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SYSTEM SPECIFIC TRIANGULATIONS FOR THE CONSTRUCTION OF CPA LYAPUNOV FUNCTIONS

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Abstract. Recently, a transformation of the vertices of a regular triangulation of $\mathbb{R}^n$ with vertices in the lattice $\mathbb{Z}^n$ was introduced, which distributes the vertices with approximate rotational symmetry properties around the origin. We prove that the simplices of the transformed triangulation are (h, d)-bounded, a type of non-degeneracy particularly useful in the numerical computation of Lyapunov functions for nonlinear systems using the CPA (continuous piecewise affine) method. Additionally, we discuss and give examples of how this transformed triangulation can be used together with a Lyapunov function for a linearization to compute a Lyapunov function for a nonlinear system with the CPA method using considerably fewer simplices than when using a regular triangulation.

1. Introduction.

A Lyapunov function, introduced by Lyapunov in 1892 [18], is an indispensable tool in the stability analysis of dynamical systems. It is a real-valued function defined on a subset of the state space that is decreasing along solutions of an ordinary differential equation. Through its minima and sublevel sets, attractors and basins of attraction can be localized. Lyapunov stability theory is discussed in practically all textbooks and monographs on linear and nonlinear systems, cf. e.g. [28, 27, 15] or [22, 26, 17] for a more modern treatment. The canonical candidate for a Lyapunov function for a physical system is its (free) energy. In particular, a dissipative physical system must approach a state of a local minimum of the energy. For general dynamical systems, however, there is no analytical method to obtain a Lyapunov function.

For this reason, various methods for the numerical generation of Lyapunov functions have emerged. To name a few, in [25, 24] the numerical generation of rational Lyapunov functions was studied, in [20, 2] sum-of-squared (SOS) polynomial Lyapunov functions were parameterized using semi-definite optimization, see also

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[21, 16] for other approaches using polynomials, and in [4] a Zubov type PDE was approximately solved using collocation. For more numerical approaches cf. the review [8].

In [19] linear programming was used to parameterize continuous and piecewise affine (CPA) Lyapunov functions. In this approach, a subset of the state space is first triangulated, i.e. subdivided into simplices, and then a number of constraints are derived for a given nonlinear system, such that a feasible solution to the resulting linear programming problem allows for the parametrization of a CPA Lyapunov function for the system. In [10, 11, 6] it was proved that this approach always succeeds in computing a Lyapunov function for a general nonlinear system with an exponentially stable equilibrium, if the simplices are small enough and nondegenerate. In more detail, a sequence of triangulations $\mathcal{T}_k$ of $\mathbb{R}^n$ is needed, such that $\mathcal{T}_k$ is (h_k, d)-bounded for a fixed d and $h_k \to 0$ as $k \to \infty$. Recall, that $h_k > 0$ is an upper bound on the diameters and $d > 0$ is an upper bound on the degeneracy, as defined in Definition 2.8, of the simplices of the triangulation. Given any triangulation $\mathcal{T}_1$ of (the whole of) $\mathbb{R}^n$ that is (h_1, d)-bounded, such a sequence can be constructed by uniformly scaling down the simplices in $\mathcal{T}_1$, i.e. $\mathcal{T}_k := (h_k/h_1)\mathcal{T}_1$ with $h_k \to 0$. The standard triangulation, see Definition 2.3, is an example of a regular triangulation of $\mathbb{R}^n$ that is (h, d)-bounded, and thus, can be used to generate such a sequence. However, its simplices are rectangular and not adaptable to the system at hand, which in general has more of an approximate elliptical symmetry close to the equilibrium.

Let us make this last point clearer: Figure 1, left, shows the regular rectangular triangulation $\mathcal{T}_K^{\text{std}}$ in two dimensions, with a triangle fan at the origin as defined in Definition 2.4. In [6] it was shown that by uniformly decreasing the size of the triangles (simplices) and adding triangles (simplices) to the triangle fan at the origin, one can always compute a CPA Lyapunov function for a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, with an exponentially stable equilibrium at the origin. The domain of this Lyapunov function can be chosen as any compact subset of the equilibrium's basin of attraction.

So while this type of triangulation will work for the computation of a Lyapunov function if both the fan at the origin and the simplices are sufficiently fine, a triangulation in the shape of an ellipse as shown in Figure 2 is more appropriate and requires fewer simplices, as we will show in this paper. To transform the regular, rectangular triangulation, we first map the vertices of Figure 1, left, to a corresponding transformed triangulation with approximate rotational symmetry as shown in Figure 1, right, discussed in Section 2.2. Afterwards, we use a linear transformation of the vertices to obtain the triangulation shown in Figure 2.

Since sublevel sets of Lyapunov functions are forward invariant for the system, the triangulation must have enough structure to support Lyapunov functions that are affine on each triangle and have sublevel sets that are forward invariant for the dynamics. For this, one might need small triangles. We know that a Lyapunov function for the linearized system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = D\mathbf{f}(\mathbf{0})$ is the Jacobian matrix of $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ at the origin, is also a Lyapunov function for the nonlinear system in a neighbourhood of the origin. Using this fact, we can easily construct sets that are forward invariant for the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, by solving the continuous-time Lyapunov equation $A^{\mathrm{T}}P + PA = -Q$ for a symmetric and positive definite matrix $Q \in \mathbb{R}^{n\times n}$. The solution $P \in \mathbb{R}^{n\times n}$ is then a symmetric and positive definite matrix and $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}P\mathbf{x}$ is a global Lyapunov function for the linearized system

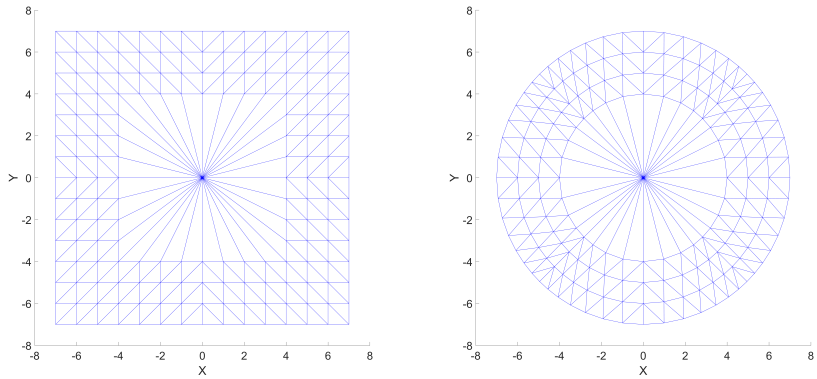


FIGURE 1. Left: The triangulation $\mathcal{T}_K^{\text{std}}$, $K = 4$, with a triangle fan at the origin. Right: The transformed approximately rotationally symmetric triangulation $\mathcal{T}_{\Phi,K}$.

and a local Lyapunov function for the nonlinear system. In particular, all sublevel sets $V^{-1}([0,r]) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^{\mathrm{T}}P\mathbf{x} \leq r\}$ for small enough $r \geq 0$ are forward invariant for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Since $V^{-1}([0,r]) = P^{-\frac{1}{2}}\overline{B_r}$, $B_r := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < r\}$, it appears sensible to use a triangulation that reflects this shape. Now consider Figure 2, where the vertices of the triangulation in Figure 1, right, have been mapped by a positive definite matrix $P^{-\frac{1}{2}}$, corresponding to the quadratic Lyapunov function $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}P\mathbf{x}$. As expected, the triangulation in Figure 1, right, which has an approximate rotational symmetry, is mapped to a triangulation which is well adapted to the forward invariant sets $V^{-1}([0,r])$. In Section 3 we will see that this can be used to compute CPA Lyapunov functions using much fewer triangles than when using a rectangular grid. In higher dimensions analogous arguments hold, the only difference is that the triangulations consist of n-simplices rather than triangles.

In [1] it was shown how to obtain a triangulation $\mathcal{T}_\Phi$ with an approximate rotational symmetry from the rectangular triangulation $\mathcal{T}_{\text{std}}$ in Definition 2.3. The approximate rotational symmetry corresponds to an approximate spherical symmetry and it is simple to obtain a triangulation with approximate elliptical symmetry from it, using a linear transformation, cf. Lemma 2.11. In this paper we prove that the simplices in the triangulation $\mathcal{T}_\Phi$ are (h,d)-bounded for constants $h, d > 0$. Note that since $\mathcal{T}_\Phi$ is a triangulation by [1], it is obvious that any *finite* collection of simplices from $\mathcal{T}_\Phi$ is (h,d)-bounded for some constants $h, d > 0$. The difficulty is showing that there is a $d > 0$ bounding the degeneracy of *all* $S \in \mathcal{T}_\Phi$. Further, we show that for a given $K \in \mathbb{N}_+$ the triangulation $\mathcal{T}_{\Phi,K}$, including a triangle fan at the origin, is (h,d)-bounded for some constants $h, d > 0$. Finally, we will show that this results in a considerable advantage for the CPA algorithm to compute Lyapunov functions, cf. Section 3.

1.1. **Prerequisites and Notation.** The set $\mathbb{N}_+$ denotes the natural numbers larger than zero and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}_+$. For a vector $\mathbf{x} \in \mathbb{R}^n$ and $p \geq 1$ we define the norm $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$. We also define $\|\mathbf{x}\|_\infty = \max_{i \in \{1,\ldots,n\}} |x_i|$. We will

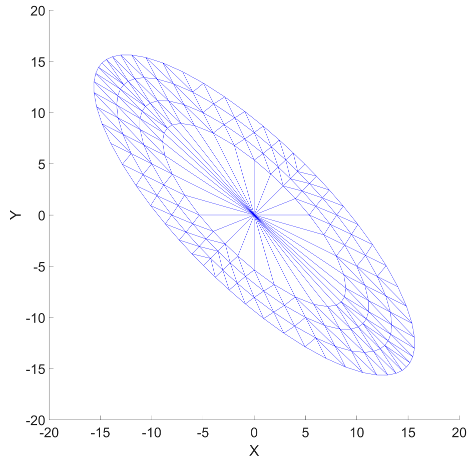


FIGURE 2. The vertices of the triangulations of Figure 1, right, are mapped by the linear transformation $\mathbf{x} \mapsto P^{-\frac{1}{2}}\mathbf{x}$, where $P^{-\frac{1}{2}}$ is a symmetric and positive definite matrix. This triangulation is adapted to the structure of the system with a local Lyapunov function $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}P\mathbf{x}$.

repeatedly use the Hölder inequality $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q$, where $p^{-1} + q^{-1} = 1$, and the norm equivalence relation

$$\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq n^{q^{-1}-p^{-1}} \|\mathbf{x}\|_p \quad \text{for } p > q.$$

The *induced matrix norm* $\| \cdot \|_p$ is defined by $\|A\|_p = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p$. Clearly $\|A\mathbf{x}\|_p \leq \|A\|_p \|\mathbf{x}\|_p$. For a matrix A we write A^{T} for its transpose. Recall that $\|A\|_1 = \|A^{\mathrm{T}}\|_\infty = \max_i \|\mathbf{a}_i\|_1$, where $\mathbf{a}_i$ are the column vectors of A, and the norm equivalences

$$\frac{1}{\sqrt{n}}\|A\|_p \leq \|A\|_2 \leq \sqrt{n}\,\|A\|_p \quad \text{for } A \in \mathbb{R}^{n\times n} \text{ and } p \in \{1, \infty\}.$$

The condition number κ_p of a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ with respect to the norm $\| \cdot \|_p$ is defined as $\kappa_p(A) := \|A\|_p \|A^{-1}\|_p$. The set of m-times continuously differentiable functions from an open set U to a set V is denoted by $C^m(U, V)$ or simply C^m if there is no danger of confusion. We denote the closure of a set U by $\overline{U}$ and its interior by U°.

We utilize a bold-face font for vectors, e.g. $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{x}$ may also be viewed as a single-column matrix, i.e. $\mathbf{x} \in \mathbb{R}^{n\times 1}$.

A diagonal matrix with entries $\mathbf{a} = (a_1, a_2, \ldots, a_n)^{\mathrm{T}}$ on its diagonal is denoted by $\mathrm{diag}(\mathbf{a})$. We denote by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ the standard orthonormal basis of $\mathbb{R}^n$ and use the Kronecker delta symbol: $\delta_{ij} = \mathbf{e}_i^{\mathrm{T}}\mathbf{e}_j$. Also, we denote by I the identity matrix in $\mathbb{R}^{n\times n}$.

There are a few closely related identities in Linear Algebra concerning a rank 1 correction $A + \mathbf{u}\mathbf{v}^{\mathrm{T}}$ of an invertible matrix A, which will be useful to us and we therefore state these in the form we need, cf. [23].

Lemma 1.1 (Sherman-Morrison)**.** *Let $A \in \mathbb{R}^{n\times n}$ be invertible and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then*

$$\left(A + \mathbf{u}\mathbf{v}^{\mathrm{T}}\right)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}A^{-1}}{1 + \mathbf{v}^{\mathrm{T}}A^{-1}\mathbf{u}},$$

provided $1 + \mathbf{v}^{\mathrm{T}}A^{-1}\mathbf{u} \neq 0$. Furthermore, we have the following identity:

$$\det\left(A + \mathbf{u}\mathbf{v}^{\mathrm{T}}\right) = \left(1 + \mathbf{v}^{\mathrm{T}}A^{-1}\mathbf{u}\right)\det A.$$

2. **Triangulations of $\mathbb{R}^n$.** We need several definitions before we can state and prove our results.

Definition 2.1. *We define the following*:

i) The **convex-combination** of vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n$ is given by

$$\mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\} := \left\{\mathbf{y} \in \mathbb{R}^n \,:\, \mathbf{y} = \sum_{i=0}^{m} \lambda_i \mathbf{x}_i,\, \forall\, i : \, 0 \leq \lambda_i \leq 1, \sum_{i=0}^{m} \lambda_i = 1\right\}.$$

ii) The vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n$ are said to be **affinely-independent** if

$$\sum_{i=0}^{m} \lambda_i = 0 \quad \text{and} \quad \sum_{i=0}^{m} \lambda_i \mathbf{x}_i = \mathbf{0} \quad \text{implies} \quad \lambda_0 = \lambda_1 = \cdots = \lambda_m = 0.$$

iii) If $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n$ are affinely-independent, the set $S = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}$ is called an m**-simplex**. The vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m$ are called the **vertices** of S. The set of vertices for an m-simplex is sometimes denoted by $\mathrm{ve}\, S = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}$. In $\mathbb{R}^n$ an n-simplex is often referred to as just a simplex.

iv) For an m-simplex S, define its **diameter** as:

$$\mathrm{diam}(S) := \max_{\mathbf{x},\mathbf{y}\in S} \|\mathbf{x} - \mathbf{y}\|_2.$$

Note that we do not consider degenerate simplices in this paper. Therefore all simplices are proper simplices in the terminology of [1].

We now define a *triangulation*. For our purposes it is advantageous to have the order of the vertices of every simplex in the triangulation fixed, similar to [7]. The reason for this becomes clear in Section 2.1, where we introduce shape-matrices of simplices. For an n-tuple of vertices $C = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ we define $\mathrm{co}\, C = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$.

Definition 2.2 (Triangulation)**.** *A* triangulation $\mathcal{T} = \{S_\nu\}_{\nu\in I}$ *in $\mathbb{R}^n$ is a set of n-simplices S_ν with ordered vertices $C_\nu = (\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)$ for all $\nu \in I$, such that*

$$S_\mu \cap S_\nu = \mathrm{co}\,\mathrm{ve}\, S_\mu \cap \mathrm{co}\,\mathrm{ve}\, S_\nu = \mathrm{co}(\mathrm{ve}\, S_\mu \cap \mathrm{ve}\, S_\nu). \tag{2.1}$$

The domain of $\mathcal{T}$ is defined as

$$\mathcal{D}_\mathcal{T} := \bigcup_{\nu\in I} S_\nu$$

and its complete set of vertices is denoted by

$$\mathcal{V}_\mathcal{T} := \bigcup_{\nu\in I} \mathrm{ve}\, S_\nu.$$

Further, we define the diameter of $\mathcal{T}$ as

$$\mathrm{diam}(\mathcal{T}) := \sup_{S\in\mathcal{T}} \mathrm{diam}(S).$$

We recall the definition of the standard triangulation $\mathcal{T}_{\mathrm{std}}$ as given in [1].

Definition 2.3 (The Standard Triangulation of $\mathbb{R}^n$)**.** *The* Standard Triangulation *is a triangulation $\mathcal{T}_{\text{std}} = \{S_\nu\}_{\nu\in I}$ with indices $\nu = (\mathbf{z},\sigma,\mathbf{J}) \in \mathbb{N}_+^n \times Sym(n) \times \{-1,+1\}^n =: I$ and vertices $C_\nu = (\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)$ given by:*

$$\mathbf{x}_k^\nu = R_{\mathbf{J}}\left(\mathbf{z} + \sum_{l=1}^{k} \mathbf{e}_{\sigma(l)}\right) = R_{\mathbf{J}}\mathbf{z} + R_{\mathbf{J}}\,\mathbf{u}_k^\sigma. \tag{2.2}$$

Here, $\mathbf{J} = (J_1, J_2, \ldots, J_n)^{\mathrm{T}} \in \{-1,+1\}^n$ and $R_{\mathbf{J}} = \operatorname{diag}(\mathbf{J}) \in \mathbb{R}^{n\times n}$ is a matrix corresponding to the reflection specified by $\mathbf{J} \in \{-1,+1\}^n$ and $\mathbf{u}_k^\sigma = \sum_{l=1}^k \mathbf{e}_{\sigma(l)}$. Further, $Sym(n)$ denotes the set of the permutations $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$.

We are interested in a specific variation of the standard triangulation, denoted by $\mathcal{T}_K^{\text{std}}$, which has a triangle fan at the origin. For a visual representation see Figure 1, left.

Definition 2.4. 1. *Fix a $K \in \mathbb{N}_+$ and consider $S_\nu \in \mathcal{T}_{\text{std}}$ such that $\|\mathbf{x}_0^\nu\|_\infty = K-1$ and $\|\mathbf{x}_k^\nu\|_\infty = K$ for $k = 1,2,\ldots,n$. A triangulation $\mathcal{T}_K^{fan}$ of $[-K,K]^n$ is now obtained by taking every such simplex S_ν and substitute $\mathbf{x}_0^\nu$ by $\mathbf{0}$. We refer to this the triangulation with the origin as a vertex as a* **triangle fan** *or a* **simplicial-fan**.

2. *We define*

$$\mathcal{T}_K^{\text{std}} := \{S_\nu \in \mathcal{T}_{\text{std}} \colon S_\nu \cap (-K,K)^n = \emptyset\} \cup \mathcal{T}_K^{fan}.$$

Note that $\mathcal{T}_1^{\text{std}} = \mathcal{T}_{\text{std}}$. However, $\mathcal{T}_K^{\text{std}}$ with $K \geq 2$ is the standard triangulation, but with a triangle fan at the origin; see Figure 1, left, for an example. For a proof of the fact, that $\mathcal{T}_K^{\text{std}}$ is a triangulation of $\mathbb{R}^n$ in the sense of Definition 2.2 and that our enumeration of the vertices is well defined see [5].

Remark 2.5. *Because $\mathcal{T}_K^{\text{std}} := \{S_\nu \in \mathcal{T}_{\text{std}} \colon S_\nu \cap (-K,K)^n = \emptyset\} \cup \mathcal{T}_K^{fan}$ and for every $S \in \mathcal{T}_K^{fan}$ with an n-tuple of vertices $C = (\mathbf{0}, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ there is exactly one $\nu \in I$, such that $\mathbf{x}_i = \mathbf{x}_i^\nu$ for $i = 1,2,\ldots,n$, we write by slight abuse of notation, $\mathcal{T}_K^{\text{std}} = \{S_\nu\}_{\nu\in I_K}$. The reason for this is the following. For an $S_\nu \in \mathcal{T}_{\text{std}}$ with $\|\mathbf{x}_0^\nu\|_\infty \geq K$ we have $S_\nu \cap (-K,K)^n = \emptyset$ and thus $S_\nu \in \mathcal{T}_K^{\text{std}}$. Denote the set of all such $\nu \in I$ by I_K^*. For an $S_\nu \in \mathcal{T}_{\text{std}}$, such that $\|\mathbf{x}_0^\nu\|_\infty = K-1$ and $\|\mathbf{x}_i^\nu\|_\infty = K$ for $i = 1,2,\ldots,n$, we have the corresponding simplex $S_\nu^* \in \mathcal{T}_K^{fan}$ with ordered tuple of vertices $(\mathbf{0}, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)$. Denote the set of such $\nu \in I$ by I_K^0. Since $I_K^0 \cap I_K^* = \emptyset$ it is convenient to set $I_K := I_K^0 \cup I_K^*$ and simply write S_ν^* as S_ν. That is, define for $\mathcal{T}_K^{\text{std}}$:*

$$\textit{for } \nu \in I_K^0 \textit{ set } \; C_\nu := (\mathbf{0}, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)\,, \; \mathbf{x}_0^\nu := \mathbf{0}, \quad \textit{and} \;\; S_\nu := \operatorname{co} C_\nu.$$

Then we can write $\mathcal{T}_K^{\text{std}} = \{S_\nu\}_{\nu\in I_K}$.

It should be noted that the computational complexity of actually generating the triangulation $\mathcal{T}_K^{\text{std}}$ in a cube $[-N,N]^n \subset \mathbb{R}^n$, $N \in \mathbb{N}_+$ and $N \geq K$, is just the number of its simplices, i.e. $\mathcal{O}(n!(2N)^n)$.

2.1. **Shape-Matrix of a Simplex.** We now define the shape-matrix of a simplex, the vertices of which are in a particular order. This is needed to define (h,d)-boundedness for a triangulation. Further, we explain why shape-matrices are of fundamental importance to our application of computing CPA Lyapunov functions.

Definition 2.6. *For an n-simplex S of a triangulation with vertices $C_\nu = (\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$ we define its* **shape-matrix** X_S *as*

$$X_S := \begin{pmatrix} (\mathbf{x}_1 - \mathbf{x}_0)^{\mathrm{T}} \\ (\mathbf{x}_2 - \mathbf{x}_0)^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_n - \mathbf{x}_0)^{\mathrm{T}} \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

For $i = 1, 2, \dots, n$ we thus write the components of the vector $\mathbf{x}_i - \mathbf{x}_0$ in the i-th row of X_S.

Notice, that because S in the definition of a shape-matrix is an n-simplex, its vertices $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent vectors, so the shape matrix X_S is nonsingular.

Remark 2.7. *Important for our application is not the shape-matrix itself but the quantity $\|X_S^{-1}\|_p$, where usually $p = 2$, but for some applications $p = 1$ or $p = \infty$ are more appropriate. Because all norms on the finite dimensional vector space $\mathbb{R}^{n\times n}$ are equivalent there is no fundamental difference between these norms. Note that the quantity $\|X_S^{-1}\|_p$ depends on the order of the vertices of the simplex S, this is why we fixed the order of the vertices in our definition of a triangulation.*

It is tempting to assume that the quantity $\|X_S^{-1}\|_p$ could be related to a quantity that does not depend on the order of the vertices of the simplex S. For example the determinant of X_S seems like a good candidate, because $|\det X_S|/n!$ is well known to be the volume of the simplex and does not depend on the choice of $\mathbf{x}_0$ or the order of the differences $\mathbf{x}_k - \mathbf{x}_0$ in the shape-matrix. However, as e.g. shown in [9, §2.6.3], *there is no correlation between $\|X_S^{-1}\|_p$ and $|\det X_S|$.*

Let us explain in detail why the quantity $\|X_S^{-1}\|_p$ is of so much interest in our application: To prove that the algorithm in [6] always succeeds in computing a CPA Lyapunov functions for any system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, with an exponentially stable equilibrium at the origin, one uses the fact that there exists a C^2 Lyapunov function W for the system. This function W is used to prove that the linear programming problem in the algorithm has a feasible solution for a suitable triangulation.

In the proof in [6] W is approximated on $S = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$ by its interpolation W_{CPA} on S: With

$$\mathbf{x} = \sum_{i=0}^{n} \lambda_i^{\mathbf{x}} \mathbf{x}_i \in S$$

as the unique convex combination of the vertices, we set

$$W_{\mathrm{CPA}}(\mathbf{x}) = \sum_{i=0}^{n} \lambda_i^{\mathbf{x}} W(\mathbf{x}_i).$$

While this obviously approximates the values of W well on a simplex S with a small diameter $h := \mathrm{diam}(S)$, e.g. using

$$|W(\mathbf{x}) - W_{\mathrm{CPA}}(\mathbf{x})| \le \sum_{i=0}^{n} \lambda_i^{\mathbf{x}} \, |W(\mathbf{x}) - W_{\mathrm{CPA}}(\mathbf{x}_i)| \le h \cdot \max_{\mathbf{z}\in S} \|\nabla W(\mathbf{z})\|_2,$$

this is not sufficient for the proof, because we additionally need ∇W_{CPA} to closely approximate ∇W at the vertices $\mathbf{x}_i$, where $X_S = (\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)^{\mathrm{T}}$ is the shape-matrix of S, cf. Steps 7 and 8 in the proof of Theorem 5 in [6].

It is not difficult to show that ∇W_{CPA} is the constant vector $X_S^{-1}\mathbf{w}$, $\mathbf{w} = (W(\mathbf{x}_1) - W(\mathbf{x}_0), W(\mathbf{x}_2) - W(\mathbf{x}_0), \ldots, W(\mathbf{x}_n) - W(\mathbf{x}_0))^{\mathrm{T}}$ for all $\mathbf{x} \in S^\circ$, cf. Remark 9 in [6]. From this one obtains by Taylor expansion the relation, cf. (19) in [6]

$$[\mathbf{w} - X_S \nabla W(\mathbf{x}_0)]_i = (\mathbf{x}_i - \mathbf{x}_0)^{\mathrm{T}} H_W(\mathbf{z}_i)(\mathbf{x}_i - \mathbf{x}_0)$$

for $i = 1, 2, \ldots, n$, where H_W is the Hessian matrix of W and the $\mathbf{z}_i$ are points in S. It follows that

$$\|\nabla W_{\mathrm{CPA}} - \nabla W(\mathbf{x}_i)\|_p \le \|X_S^{-1}\mathbf{w} - \nabla W(\mathbf{x}_0)\|_p + \|\nabla W(\mathbf{x}_i) - \nabla W(\mathbf{x}_0)\|_p.$$

The second term on the right-hand side is small if the diameter of the simplex S is small because $W \in C^2$. The first term, however, is not necessarily small for a simplex S with a small diameter h.

Nonetheless, we can bound it by

$$\|X_S^{-1}\mathbf{w} - \nabla W(\mathbf{x}_0)\|_p \le \|X_S^{-1}\|_p \|\mathbf{w} - X_S \nabla W(\mathbf{x}_0)\|_p \le h^2 \cdot \|X_S^{-1}\|_p C_p,$$

where C_p is a constant depending on the second order derivatives of W. As a consequence, we need a sequence of finite triangulations $\mathcal{T}_k$ where the simplices become smaller, i.e. $h_k \to 0$ as $k \to \infty$, but also such that $\sup_{S \in \mathcal{T}_k} \operatorname{diam}(S)^2 \cdot \|X_S^{-1}\|_p \to 0$ as $k \to \infty$.

A second observation is that when we scale down the simplex S, i.e. multiply the vertices of S with a number $0 < \gamma < 1$, then $\operatorname{diam}(\gamma S) = \gamma \operatorname{diam}(S)$ and $\|X_{\gamma S}^{-1}\|_p = \gamma^{-1}\|X_S^{-1}\|_p$. This leads to the following strategy of obtaining a suitable sequence of triangulations $\mathcal{T}_k$ for proving that the algorithm in [6] succeeds in computing a Lyapunov function on any compact set $\mathcal{C}$, that is contained in the basin of attraction of the equilibrium at the origin. For simplicity we ignore some adaptations that have to be made close to the equilibrium, but do not change the main idea:

We know that $\operatorname{diam}(\mathcal{T}_{\mathrm{std}}) = \sqrt{n}$ and $\sup_{S \in \mathcal{T}_{\mathrm{std}}} \|X_S^{-1}\|_p \le 2$ for $p = 1, 2, \infty$, cf. Remark 2 in [14]. Define $\mathcal{T}_k := \{\gamma^k S : S \in \mathcal{T}_{\mathrm{std}} \text{ and } (\gamma^k S) \cap \mathcal{C}^\circ \ne \emptyset\}$ (which ensures that the triangulation is finite) for $k \in \mathbb{N}_0$ and some $0 < \gamma < 1$. Then

$$\operatorname{diam}(\mathcal{T}_k) = \gamma^k \sqrt{n} \quad \text{and} \quad \sup_{\gamma^k S \in \mathcal{T}_k} \operatorname{diam}(\gamma^k S) \|X_{\gamma^k S}^{-1}\|_p \le \gamma^k \sqrt{n} \cdot \gamma^{-k} 2 = 2\sqrt{n},$$

i.e. $\operatorname{diam}(\mathcal{T}_k)$ converges to zero as $k \to \infty$ and $\sup_{\gamma^k S \in \mathcal{T}_k} \operatorname{diam}(\gamma^k S)\|X_{\gamma^k S}^{-1}\|_p \le d = 2\sqrt{n}$ for all $k \in \mathbb{N}_0$.

With the same argumentation, any sequence of triangulations $\mathcal{T}_k$ such that

- $\operatorname{diam}(\mathcal{T}_k) \to 0$ for $k \to \infty$ and
- there is a bound d such that $\sup_{S \in \mathcal{T}_k} \operatorname{diam}(S)\|X_S^{-1}\|_p \le d$ holds for all $k \in \mathbb{N}_0$

can be used in the proof in [6].

This leads us to the following definition introduced in [7], where we have fixed $p = 2$ in $\|X^{-1}\|_p$ for brevity.

Definition 2.8. *We define the* **degeneracy** *of the triangulation $\mathcal{T}$ to be the quantity*

$$\sup_{S \in \mathcal{T}} \operatorname{diam}(S)\|X_S^{-1}\|_2,$$

where X_S is the shape-matrix of S. We say that the triangulation $\mathcal{T}$ is (h, d)- **bounded** *for constants $h, d > 0$, if $\operatorname{diam}(\mathcal{T}) < h$ and the degeneracy of $\mathcal{T}$ is bounded by d, i.e. $\sup_{S \in \mathcal{T}} \operatorname{diam}(S)\|X_S^{-1}\|_2 \le d$.*

Hence, a sequence $\mathcal{T}_k$ of triangulations with the properties above can be obtained by scaling a fixed (h, d)-bounded triangulation $\mathcal{T}$ of $\mathbb{R}^n$ and restricting it to $\mathcal{C}$, $\mathcal{T}_k := \{\gamma^k S : S \in \mathcal{T} \text{ and } (\gamma^k S) \cap \mathcal{C}^\circ \neq \emptyset\}$ with $0 < \gamma < 1$, similar to the standard triangulation above.

A useful fact we need later is given by the following lemma.

Lemma 2.9. *The set of the shape-matrices of $\mathcal{T}_{\mathrm{std}}$ is finite. For any $K \in \mathbb{N}_+$ the set of the shape matrices of $\mathcal{T}_K^{\mathrm{std}}$ is finite.*

Proof. As the number of the simplices in the triangle fan is clearly finite for any $K \in \mathbb{N}_+$, it suffices to prove the first proposition of the lemma. Notice that S_ν and $S_{\nu*}$ with $\nu = (\mathbf{z}, \sigma, \mathbf{J})$ and $\nu = (\mathbf{z}^*, \sigma^*, \mathbf{J}^*)$ have the same shape matrix if $\sigma = \sigma^*$ and $\mathbf{J} = \mathbf{J}^*$ and that $\sigma \in \mathrm{Sym}(n)$ and $\mathbf{J} \in \{-1, 1\}^n$. As there are $n!$ different permutations in $\mathrm{Sym}(n)$ and 2^n different vectors in $\{-1, 1\}^n$, there can be no more than $2^n n!$ different shape-matrices for $\mathcal{T}_{\mathrm{std}}$. $\square$

2.2. **Mapping the Standard Triangulation.** We will now study new triangulations, obtained by mapping the vertices of the simplices in $\mathcal{T}_K^{\mathrm{std}} = \{S_\nu\}_{\nu \in I_K}$, but retaining the triangulation structure through the specification of the vertex-sets $\{C_\nu\}_{\nu \in I_K}$. To be more precise, for $\mathcal{T}_K^{\mathrm{std}} = \{\mathrm{co}\, C_\nu\}_{\nu \in I_K}$ we will consider the set of simplices given by $\mathcal{T}_{\Phi,K} = \{\mathrm{co}\, \Phi(C_\nu)\}_{\nu \in I_K}$, where the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ performs the rearrangement of the vertices. Note that $C_\nu = (\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)$ and

$$\Phi(C_\nu) := (\Phi(\mathbf{x}_0^\nu), \Phi(\mathbf{x}_1^\nu), \ldots, \Phi(\mathbf{x}_n^\nu)).$$

For nonlinear Φ we have in general $\mathrm{co}\, \Phi(C_\nu) \neq \Phi(\mathrm{co}\, C_\nu)$ and so the question arises, when $\mathcal{T}_{\Phi,K}$ is in fact a triangulation in the sense of Definition 2.2. In [1] the following mappings were considered:

Definition 2.10. *Define*

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi(\mathbf{x}) = \rho(\|\mathbf{x}\|_\infty) \cdot \mathbf{F}(\mathbf{x}), \tag{2.3}$$

where

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{F}(\mathbf{x}) = \frac{\|\mathbf{x}\|_\infty}{\|\mathbf{x}\|_2}\mathbf{x} \quad \textit{if } \mathbf{x} \neq \mathbf{0} \textit{ and } \mathbf{F}(\mathbf{0}) = \mathbf{0}. \tag{2.4}$$

and $\rho : [0, \infty) \to [0, \infty)$ is non-decreasing with $\rho^{-1}(0) = \{0\}$.

With $\mathcal{T}_{\mathrm{std}} = \{\mathrm{co}\, C_\nu\}_{\nu \in I}$ and $C_\nu = (\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)$ as before, it was then proved that the set of simplices $\mathcal{T}_\Phi = \{\mathrm{co}\, \Phi(C_\nu)\}_{\nu \in I}$, where $\Phi(C_\nu) := (\Phi(\mathbf{x}_0^\nu), \Phi(\mathbf{x}_1^\nu), \ldots, \Phi(\mathbf{x}_n^\nu))$, is a triangulation of $\mathbb{R}^n$.

Here, we will prove the corresponding result including the triangular fan, i.e. with $\mathcal{T}_K^{\mathrm{std}} = \{\mathrm{co}\, C_\nu\}_{\nu \in I_K}$, we will show that the set

$$\mathcal{T}_{\Phi,K} = \Phi\left(\mathcal{T}_K^{\mathrm{std}}\right) := \{\mathrm{co}\, \Phi(C_\nu)\}_{\nu \in I_K} = \{\mathrm{co}\{\Phi(\mathbf{x}_0^\nu), \Phi(\mathbf{x}_1^\nu), \ldots, \Phi(\mathbf{x}_n^\nu)\}\}_{\nu \in I_K}$$

is a triangulation of $\mathbb{R}^n$. Note, that for $K = 1$ this is just the result from [1]. For $K \geq 2$ the vertex sets C_ν are different and we require a proof.

Further, and this is the main contribution of this paper, we will prove that with $\rho(0) = 0$ and $\rho(x) = 1$ for all $x > 0$, the triangulation $\mathcal{T}_{\Phi,K}$ is (h, d)-bounded with appropriate $h, d > 0$. This means that we can use sequences of triangulations $\mathcal{T}_k := \gamma^k P^{-\frac{1}{2}}(\mathcal{T}_{\Phi,K})$, $0 < \gamma < 1$, adapted to the system at hand with a local quadratic Lyapunov function $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} P \mathbf{x}$, in the algorithm from [6], cf. Section 3.

Note that here $\gamma^k P^{-\frac{1}{2}}(\mathcal{T}_{\Phi,K})$ stands for the triangulation $\mathcal{T}_{\Psi,K} := \{\Psi(C_\nu)\}_{\nu\in I_K}$ with $\Psi := \gamma^k P^{-\frac{1}{2}}\Phi$. The reason why we can use this triangulation in the algorithm is given in the following lemma.

Lemma 2.11. *Assume* $\mathcal{T} = \{\operatorname{co} C_\nu\}_{\nu\in I}$ *is an* (h,d)*-bounded triangulation and* $A \in \mathbb{R}^{n\times n}$ *is nonsingular. Then* $A\mathcal{T} := \{\operatorname{co} AC_\nu\}_{\nu\in I}$ *is an* (h^*,d^*)*-bounded triangulation, with* $h^* := \|A\|_2 h$ *and* $d^* := \kappa_2(A)hd$*.*

Proof. Let $S = \operatorname{co} C$ be an arbitrary simplex in $\mathcal{T}$. Since the vectors in $C = (\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$ are affinely independent, so are the vectors in $AC = (A\mathbf{x}_0, A\mathbf{x}_1, \dots, A\mathbf{x}_n)$, because A is nonsingular. Now

$$\operatorname{diam}(S) = \max_{\mathbf{x},\mathbf{y}\in S} \|\mathbf{x}-\mathbf{y}\|_2 = \max_{i,j\in\{0,1,\dots,n\}} \|\mathbf{x}_i - \mathbf{x}_j\|_2 < h$$

and

$$\operatorname{diam}(AS) = \operatorname{diam}(A \operatorname{co} C) = \operatorname{diam}(\operatorname{co} AC) = \max_{i,j\in\{0,1,\dots,n\}} \|A\mathbf{x}_i - A\mathbf{x}_j\|_2 < \|A\|_2 h.$$

Thus $\operatorname{diam}(A\mathcal{T}) < \|A\|_2 h = h^*$.

Further, for the shape-matrices we easily see that $X_{\operatorname{co} AC} = AX_{\operatorname{co} C} = AX_S$ and thus

$$\|X^{-1}_{\operatorname{co} AC}\|_2 = \|X_S^{-1}A^{-1}\|_2 \le \|A^{-1}\|_2 \|X_S^{-1}\|_2,$$

and it follows that

$$\operatorname{diam}(\operatorname{co} AC) \cdot \|X^{-1}_{\operatorname{co} AC}\|_2 < \|A\|_2 h \cdot \|A^{-1}\|_2 \|X_S^{-1}\|_2 \le \kappa_2(A)hd = d^*.$$

□

In Section 3 we will show with examples, that one can compute a CPA Lyapunov function for a nonlinear system using fewer simplices, when one uses a triangulation adapted to the system in this way. Note that the actual transformation of a finite subset of $\mathcal{T}_K^{\text{std}}$ to simplices in $\mathcal{T}_{\Phi,K}$ is simply done by mapping the vertices and is thus computationally very efficient.

2.3. Bound on Degeneracy of the $\mathcal{T}_\Phi$ Triangulation.

We state and prove a few lemmas before we state and prove main results of this paper in Theorem 2.19.

Lemma 2.12. *For* $i = 1, 2, \dots, n$ *consider*

$$\mathbf{G}_i \colon \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{G}_i(\mathbf{x}) = \frac{|x_i|}{\|\mathbf{x}\|_2}\mathbf{x} \quad \text{for } \mathbf{x} \neq \mathbf{0} \text{ and } \mathbf{G}_i(\mathbf{0}) = \mathbf{0}.$$

Then:

i) $D\mathbf{G}_i(\mathbf{x})$ *is nonsingular and* $\|D\mathbf{G}_i(\mathbf{x})\|_2 \le 3$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *such that* $x_i \neq 0$*.*

ii) For every $i = 1, 2, \dots, n$ *we have*

$$\lim_{h\to 0} \sup_{|z_i| = \|\mathbf{z}\|_\infty = \|\mathbf{u}\|_\infty = 1} \left\| \frac{\mathbf{G}_i(\mathbf{z}+h\mathbf{u}) - \mathbf{G}_i(\mathbf{z})}{h} - D\mathbf{G}_i(\mathbf{z})\mathbf{u} \right\|_\infty = 0.$$

Proof. Let $\mathbf{x} \in \mathbb{R}^n$, $x_i \neq 0$. Then

$$D\mathbf{G}_i(\mathbf{x}) = \frac{|x_i|}{\|\mathbf{x}\|_2}\left(I + \mathbf{x}\left(-\frac{\mathbf{x}}{\|\mathbf{x}\|_2^2} + \frac{1}{x_i}\mathbf{e}_i\right)^{\mathrm{T}}\right) \tag{2.5}$$

and as $|x_i| \neq 0$ it is enough to show that

$$I + \mathbf{x}\left(-\frac{\mathbf{x}}{\|\mathbf{x}\|_2^2} + \frac{1}{x_i}\mathbf{e}_i\right)^{\mathrm{T}} = I + \mathbf{w}\mathbf{v}^{\mathrm{T}} \quad \text{with} \quad \mathbf{w} = \mathbf{x} \quad \text{and} \quad \mathbf{v} = -\frac{\mathbf{x}}{\|\mathbf{x}\|_2^2} + \frac{1}{x_i}\mathbf{e}_i$$

is nonsingular to establish the nonsingularity of $D\mathbf{G}_i(\mathbf{x})$. By Lemma 1.1, the nonsingularity of this matrix is equivalent to $1+\mathbf{v}^{\mathrm{T}}\mathbf{w}\neq 0$, which follows from

$$1+\mathbf{v}^{\mathrm{T}}\mathbf{w} = 1+\left(-\frac{\mathbf{x}}{\|\mathbf{x}\|_2^2}+\frac{\mathbf{e}_i}{x_i}\right)^{\mathrm{T}}\mathbf{x} = 1-\frac{\mathbf{x}^{\mathrm{T}}\mathbf{x}}{\|\mathbf{x}\|_2^2}+\frac{x_i}{x_i} = 1-1+1 = 1.$$

From the formula (2.5) we further get

$$\|D\mathbf{G}_i(\mathbf{x})\|_2 \le \frac{|x_i|}{\|\mathbf{x}\|_2}\left(\|I\|_2+\frac{\|\mathbf{x}\mathbf{x}^{\mathrm{T}}\|_2}{\|\mathbf{x}\|_2^2}\right)+\frac{\|\mathbf{x}\mathbf{e}_i^{\mathrm{T}}\|_2}{\|\mathbf{x}\|_2}\le 3,$$

because

$$\|\mathbf{a}\mathbf{b}^{\mathrm{T}}\|_2^2 = \max_{\|\mathbf{y}\|_2=1}\mathbf{y}^{\mathrm{T}}\mathbf{b}\mathbf{a}^{\mathrm{T}}\mathbf{a}\mathbf{b}^{\mathrm{T}}\mathbf{y} = \max_{\|\mathbf{y}\|_2=1}(\mathbf{b}\cdot\mathbf{y})^2\|\mathbf{a}\|_2^2,$$

i.e. $\|\mathbf{x}\mathbf{x}^{\mathrm{T}}\|_2=\|\mathbf{x}\|_2^2$ and $\|\mathbf{x}\mathbf{e}_i^{\mathrm{T}}\|_2=\|\mathbf{x}\|_2$. Thus proposition i) holds true.

To prove proposition ii) observe that for all $\mathbf{x}\in\mathbb{R}^n$ with $x_i\neq 0$ we have

$$\begin{aligned}\frac{\partial^2(\mathbf{G}_i)_j}{\partial x_r\partial x_s}(\mathbf{x}) \quad = \quad & -\delta_{si}\frac{\mathrm{sign}(x_i)}{\|\mathbf{x}\|_2^3}x_rx_j-\delta_{ri}\frac{\mathrm{sign}(x_i)}{\|\mathbf{x}\|_2^3}x_sx_j+\delta_{si}\frac{\mathrm{sign}(x_i)}{\|\mathbf{x}\|_2}\delta_{jr}\\ & +\delta_{ir}\frac{\mathrm{sign}(x_i)}{\|\mathbf{x}\|_2}\delta_{js}-\frac{|x_i|}{\|\mathbf{x}\|_2^3}x_s\delta_{jr}-\frac{|x_i|}{\|\mathbf{x}\|_2^3}x_r\delta_{js}\\ & +3\frac{|x_i|x_sx_r}{\|\mathbf{x}\|_2^5}x_j-\frac{|x_i|}{\|\mathbf{x}\|_2^3}\delta_{sr}x_j.\end{aligned}$$

Thus, for every $i=1,2,\dots,n$ the functions $\dfrac{\partial^2(\mathbf{G}_i)_j}{\partial x_r\partial x_s}$, $j,r,s=1,2,\dots,n$, are continuous on the compact set

$$C_i^* := \left\{\mathbf{x}\in\mathbb{R}^n : \frac{1}{2}\le |x_i| \quad \text{and} \quad \|\mathbf{x}\|_\infty \le \frac{3}{2}\right\}.$$

Hence, there is a constant G^* such that

$$\max_{i,j=1,2,\dots,n}\ \sup_{\mathbf{x}\in C_i^*}\sum_{r,s=1,2,\dots,n}\left|\frac{\partial^2(\mathbf{G}_i)_j}{\partial x_s\partial x_r}(\mathbf{x})\right|\le G^*.$$

But then by Taylor's theorem for all $i=1,2,\dots,n$, all $\mathbf{z}$ and $\mathbf{u}$ such that $|z_i|=\|\mathbf{z}\|_\infty=\|\mathbf{u}\|_\infty=1$, and all $0<h<1/2$ there exists a $\vartheta_{h,i}$ between 0 and 1 such that

$$\left\|\frac{\mathbf{G}_i(\mathbf{z}+h\mathbf{u})-\mathbf{G}_i(\mathbf{z})}{h}-D\mathbf{G}_i(\mathbf{z})\mathbf{u}\right\|_\infty = h\cdot\max_{j=1,2,\dots,n}\left|\sum_{r,s=1,2,\dots,n}\frac{u_ru_s}{2}\cdot\frac{\partial^2(\mathbf{G}_i)_j}{\partial x_r\partial x_s}(\mathbf{z}+\vartheta_{h,i}h\mathbf{u})\right|,$$

because $\mathbf{z}+\vartheta_{h,i}h\mathbf{u}\in C_i^*$.

Hence

$$\sup_{|z_i|=\|\mathbf{z}\|_\infty=\|\mathbf{u}\|_\infty=1}\left\|\frac{\mathbf{G}_i(\mathbf{z}+h\mathbf{u})-\mathbf{G}_i(\mathbf{z})}{h}-D\mathbf{G}_i(\mathbf{z})\mathbf{u}\right\|_\infty\le\frac{hG^*}{2},$$

from which proposition ii) immediately follows. □

Lemma 2.13. *For every $S\in\mathcal{T}_{\mathrm{std}}$ there is an $i\in\{1,2,\dots,n\}$ such that for every $\mathbf{x}\in S$ we have $|x_i|=\|\mathbf{x}\|_\infty$. In particular, $\mathbf{F}(\mathbf{x})=\mathbf{G}_i(\mathbf{x})$, where $\mathbf{G}_i$ is defined as in Lemma 2.12, for all $\mathbf{x}\in S$.*

Proof. By the definition of $\mathcal{T}_{\rm std}$ we have $S = S_\nu$ for some $\nu = (\mathbf{z}, \sigma, \mathbf{J}) \in \mathbb{N}_+^n \times \mathrm{Sym}(n) \times \{-1, +1\}^n$. Let $i_1 < i_2 < \ldots < i_k$ be those elements of $\{1, 2, \ldots, n\}$ such that $|z_{i_j}| = \|\mathbf{z}\|_\infty$. We claim that $\|\mathbf{x}\|_\infty = |x_{\sigma(i_k)}|$ for all $\mathbf{x} \in S$, from which the proposition of the lemma follows. To see this just note that the simplex S_ν, considered as a set, is the image of the simplex $\big\{\mathbf{x} \in \mathbb{R}^n : 0 \le x_{\sigma(1)} \le \ldots \le x_{\sigma(n)} \le 1\big\}$ under the mapping $\mathbf{x} \mapsto R_{\mathbf{J}}(\mathbf{z}+\mathbf{x})$. Thus $|J_{\sigma(i_k)} z_{\sigma(i_k)} + J_{\sigma(i_k)} x_{\sigma(i_k)}| \ge |J_j z_j + J_j x_j|$ for $j = 1, 2, \ldots, n$ because either $|J_{\sigma(i_k)} z_{\sigma(i_k)}| \ge |J_j z_j| + 1$ or $|J_{\sigma(i_k)} z_{\sigma(i_k)}| = |J_j z_j|$ and $|J_{\sigma(i_k)} x_{\sigma(i_k)}| \ge |J_j x_j|$. □

Now a few simple facts on the minimum eigenvalues $\lambda_{\min}(A)$ of symmetric matrices $A \in \mathbb{R}^{n\times n}$.

Lemma 2.14. *Let $A, B \in \mathbb{R}^{n\times n}$. Then $\lambda_{\min}((AB)^{\rm T} AB) \ge \lambda_{\min}(A^{\rm T}A) \cdot \lambda_{\min}(B^{\rm T}B)$.*

Proof. If either A or B or both are singular, then so is AB and both sides of the inequality are zero so it is trivially fulfilled. If A and B are nonsingular, then so is AB and we have $1/\lambda_{\min}((AB)^{\rm T}AB) = \|B^{-1}A^{-1}\|_2^2$, $1/\lambda_{\min}(A^{\rm T}A) = \|A^{-1}\|_2^2$, and $1/\lambda_{\min}(B^{\rm T}B) = \|B^{-1}\|_2^2$. Because $\|\cdot\|_2$ is a sub-multiplicative matrix norm we further have $\|B^{-1}A^{-1}\|_2 \le \|B^{-1}\|_2\|A^{-1}\|_2$, from which the inequality follows. □

Lemma 2.15. *For any $h > \sqrt{n}$ the triangulation $\mathcal{T}_{\rm std}$ of $\mathbb{R}^n$ is $(h, 2\sqrt{n})$-bounded. In particular we have for any $S \in \mathcal{T}_{\rm std}$ that $\|X_S\|_2 \le n\sqrt{n}$ and $\lambda_{\min}(X_S^{\rm T} X_S) = 1/\|X_S^{-1}\|_2^2 \ge 1/4$.*

Proof. The diameter of a simplex $S \in \mathcal{T}_{\rm std}$ is clearly bounded by $\sqrt{n}$ and by

$$\|X_S\|_2 \le \sqrt{n}\|X_S\|_\infty = \sqrt{n} \max_{i=1,2,\ldots,n} \|\mathbf{x}_i - \mathbf{x}_0\|_1 \le \sqrt{n}\sqrt{n} \max_{i=1,2,\ldots,n} \|\mathbf{x}_i - \mathbf{x}_0\|_2 \le n\sqrt{n}.$$

That $\|X_S^{-1}\|_2 \le 2$ was shown in [14, Remark 2], hence, $d = 2\sqrt{n}$ is sufficiently large. □

The next lemma is a simple consequence of the eigenvalues of a matrix being continuous functions of its entries.

Lemma 2.16. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^{n\times n}$ and let $\mathcal{B} \subset \mathbb{R}^{n\times n}$ be a compact subset of $\mathbb{R}^{n\times n}$ in the norm topology. Assume that there is an $\beta > 0$ such that $\min_{B\in\mathcal{B}} \lambda_{\min}(B^{\rm T}B) \ge \beta$. Then there is a $\delta > 0$ such that $\min_{B\in\mathcal{B}} \|A - B\| \le \delta$ implies $\lambda_{\min}(A^{\rm T}A) \ge \beta/2$.*

Proof. Assume on the contrary that there is sequence $(A_k)_{k\in\mathbb{N}_+}$ of $\mathbb{R}^{n\times n}$ matrices such that $\min_{B\in\mathcal{B}} \|A_k - B\| \le 1/k$, but $\lambda_{\min}(A_k^{\rm T}A_k) < \beta/2$, for all $n \in \mathbb{N}_+$. Then there is a sequence $(B_k)_{k\in\mathbb{N}_+}$ of matrices in $\mathcal{B}$ such that $\|A_k - B_k\| \le 1/k$ for all $k \in \mathbb{N}_+$ and a subsequence $(B_{k_l})_{l\in\mathbb{N}_+}$ of this sequence such that B_{k_l} converges to a $\widetilde{B} \in \mathcal{B}$. Because $\|A_{k_l} - \widetilde{B}\| \le \|A_{k_l} - B_{k_l}\| + \|B_{k_l} - \widetilde{B}\|$ we thus have a sequence $(A_{k_l})_{l\in\mathbb{N}_+}$ of matrices converging to $\widetilde{B}$, but $\lambda_{\min}(A_{k_l}^{\rm T}A_{k_l}) < \beta/2$ for all $k \in \mathbb{N}_+$, which is contradictory to $\lambda_{\min}(\widetilde{B}^{\rm T}\widetilde{B}) \ge \beta$ and $\lambda_{\min} : \mathbb{R}^{n\times n} \to \mathbb{R}$ being continuous. □

Lemma 2.17. *Consider the mapping $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ from Definition 2.3. Then there are constants $\alpha_{\mathbf{F}}, \beta_{\mathbf{F}} > 0$ such that*

$$\mathrm{diam}(S_{\mathbf{F}}) < \alpha_{\mathbf{F}} \quad \textit{and} \quad \|X_{S_{\mathbf{F}}}^{-1}\|_2 \le \beta_{\mathbf{F}} \quad \textit{for all} \quad S \in \mathcal{T}_{\rm std},$$

where $S_{\mathbf{F}} = \mathrm{co}\,(\mathbf{F}(\mathbf{x}_0), \mathbf{F}(\mathbf{x}_1), \ldots, \mathbf{F}(\mathbf{x}_n))$ is the simplex $S = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{T}_{\rm std}$ mapped by $\mathbf{F}$ and $X_{S_{\mathbf{F}}}$ is the corresponding shape-matrix.

Proof. For $i = 1, 2, \ldots, n$ we can define

$$\min_{|x_i|=\|\mathbf{x}\|_\infty=1} \lambda_{\min}(D\mathbf{G}_i(\mathbf{x})D\mathbf{G}_i(\mathbf{x})^{\mathrm{T}}) =: \beta_i,$$

because $D\mathbf{G}_i$ and $\lambda_{\min}$ are continuous functions on the compact set $C_i := \{\mathbf{x} \in \mathbb{R}^n : |x_i| = \|\mathbf{x}\|_\infty = 1\}$. By Lemma 2.12 $D\mathbf{G}_i(\mathbf{x})$ is nonsingular for all $\mathbf{x}$ with $x_i \neq 0$. Hence $D\mathbf{G}_i(\mathbf{x})D\mathbf{G}_i(\mathbf{x})^{\mathrm{T}}$ is nonsingular for all such $\mathbf{x}$ and therefore $\beta_i > 0$. Define

$$\beta := \min\{\beta_1, \ldots, \beta_n\} = \min_{i=1,2,\ldots,n} \left[\min_{|x_i|=\|\mathbf{x}\|_\infty=1} \lambda_{\min}(D\mathbf{G}_i(\mathbf{x})D\mathbf{G}_i(\mathbf{x})^{\mathrm{T}}) \right] > 0. \quad (2.6)$$

Observe from the definition of $\mathbf{G}_i$ and formula (2.5) that $\mathbf{G}_i(\frac{1}{R}\mathbf{x}) = \frac{1}{R}\mathbf{G}_i(\mathbf{x})$ and $D\mathbf{G}_i(\frac{1}{R}\mathbf{x}) = D\mathbf{G}_i(\mathbf{x})$ for all $R > 0$ and all $\mathbf{x} \in \mathbb{R}^n$ with $x_i \neq 0$. Let $X_{S_1}, X_{S_2}, \ldots, X_{S_N}$ be a finite enumeration of the different shape-matrices of the simplices in $\mathcal{T}_{\mathrm{std}}$, cf. Lemma 2.9. By Lemma 2.14 and Lemma 2.15 we have

$$\begin{aligned}
&\min_{i=1,2,\ldots,n} \left[\min_{\substack{|x_i|=\|\mathbf{x}\|_\infty=1 \\ j=1,2,\ldots,N}} \lambda_{\min}\left([X_{S_j} D\mathbf{G}_i(\mathbf{x})^{\mathrm{T}}]^{\mathrm{T}} X_{S_j} D\mathbf{G}_i(\mathbf{x})^{\mathrm{T}}\right) \right] \\
&\qquad \geq \min_{i=1,2,\ldots,n} \left[\min_{\substack{|x_i|=\|\mathbf{x}\|_\infty=1 \\ j=1,2,\ldots,N}} \lambda_{\min}\left(X_{S_j}^{\mathrm{T}} X_{S_j}\right) \lambda_{\min}\left(D\mathbf{G}_i(\mathbf{x})D\mathbf{G}_i(\mathbf{x})^{\mathrm{T}}\right) \right] \\
&\qquad \geq \frac{\beta}{4},
\end{aligned}$$

with $\beta > 0$ from (2.6), and that

$$\mathcal{B} := \bigcup_{i=1,2,\ldots,n} \left\{ X_{S_j} D\mathbf{G}_i(\mathbf{x})^{\mathrm{T}} \in \mathbb{R}^{n\times n} : |x_i| = \|\mathbf{x}\|_\infty = 1 \text{ and } j = 1, 2, \ldots, N \right\} \subset \mathbb{R}^{n\times n}$$

is compact in the norm topology.

Hence, by Lemma 2.16 there is a $\delta > 0$ such that

$$\min_{B\in\mathcal{B}} \|A - B\|_\infty \leq \delta \quad \text{implies} \quad \lambda_{\min}(A^{\mathrm{T}}A) \geq \frac{\beta}{8}. \quad (2.7)$$

Clearly we may assume $\delta \leq 1$:

By Lemma 2.12 ii) there is an $R^* > 0$ so large that for any $0 < |h| < 1/R^*$ we have

$$\max_{i=1,2,\ldots,n} \sup_{|z_i|=\|\mathbf{z}\|_\infty=\|\mathbf{u}\|_\infty=1} \left\| \frac{\mathbf{G}_i(\mathbf{z} + h\mathbf{u}) - \mathbf{G}_i(\mathbf{z})}{h} - D\mathbf{G}_i(\mathbf{z})\mathbf{u} \right\|_\infty < \frac{\delta}{n}. \quad (2.8)$$

We now show that for every $S \in \mathcal{T}_{\mathrm{std}}$ such that $S \cap \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty \leq R^*\} = \emptyset$ we have

$$\lambda_{\min}(X_{S_{\mathbf{F}}}^{\mathrm{T}} X_{S_{\mathbf{F}}}) \geq \frac{\beta}{8}.$$

To do this fix an arbitrary $S := S_\nu = \mathrm{co}\,(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu) \in \mathcal{T}_{\mathrm{std}}$ such that $S \cap \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty \leq R^*\} = \emptyset$. Recall that with $\nu = (\mathbf{z}, \sigma, \mathbf{J})$ we have

$$S_\nu = \mathrm{co}\,(R_{\mathbf{J}}\mathbf{z}, R_{\mathbf{J}}\mathbf{z} + R_{\mathbf{J}}\mathbf{u}_1^\sigma, R_{\mathbf{J}}\mathbf{z} + R_{\mathbf{J}}\mathbf{u}_2^\sigma, \ldots, R_{\mathbf{J}}\mathbf{z} + R_{\mathbf{J}}\mathbf{u}_n^\sigma)$$

and thus $R := \|R_{\mathbf{J}}\mathbf{z}\|_\infty > R^*$. By Lemma 2.13 there is an $i \in \{1, 2, \ldots, n\}$ such that $|x_i| = \|\mathbf{x}\|_\infty$ for all $\mathbf{x} \in S_\nu$. Obviously $x_i \neq 0$ and $\mathbf{F}(\mathbf{x}) = \mathbf{G}_i(\mathbf{x})$ for all $\mathbf{x} \in S_\nu$.

The shape-matrices X_S and $X_{S_\mathbf{F}}$ are now given by $X_S = (R_\mathbf{J}\mathbf{u}_1^\sigma, R_\mathbf{J}\mathbf{u}_2^\sigma, \ldots, R_\mathbf{J}\mathbf{u}_n^\sigma)^\mathrm{T}$ and

$$X_{S_\mathbf{F}} = \begin{pmatrix} [\mathbf{F}(R_\mathbf{J}\mathbf{z} + R_\mathbf{J}\mathbf{u}_1^\sigma) - \mathbf{F}(R_\mathbf{J}\mathbf{z})]^\mathrm{T} \\ [\mathbf{F}(R_\mathbf{J}\mathbf{z} + R_\mathbf{J}\mathbf{u}_2^\sigma) - \mathbf{F}(R_\mathbf{J}\mathbf{z})]^\mathrm{T} \\ \vdots \\ [\mathbf{F}(R_\mathbf{J}\mathbf{z} + R_\mathbf{J}\mathbf{u}_n^\sigma) - \mathbf{F}(R_\mathbf{J}\mathbf{z})]^\mathrm{T} \end{pmatrix} = \begin{pmatrix} [\mathbf{G}_i(R_\mathbf{J}\mathbf{z} + R_\mathbf{J}\mathbf{u}_1^\sigma) - \mathbf{G}_i(R_\mathbf{J}\mathbf{z})]^\mathrm{T} \\ [\mathbf{G}_i(R_\mathbf{J}\mathbf{z} + R_\mathbf{J}\mathbf{u}_2^\sigma) - \mathbf{G}_i(R_\mathbf{J}\mathbf{z})]^\mathrm{T} \\ \vdots \\ [\mathbf{G}_i(R_\mathbf{J}\mathbf{z} + R_\mathbf{J}\mathbf{u}_n^\sigma) - \mathbf{G}_i(R_\mathbf{J}\mathbf{z})]^\mathrm{T} \end{pmatrix}.$$

Hence, by (2.8) and because $\|A\|_\infty = \|A^\mathrm{T}\|_1$ we have

$$\begin{aligned} &\left\|X_{S_\mathbf{F}} - X_S D\mathbf{G}_i(R_\mathbf{J}\mathbf{z})^\mathrm{T}\right\|_\infty = \left\|X_{S_\mathbf{F}}^\mathrm{T} - D\mathbf{G}_i(R_\mathbf{J}\mathbf{z})X_S^\mathrm{T}\right\|_1 \qquad (2.9)\\ &\quad = \max_{j=1,2,\ldots,n} \left\|\mathbf{G}_i(R_\mathbf{J}\mathbf{z} + R_\mathbf{J}\mathbf{u}_j^\sigma) - \mathbf{G}_i(R_\mathbf{J}\mathbf{z}) - D\mathbf{G}_i(R_\mathbf{J}\mathbf{z})R_\mathbf{J}\mathbf{u}_j^\sigma\right\|_1 \\ &\quad \le n \cdot \max_{j=1,2,\ldots,n} \left\|\mathbf{G}_i(R_\mathbf{J}\mathbf{z} + R_\mathbf{J}\mathbf{u}_j^\sigma) - \mathbf{G}_i(R_\mathbf{J}\mathbf{z}) - D\mathbf{G}_i(R_\mathbf{J}\mathbf{z})R_\mathbf{J}\mathbf{u}_j^\sigma\right\|_\infty \\ &\quad = n \cdot \max_{j=1,2,\ldots,n} \left\| \frac{\mathbf{G}_i(\frac{1}{R}R_\mathbf{J}\mathbf{z} + \frac{1}{R}R_\mathbf{J}\mathbf{u}_j^\sigma) - \mathbf{G}_i(\frac{1}{R}R_\mathbf{J}\mathbf{z})}{\frac{1}{R}} - D\mathbf{G}_i\left(\frac{1}{R}R_\mathbf{J}\mathbf{z}\right)R_\mathbf{J}\mathbf{u}_j^\sigma \right\|_\infty \\ &\quad < n \cdot \frac{\delta}{n} = \delta. \end{aligned}$$

Thus we have

$$\left\|X_{S_\mathbf{F}}\right\|_\infty < \delta + \|X_S\|_\infty \|D\mathbf{G}_i(R_\mathbf{J}\mathbf{z})\|_1 \le 1 + 3n\sqrt{n},$$

where we used $\delta \le 1$, Lemma 2.12 for the bound $\|D\mathbf{G}_i(R_\mathbf{J}\mathbf{z})\|_1 \le \sqrt{n}\|D\mathbf{G}_i(R_\mathbf{J}\mathbf{z})\|_2 \le 3\sqrt{n}$, and $\|X_S\|_\infty = \max_{i=1,2,\ldots,n} \|\mathbf{x}_i^\nu - \mathbf{x}_0^\nu\|_1 \le n$. Fix $i, j \in \{0,1,\ldots,n\}$ such that $\operatorname{diam}(S_\mathbf{F}) = \|\mathbf{F}(\mathbf{x}_i^\nu) - \mathbf{F}(\mathbf{x}_j^\nu)\|_2$. Then we have

$$\|\mathbf{F}(\mathbf{x}_i^\nu) - \mathbf{F}(\mathbf{x}_j^\nu)\|_2 \le \|\mathbf{F}(\mathbf{x}_i^\nu) - \mathbf{F}(\mathbf{x}_j^\nu)\|_1 \le \|\mathbf{F}(\mathbf{x}_i^\nu) - \mathbf{F}(\mathbf{x}_0^\nu)\|_1 + \|\mathbf{F}(\mathbf{x}_j^\nu) - \mathbf{F}(\mathbf{x}_0^\nu)\|_1 \le 2\|X_{S_\mathbf{F}}\|_\infty$$

and it follows that

$$\operatorname{diam}(S_\mathbf{F}) < 2 \cdot (1 + 3n\sqrt{n}) =: \alpha_\mathbf{F}.$$

A further consequence of (2.9) is, by (2.7), that

$$\lambda_{\min}\left(X_{S_\mathbf{F}}^\mathrm{T} X_{S_\mathbf{F}}\right) \ge \frac{\beta}{8}.$$

Because there is only a finite number of simplices $S \in \mathcal{T}_{\mathrm{std}}$ such that $S \cap \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty \le R^*\} \neq \emptyset$, say $S^1, S^2, \ldots, S^M$, and all $S_\mathbf{F}^j$ are (nondegenerate) simplices by [1], i.e. $X_{S_\mathbf{F}^j}$ nonsingular, we have with

$$\beta^* := \min\left\{\frac{\beta}{8}, \lambda_{\min}\left(X_{S^1}^\mathrm{T} X_{S^1}\right), \lambda_{\min}\left(X_{S^2}^\mathrm{T} X_{S^2}\right), \ldots, \lambda_{\min}\left(X_{S^M}^\mathrm{T} X_{S^M}\right)\right\} > 0$$

that for every $S \in \mathcal{T}_{\mathrm{std}}$ that

$$\lambda_{\min}\left(X_{S_\mathbf{F}}^\mathrm{T} X_{S_\mathbf{F}}\right) \ge \beta^*.$$

Using $\lambda_{\min}\left(X_{S_\mathbf{F}}^\mathrm{T} X_{S_\mathbf{F}}\right) = 1/\|X_{S_\mathbf{F}}\|_2^2$ and setting $\beta_\mathbf{F} := 1/\sqrt{\beta^*}$ concludes the proof. □

A direct corollary from Lemma 2.17 is that $\mathcal{T}_\Phi$ is not only a triangulation as proved in [1], but an (h,d)-bounded triangulation when $\Phi = \mathbf{F}$.

Corollary 2.18. *Consider the mapping Φ from Definition 2.3, where $\rho(0) = 1$ and $\rho(x) = 1$ for $x > 0$. Then $\Phi = \mathbf{F}$ and there are constants $h, d > 0$ such that the triangulation $\mathcal{T}_\Phi = \Phi(\mathcal{T}_{\mathrm{std}}) = \mathbf{F}(\mathcal{T}_{\mathrm{std}})$ is (h,d)-bounded.*

Proof. Follows immediately from Lemma 2.17 with $h = \alpha_{\mathbf{F}}$ and $d = \alpha_{\mathbf{F}}\beta_{\mathbf{F}}$. □

We finish this section with the main results of the paper, which are a more general version of the last corollary where we consider $\mathcal{T}_{\Phi,K}$ for a $K > 1$; recall that $\mathcal{T}_1^{\text{std}} = \mathcal{T}_{\text{std}}$. Here we also must prove that $\mathcal{T}_{\Phi,K}$ actually is a triangulation.

Theorem 2.19. *Consider the mapping Φ from* (2.3)*, where $\rho(0) = 1$ and $\rho(x) = 1$ for $x > 0$. Let $K \in \mathbb{N}_+$. Then $\mathbf{F} = \Phi$, $\mathcal{T}_{\Phi,K}$ is a triangulation in the sense of Definition 2.2 and there are constants $h, d > 0$ such that the triangulation $\mathcal{T}_{\Phi,K} = \Phi(\mathcal{T}_K^{\text{std}}) = \mathbf{F}(\mathcal{T}_K^{\text{std}})$ is (h,d)-bounded.*

Proof. Recall the definition of $\mathcal{T}_K^{\text{std}}$ as the (disjoint) union of $\mathcal{T}_K^* := \{S_\nu \in \mathcal{T}_{\text{std}} : S_\nu \cap (-K,K)^n = \emptyset\}$ and $\mathcal{T}_K^{\text{fan}}$ in Definition 2.4. Surely $\mathcal{T}_{\Phi,K}^* := \Phi(\mathcal{T}_K^*)$ is a triangulation in the sense of Definition 2.4, cf. the discussion after Corollary 2.18 in [1].

Consider a simplex $S_\nu = \text{co}(\mathbf{0}, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu) \in \mathcal{T}_K^{\text{fan}}$. Since its vertices are affinely independent, the vectors $\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu$ are linearly independent and so are the vectors $\Phi(\mathbf{x}_i^\nu) = (\|\mathbf{x}_i^\nu\|_\infty / \|\mathbf{x}_i^\nu\|_2)\mathbf{x}_i^\nu$, $i = 1, 2, \ldots, n$, because they are parallel to the $\mathbf{x}_i^\nu$s. It follows that $\Phi(S) = \text{co}(\Phi(\mathbf{0}), \Phi(\mathbf{x}_1^\nu), \ldots, \Phi(\mathbf{x}_n^\nu))$ is a simplex.

We now show similarly as in [5], where it was shown that $\mathcal{T}_K^{\text{std}}$ is a triangulation, that $\mathcal{T}_{\Phi,K} = \Phi(\mathcal{T}_K^{\text{std}})$ is a triangulation. Fix an arbitrary $\mathbf{x} \neq \mathbf{0}$ and consider the half line $\{t\mathbf{x} : t > 0\}$ that must intersect with the boundary of $\mathcal{T}_K^{\text{fan}}$. Hence, there is a $t^* > 0$ and a $\nu = (\mathbf{z}, \sigma, \mathbf{J}) \in I$ with $\|\mathbf{z}\|_\infty = \|\mathbf{x}_0^\nu\|_\infty = K - 1$ and $\|\mathbf{x}_i^\nu\|_\infty = K$ for $i = 1, 2, \ldots, n$, such that $t^*\mathbf{x} \in \text{co}\{\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu\}$. Since Φ is radial, i.e. $\Phi(\mathbf{x}) = c_{\mathbf{x}}\mathbf{x}$ for some $c_{\mathbf{x}} > 0$, it follows that there is a $t > 0$ such that $t\mathbf{x} \in \text{co}\{\Phi(\mathbf{x}_1^\nu), \Phi(\mathbf{x}_2^\nu), \ldots, \Phi(\mathbf{x}_n^\nu)\}$, i.e.

$$t\mathbf{x} = \sum_{i=1}^n \lambda_i \Phi(\mathbf{x}_i^\nu), \quad \lambda_i \geq 0, \quad \text{and} \quad \sum_{i=1}^n \lambda_i = 1.$$

If $t \geq 1$, then the vector $\mathbf{x}$ is in the domain of $\mathcal{T}_{\Phi,K}^{\text{fan}}$, where it can be written as

$$\mathbf{x} = \left(1 - \frac{1}{t}\right)\mathbf{0} + \sum_{i=1}^n \frac{\lambda_i}{t}\Phi(\mathbf{x}_i^\nu).$$

It is simple to show that $\mathbf{x}$ is an inner point, cf. Definition 2.4, of exactly one simplex in the simplicial complex associated to $\mathcal{T}_{\Phi,K}^{\text{fan}}$. As the same obviously holds true for $\mathbf{x} = \mathbf{0}$, it follows that $\mathcal{T}_{\Phi,K}^{\text{fan}}$ is a triangulation, cf. Lemma 2.5 in [1].

If $t \leq 1$, then the vector $\mathbf{x}$ is in the domain of $\mathcal{T}_{\Phi,K}^*$ and by Corollary 2.18 and Lemma 2.5 in [1] it is the inner point of exactly one simplex in the simplicial complex associated to the triangulation $\mathcal{T}_{\Phi,K}^*$.

If $t = 1$ we have from the formula above for the case $t \geq 1$ and because $\mathcal{T}_{\Phi,K}^{\text{fan}}$ is a triangulation, that

$$\mathbf{x} = \sum_{j=1}^m \lambda_{i_j}\Phi(\mathbf{x}_{i_j}^\nu), \quad \sum_{j=1}^m \lambda_{i_j} = 1, \quad \lambda_{i_j} > 0,$$

for exactly one simplex $\text{co}\left\{\Phi(\mathbf{x}_{i_1}^\nu), \Phi(\mathbf{x}_{i_2}^\nu), \ldots, \Phi(\mathbf{x}_{i_m}^\nu)\right\}$ of the simplicial complex associated to $\mathcal{T}_{\Phi,K}^{\text{fan}}$. Since this simplex is also in the simplicial complex associated to $\mathcal{T}_{\Phi,K}^*$, and $\mathbf{x}$ cannot be an inner point of any further simplex in the simplicial complex associated to $\mathcal{T}_{\Phi,K}^*$ because $\mathcal{T}_{\Phi,K}^*$ is a triangulation, it follows that $\mathbf{x}$ is an inner point of exactly one simplex in the simplicial complex associated to $\mathcal{T}_{\Phi,K}^{\text{std}}$.

Finally note that if $t > 1$, then $\mathbf{x}$ is not in the domain of $\mathcal{T}^*_{\Phi,K}$, and if $t < 1$, then $\mathbf{x}$ is not in the domain of $\mathcal{T}^{\text{fan}}_{\Phi,K}$.

Thus every $\mathbf{x} \in \mathbb{R}^n$ is an inner point of exactly one simplex in the simplicial complex associated to $\mathcal{T}_{\Phi,K}$, from which it follows by Lemma 2.5 in [1] that $\mathcal{T}_{\Phi,K}$ is a triangulation.

That the triangulation $\mathcal{T}_{\Phi,K}$ is (h,d)-bounded can now be proved exactly as in Lemma 2.17. Just make sure that $R^* > K$ so that the triangle fan $\mathcal{T}^{\text{fan}}_K$ of $\mathcal{T}^{\text{std}}_K$ is within the area where we enumerate the simplices and note that there are only finitely many (nondegenerate) simplices in the triangle fan $\Phi(\mathcal{T}^{\text{fan}}_K)$ because $\mathcal{T}_{\Phi,K}$ is a triangulation. $\square$

3. **Examples.**

We show by examples that the CPA algorithm can compute Lyapunov functions for nonlinear system using much fewer triangles, when the triangulation $\mathcal{T}^{\text{std}}_K$ is mapped to adapt it to the symmetry of the system. We consider the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y + \alpha x(x^2+y^2-1) \\ \\ x + \beta y(x^2+y^2-1) \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ \\ f_2(x,y) \end{pmatrix}, \tag{3.1}$$

dependent on the parameters α and β. The second order derivatives $f_i(x,y)$ are

$$\frac{\partial^2 f_1}{\partial x^2}(x,y) = 6\alpha x, \quad \frac{\partial^2 f_1}{\partial x \partial y}(x,y) = 2\alpha y, \quad \frac{\partial^2 f_1}{\partial y^2}(x,y) = 2\alpha x$$

$$\frac{\partial^2 f_2}{\partial x^2}(x,y) = 2\beta y, \quad \frac{\partial^2 f_2}{\partial x \partial y}(x,y) = 2\beta x, \quad \frac{\partial^2 f_2}{\partial y^2}(x,y) = 6\beta y.$$

We can thus use the upper bounds ($x_1 := x, x_2 := y$)

$$B_\nu := 2\max\{|3\alpha x|, |\beta x|, |3\beta y|, |\alpha y|\} \geq \max_{i,r,s=1,2} \max_{(x,y)\in S_\nu} \left| \frac{\partial^2 f_i}{\partial x_r \partial x_s}(x,y) \right|$$

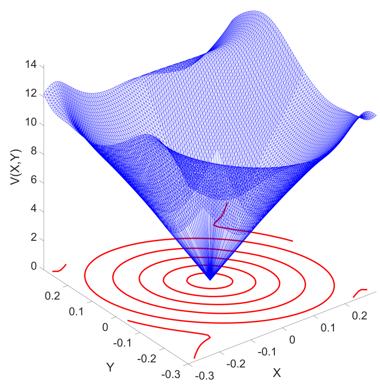
for the linear programming problem, cf. Definition 6 in [6], to compute a CPA Lyapunov function for the system.

The Jacobian at the origin is given by

$$A := D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} -\alpha & -1 \\ 1 & -\beta \end{pmatrix}, \text{ with eigenvalues } \lambda_\pm = -\frac{\alpha+\beta}{2} \pm \sqrt{\frac{(\alpha+\beta)^2 - 4(1+\alpha\beta)}{4}},$$

and from [6] we know that if the real-parts of $\lambda_\pm$ are both negative, then the linear programming problem in [6] has a feasible solution, if the triangles of the simplicial complex are small enough and there are enough triangles in the triangle fan at the origin.

For our computations we used the implementation of the CPA algorithm described in [12, 3, 13]. The computations were performed on a state of the art PC (intel i7 7700K processor, 4 cores at 4.2GHz, 64GB RAM) and each CPA Lyapunov functions was computed in less than 2 minutes. The problem is a feasibility problem; nonetheless, we minimized the objective $\max_\nu \|\nabla V_\nu\|_\infty$ in all the examples to avoid extremely steep gradients.



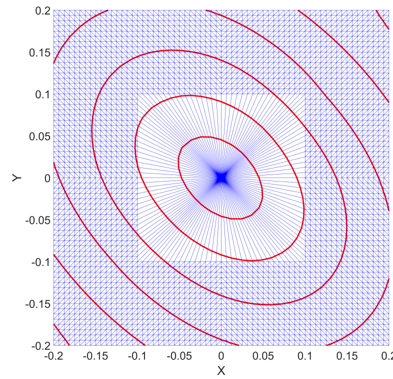


FIGURE 3. Left: CPA Lyapunov function for system (3.1) with $\alpha = 0.5$ and $\beta = -0.3$ using a rectangular grid. Right: The rectangular grid, with a triangle fan at the origin, used for the computation. Level-sets of the Lyapunov function are drawn in red on both figures.

3.1. **Experiment 1.** For the first numerical experiment we set $\alpha = 0.5$ and $\beta = -0.3$ and fixed the domain of the triangle fan at the origin of $\mathcal{T}_K^{\text{std}}$ as $[-0.1, 0.1]^2$. We increased K in powers of 2 until we obtained a solution with $K = 16$, resulting in $4 \cdot 2 \cdot 16 = 128$ triangles in the triangle fan at the origin. In Figure 3 the computed CPA Lyapunov function is plotted, together with the triangulation used and some level-sets of the Lyapunov function. Clearly, the level-sets are in the form of ellipses, indicating that a rectangular grid is not optimal for the computations. We then solved the Lyapunov equation $A^{\mathrm{T}}P + PA = -I$ and obtained the solution

$$P = \begin{pmatrix} -1.217391304347826 & 0.108695652173913 \\ 0.108695652173913 & -1.304347826086957 \end{pmatrix}.$$

By using the triangulation $\frac{1}{32}P^{-\frac{1}{2}}\mathcal{T}_{\Phi,8}$ we could compute a CPA Lyapunov function with half as many triangles in the triangle fan at the origin, covering an area of similar size as for the rectangular grid. Further, the triangles not in the triangle fan are much larger than the triangles needed in the rectangular triangulation $\mathcal{T}_K^{\text{std}}$. The CPA Lyapunov function and the corresponding triangulation are depicted in Figure 4, where one can additionally see that the triangulation is much better adapted to the elliptical shape of the level-sets of the Lyapunov function.

3.2. **Experiment 2.** For the second numerical experiment we set $\alpha = 0.5$ and $\beta = -0.4$, resulting in eigenvalues of the Jacobian at the origin closer to zero than in the previous experiment, i.e. slower convergence. It is thus more difficult to compute a CPA Lyapunov function. Here we fixed the number of the triangles in the triangle fan at the origin to 64, i.e. $K = 8$, and investigated how small we have to make the area covered by the triangle fan, both for the rectangular triangulation and the transformed triangulation. Here, in this more difficult example due to the slower convergence, the advantage of using the transformed grid is even clearer than in the first numerical experiment.

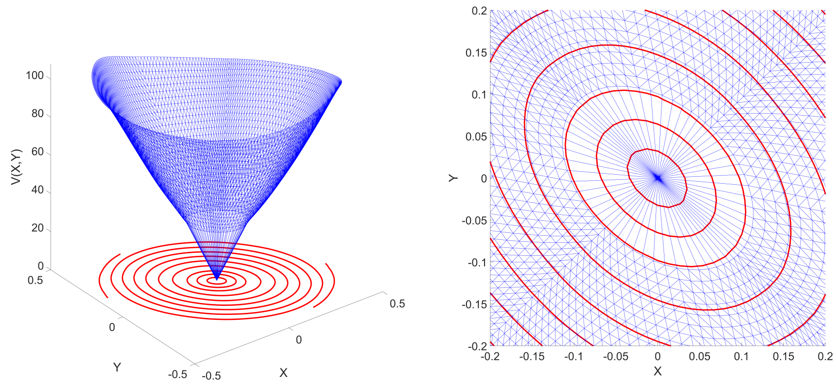


FIGURE 4. Left: CPA Lyapunov function for system (3.1) with $\alpha = 0.5$ and $\beta = -0.3$ using a transformed grid. Right: The transformed grid, with a triangle fan at the origin, used for the computation. Level-sets of the Lyapunov function are drawn in red on both figures. Note that the triangulation is much better adapted to the shape of the level-sets than when using a rectangular grid as in Figure 3.

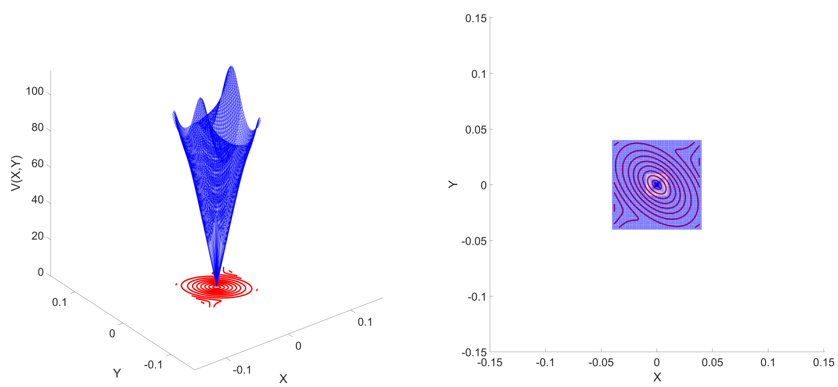


FIGURE 5. Left: CPA Lyapunov function for system (3.1) with $\alpha = 0.5$ and $\beta = -0.4$ using a rectangular grid. Right: The rectangular grid, with a triangle fan at the origin, used for the computation. Level-sets of the Lyapunov function are drawn in red on both figures.

To obtain a solution with $K = 8$, i.e. $4 \cdot 2 \cdot 8 = 64$ triangles in the fan, we can only cover the rectangle $[-0.01, 0.01]^2$ with the scaled rectangular grid, cf. Figure 5. Using the transformation we can map the rectangle $[-0.1, 0.1]^2$, cf. Figure 6. In both Figures 5 and 6 we use the same xy-plane to make the difference in the size of the area covered by the triangle fan clearly visible. To obtain an area of

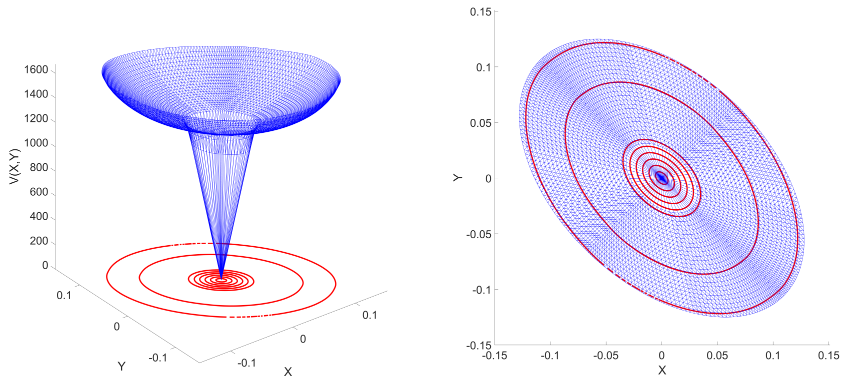


FIGURE 6. Left: CPA Lyapunov function for system (3.1) with $\alpha = 0.5$ and $\beta = -0.4$ using a transformed grid. Right: The transformed grid, with a triangle fan at the origin, used for the computation. Level-sets of the Lyapunov function are drawn in red on both figures. Note that the triangulation is much better adapted to the shape of the level-sets than when using a rectangular grid as in Figure 5. Both the area covered by the triangle fan, in both cases with 64 triangles, as well as the area covered overall are much larger than when using the rectangular grid, see Figure 5.

comparable size to the area covered by the triangle fan in Figure 6, but without using the transformation, even $4 \cdot 2 \cdot 64 = 512$ triangles in the fan are not sufficient.

4. **Conclusions.** We advanced the CPA algorithm to compute continuous and piece-wise affine (CPA) Lyapunov functions for nonlinear systems [10, 11, 6] by proving that the triangulation $\mathcal{T}_\Phi$ of $\mathbb{R}^n$ from [1] is (h,d)-bounded.

Further, we showed that a modification $\mathcal{T}_{\Phi,K}$ of $\mathcal{T}_\Phi$ with a triangle fan at the origin is an (h,d)-bounded triangulation. Note that this implies that a sufficiently scaled-down triangulation of this type is suitable for the algorithm to compute CPA Lyapunov functions for nonlinear systems. The triangulation $\mathcal{T}_{\Phi,K}$ is exceptionally well suited, because it can easily be adapted to symmetries of the system, revealed by a quadratic Lyapunov function for a linearization of the system. We demonstrated examples, where the algorithm from [6] was able to compute Lyapunov functions for nonlinear systems using considerably fewer triangles (simplices) than previously possible, and we explained in detail why this is the case.

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