{29}$$

Note that the j-th component of the vector $\mathbf{w}_\nu-X_\nu\nabla V(\mathbf{x}_0)$ is given by

$$[\mathbf{w}_\nu-X_\nu\nabla V(\mathbf{x}_0)]_j = V_{\mathbf{x}_j}-V_{\mathbf{x}_0}-[V(\mathbf{x}_j)-V(\mathbf{x}_0)]+V(\mathbf{x}_j)-V(\mathbf{x}_0)-\nabla V(\mathbf{x}_0)\cdot(\mathbf{x}_j-\mathbf{x}_0)$$

and can thus be bounded using (28) and (27),

$$|\,[\mathbf{w}_\nu-X_\nu\nabla V(\mathbf{x}_0)]_j\,| \le \varepsilon+\frac{1}{2}Hh_\nu^2.$$

Hence

$$\|\mathbf{w}_\nu-X_\nu\nabla V(\mathbf{x}_0)\|_2 \le \sqrt{n}\left(\varepsilon+\frac{1}{2}Hh_\nu^2\right) \tag{30}$$

Thus, by (26), (29) and (30), for every $i=0,1,\ldots,n$ we have

$$\begin{aligned}\|\nabla W_\nu-\nabla V(\mathbf{x}_i)\|_2 &\le \|\nabla W_\nu-\nabla V(\mathbf{x}_0)\|_2+\|\nabla V(\mathbf{x}_i)-\nabla V(\mathbf{x}_0)\|_2 \\ &\le \|X_\nu^{-1}\|_2\|\mathbf{w}_\nu-X_\nu\nabla V(\mathbf{x}_0)\|_2+Hh \\ &\le \sqrt{n}\|X_\nu^{-1}\|_2\left(\varepsilon+\frac{1}{2}Hh_\nu^2\right)+Hh \\ &\le h_\nu\|X_\nu^{-1}\|_2\sqrt{n}\left(\frac{\varepsilon}{h_\nu}+\frac{1}{2}Hh_\nu\right)+Hh \\ &\le d\sqrt{n}\left(\frac{\varepsilon}{h_\nu}+\frac{1}{2}Hh\right)+Hh \;=:\; A_\nu \\ &\le d\sqrt{n}\left(\frac{\varepsilon}{h_*}+\frac{1}{2}Hh\right)+Hh \\ &= d\sqrt{n}\frac{\varepsilon}{h_*}+\frac{d\sqrt{n}+2}{2}Hh. \end{aligned}$$

In particular,

$$\|\nabla W_\nu\|_1 \le \|\nabla W_\nu - \nabla V(\mathbf{x}_0)\|_1 + \|\nabla V(\mathbf{x}_0)\|_1 \le \sqrt{n}(A_\nu + G). \tag{31}$$

Combining all the results, we obtain

$$\begin{aligned}
&\nabla W_\nu \bullet \mathbf{f}(\mathbf{x}_i) + \|\nabla W_\nu\|_1 E_i^\nu \\
&\le \nabla V(\mathbf{x}_i) \bullet \mathbf{f}(\mathbf{x}_i) + [\nabla W_\nu - \nabla V(\mathbf{x}_i)] \bullet \mathbf{f}(\mathbf{x}_i) + \sqrt{n}(A_\nu + G)E_i^\nu \\
&\le -3m + \|\nabla W_\nu - \nabla V(\mathbf{x}_i)\|_2 \max_{\mathbf{z}\in S_\nu} \|\mathbf{f}(\mathbf{z})\|_2 + \sqrt{n}(A_\nu + G)E_i^\nu \\
&\le -3m + A_\nu F + \sqrt{n}(A_\nu + G)E_i^\nu \\
&= -3m + A_\nu \left(F + \sqrt{n}E_i^\nu\right) + \sqrt{n}GE_i^\nu \\
&\le -3m + \left(d\sqrt{n}\left(\frac{\varepsilon}{h_\nu} + \frac{1}{2}Hh\right) + Hh\right)\left(F + n^{\frac{5}{2}}Bh_\nu^2\right) + n^{\frac{5}{2}}BGh_\nu^2 \\
&\le -3m + \varepsilon \cdot d\sqrt{n}\left(\frac{F}{h_*} + n^{\frac{5}{2}}Bh\right) + \frac{d\sqrt{n}+2}{2}Hh\left(F + n^{\frac{5}{2}}Bh^2\right) + n^{\frac{5}{2}}BGh^2 \\
&\le -3m + m + m = -m < 0,
\end{aligned}$$

where we have used (25) and (24) in the last step.

6 Conclusions

The construction of a Lyapunov function is an important and non-trivial problem to determine the basin of attraction of equilibria. The CPA algorithm to compute continuous and piecewise affine (CPA) Lyapunov functions for nonlinear systems uses linear optimisation to determine the CPA Lyapunov function. However, determining the function through numerical integration and quadrature, and then just using the error estimates of the CPA method to verify the conditions of a Lyapunov function, is considerably faster. In this paper we have proven that this method always succeeds in computing and verifying a CPA Lyapunov function if the triangulation is sufficiently fine and the numerical methods sufficiently accurate.

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