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PREFACE

Dynamical systems arise from a variety of problems in physics, engineering, biology and economics, which are modelled, e.g., by ordinary or partial differential equations, stochastic differential equations, or iterations. The main goal of dynamical systems is to study the long-term behavior of solutions and to distinguish between different behavior according to initial conditions. Lyapunov functions and contraction metrics are tools to achieve this goal without the need to compute individual solutions.

Aleksandr Lyapunov generalized the idea of the energy of a dissipative physical system to the concept of Lyapunov functions, which decrease along solutions. They characterize stability, local attractors and their basins of attraction. Lyapunov functions for one asymptotically stable equilibrium are a standard tool in control theory, but their theory is much more general and the existence of complete Lyapunov functions has even been suggested as the Fundamental theorem of dynamical systems.

A contraction metric is a Riemannian metric, with respect to which the distance of adjacent solutions decreases over time. Hence, a contraction metric characterizes similar long-term behavior and stability. The study of contraction metrics is somewhat complicated by the fact that many closely related concepts like incremental stability, contracting systems and Finsler Lyapunov functions have been investigated in the literature; for many systems these concepts are indeed equivalent.

Both Lyapunov functions and contraction metrics define tailor-made metrics for the system at hand. The main difference is that the former defines the metric in the state-space of the system, while the latter defines it in the tangent space.

While Lyapunov functions and contraction metrics are classical and useful tools, it is generally close to impossible to find a suitable Lyapunov function or contraction metric for a given system *analytically*. Converse theorems, ensuring the existence in theory, have been established, but the explicit construction remains a formidable task. In recent years, various *numerical* methods have been derived for the computation of Lyapunov functions and contraction metrics in various settings, using collocation, generalized interpolation in Hilbert spaces, convex optimization and other methods. The combination of advanced numerical methods with the theory of dynamical systems has led to remarkable advances, which are the topic of this special issue.

The papers in the special issue cover a wide range of theory and applications of Lyapunov functions and contraction metrics together with their numerical computation. In the following we will give an overview of the papers.

Review on contraction analysis and computation of contraction metrics by Peter Giesl, Sigurdur Hafstein and Christoph Kawan gives an introduction to contraction analysis and its historical development. In particular, several related presentations of the relevant concepts in the literature are discussed, together with the connection between contraction metrics, contraction analysis and Lyapunov stability theory. Moreover, recent numerical construction methods are reviewed, compared and put into context.

A converse sum of squares Lyapunov function for outer approximation of minimal attractor sets of nonlinear systems by Morgan Jones and Matthew Peet considers the concept of an attractor set as a generalization of an equilibrium and seeks to determine an optimal outer approximation of this set. The authors prove a converse result, showing a characterization of attractor sets with Lyapunov functions, even when the Lyapunov function is constrained to the set of sum-of-squared polynomials. This leads to the formulation and use of a sum-of-squares approach to compute these Lyapunov functions, which is then applied to three numerical examples.

A novel moving orthonormal coordinate-based approach for region of attraction analysis of limit cycles by Eva Ahbe, Andrea Iannelli and Roy Smith deals with the determination of the region of attraction of limit cycles. They use a novel construction for a moving orthonormal coordinate system by introducing a center point and bilinear sum-of-squares optimization to compute inner estimates of the region of attraction for polynomial systems. They demonstrate the applicability of their approach for a stabilized airborne wind energy system.

Finding invariant sets and proving exponential stability of limit cycles using sum-of-squares decompositions by Elias August and Mauricio Barahona develops methods to determine positively invariant sets and to show the existence of limit cycles as well as to determine their basins of attraction. The analytical tools are sublevel sets of a Lyapunov-like function to determine positively invariant sets and a Riemannian contraction metric to show existence of a limit cycle and to determine its basin of attraction. The numerical construction of both is achieved using sum-of-squares and they apply the methods to numerous systems.

Modified refinement algorithm to construct Lyapunov functions using meshless collocation by Najla Mohammed and Peter Giesl considers the construction of Lyapunov functions for an ordinary differential equation with an exponentially stable equilibrium using meshfree collocation by approximating the solution of a PDE for the orbital derivative. First, a previous grid refinement algorithm is studied, which in many cases enables the computation of a Lyapunov function with fewer collocation points than with a regular grid. Then the grid refinement is enhanced by additionally using a clustering algorithm, which eliminates the sensitiveness of the grid refinement to the starting grid. The improvements of the new refinement algorithm, both in comparison to using a regular grid and to the previous grid refinement algorithm, are shown for three examples.

Low-Rank Kernel Approximation of Lyapunov Functions using Neural Networks by Kevin Webster investigates the approximation of Lyapunov functions for ordinary differential equations with an exponentially stable equilibrium, using a neural network with a single hidden layer. In particular, the relation of this approach to the method using a Reproducing Kernel Hilbert Space is analyzed, it is shown that an optimized neural network is functionally equivalent to a Reproducing Kernel

Hilbert Space method and that the method is able to learn an appropriate kernel. Convergence for the neural network and the learning of the kernel is established in numerical examples.

Learning dynamical systems using local stability priors by Arash Mehrjou, Andrea Iannelli and Bernhard Schölkopf introduces a numerical method to approximate a Lyapunov function for an unknown nonlinear dynamical system with an asymptotically stable equilibrium, as well as approximating the underlying vector field. The algorithm learns the ordinary differential equation and the region of attraction simultaneously from observed trajectories. The method is entirely based on data, which can be noisy, and uses Reproducing Kernel Hilbert Spaces and deep neural networks as approximation architectures. The proposed framework is applied to two well-known nonlinear test systems.

Smooth complete Lyapunov functions for multifunctions by Stefan Suhr deals with chain recurrence and complete Lyapunov functions for differential inclusions, the right-hand sides of which are given by convex semicontinuous multifunctions. After suitably defining the chain recurrent set for such systems, the existence of complete Lyapunov functions characterizing the chain recurrent sets is established. This is done by adapting a previous construction of time functions for cone fields to convex, semicontinuous multifunction differential inclusions. The complete Lyapunov function may be assumed to be smooth, i.e. infinitely differentiable.

Construction of mean-square Lyapunov-basins for random ordinary differential equations by Florian Rupp proposes a basin search algorithm for random ordinary differential equations using mean-square Lyapunov functions around the equilibria of interest. This allows to numerically estimate the effect of noise intensity on the size of the basin of attraction, and at which points stability vanishes. Some counterexamples are presented together with a benchmark example showing the usefulness of the method.

Monotonicity and discretization of Hammerstein integrodifference equations by Magdalena Nockowska-Rosiak and Christian Pötzsche considers recursions given by a Hammerstein integral operator with a kernel. The authors provide conditions on the kernel to ensure monotonicity, subhomogeneity and convexity. Moreover, they analyse whether these structural properties are preserved for numerical methods using spatial discretization techniques. Numerous theoretical and practical examples for the applications of the theory are given, in particular for models in theoretical ecology.

Contraction analysis of Volterra integral equations via realization theory and frequency-domain methods by Elena Kudryashova and Volker Reitmann deals with infinite-dimensional nonlinear Volterra equations in a Hilbert space. First, the authors reduce the problem to a state-space realization that inherits the properties of the initial integral equation. Then they demonstrate how the frequency-domain technique can be applied to study its absolute stability and instability. Finally, the authors develop a contraction method applicable to Volterra equations with periodic coefficients, and use it to obtain conditions for the existence of a globally stable periodic solution.

We are greatly indebted to Editor-in-Chief Robert McLachlan for the opportunity to serve as guest editors for this special issue on *Computations of Lyapunov functions and contraction metrics*, a research area in which we have invested most of our professional careers. Further, we would like to thank the numerous contributors, that delivered a plethora of interesting original work for the issue, including theoretical results, numerical methods and many examples. We are certain that this special issue advances the field of numerics in dynamical systems and hope that its contributions are of interest to many readers.

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