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Automatic Determination of Connected Sublevel Sets of CPA Lyapunov Functions*

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Abstract. Lyapunov functions are an important tool to determine the basin of attraction of equilibria. In particular, the connected component of a sublevel set, which contains the equilibrium, is a forward invariant subset of the basin of attraction. One method to compute a Lyapunov function for a general nonlinear autonomous differential equation constructs a Lyapunov function, which is continuous and piecewise affine (CPA) on each simplex of a fixed triangulation. In this paper we propose an algorithm to determine the largest connected sublevel set of such a CPA Lyapunov function and prove that it determines the largest subset of the basin of attraction that can be obtained by this Lyapunov function.

Key words. Lyapunov function, level sets, CPA method

AMS subject classifications. 37B25, 37M99, 34D20

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1. Introduction.

Within the theory of dynamical systems, Lyapunov functions are a well-known method for proving stability and finding the basin of attraction of an equilibrium. In the classical Lyapunov stability theory [19] the stability of one attractor of a differential equation is considered and is studied in most textbooks on nonlinear differential equations; see, e.g., [13, 17, 34].

Given an autonomous system $\dot{x} = f(x)$, $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, with equilibrium x_0, a strict Lyapunov function is a function $V \in C^0(D, \mathbb{R})$, where D is an open neighborhood of x_0, which has a minimum at the equilibrium and is strictly decreasing along solutions of the ODE within D apart from the equilibrium. The existence of a strict Lyapunov function implies that the equilibrium is asymptotically stable, and it provides a lower bound on its basin of attraction by sublevel sets, which are compact and contained in D.

Converse theorems, proving the existence of Lyapunov functions with various properties, have been obtained in the last 70 years; for a review, see [16]. If $f(x) = Ax$ is linear and the equilibrium at the origin is exponentially stable, then one can construct a Lyapunov function $V(x) = x^TQx$, where the positive definite matrix $Q \in \mathbb{R}^{n\times n}$ is the solution of the Lyapunov equation $A^TQ + QA = -P$ and $P \in \mathbb{R}^{n\times n}$ is an arbitrary positive definite matrix. Note that this method also works locally for nonlinear systems: Letting $A = Df(x_0)$, the function $V(x) = (x - x_0)^TQ(x - x_0)$ is a strict Lyapunov function for the nonlinear system; however, in general only in a small neighborhood D of x_0.

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Over the last 20 years many different methods of finding Lyapunov functions and, therefore, estimating the basin of attraction have been considered; for a review, see [30] and for a recent review of numerical methods, see [10]. One class of methods involves solving the Zubov equation [7, 33].

A large class of methods is based on SOS (sum of squares), which uses semidefinite programming to parameterize polynomial Lyapunov functions [2, 5, 21, 22, 23, 24, 25, 26]. The positive definiteness of polynomials is relaxed to checking that a polynomial is the sum of squared polynomials. This method has originally been used to check either (local) stability or global stability in the entire phase space.

Often the basin of attraction is not the entire phase space, but one is still interested in obtaining a good lower estimate. Sublevel sets of Lyapunov functions can be used to determine a subset of the basin of attraction. However, there are additional assumptions to check. One possibility is to assume that the function is defined on $\mathbb{R}^n$ and is radially unbounded, but that $\dot{v}(x) < 0$ does not hold everywhere, where $\dot{v}$ denotes the orbital derivative, i.e., the derivative along solutions of the ODE. Alternatively, one can consider a Lyapunov function which is defined in a set $B \subseteq \mathbb{R}^n$ such that $\dot{v}(x) < 0$ holds for all $x \in B \setminus \{0\}$, where 0 is the equilibrium, but then one needs to find the connected component of the sublevel set that contains the equilibrium and show that it does not intersect ∂B.

If the Lyapunov function is assumed to be radially unbounded, then the largest sublevel set (largest estimate of the domain of attraction, LEDA) is given by the level $\gamma = \inf_{x\in\mathbb{R}^n} v(x)$ such that $\dot{v}(x) = 0$ [4]; note that this is false if the Lyapunov function is not radially unbounded as the example in Figure 1 shows. If, more specifically, the Lyapunov function is assumed to be of the form $v(x) = x^T Qx$ with positive definite matrix Q, then the γ above can be found using a generalized bisection [28, 31].

Another approach is to fix a positive definite polynomial $p(x)$ and find the largest sublevel set $\{x \in \mathbb{R}^n \mid p(x) \leq \gamma\}$ such that there exists a Lyapunov function V within a certain set of functions, often polynomials up to a certain degree, such that the sublevel set is contained in a bounded sublevel set of fixed level [32]. This approach can be generalized by considering the maximum of finitely many Lyapunov functions [30].

In [27], Lyapunov functions are considered within a (bounded) set B, and additionally also for a general target set instead of an equilibrium. In this case one needs additional assumptions such as that the sublevel does not intersect the boundary of B. To find a sublevel set, it is proposed to guess a level and check whether there is a connected component of the sublevel set that does not intersect ∂B [27]. The other proposed method is to find the minimum of $v(x)$ on the boundary ∂B and use the corresponding sublevel set; see [29, Remark 5]. However, as the example in Figure 1 shows, it can contain points outside of the basin of attraction.

In this paper we consider the continuous and piecewise affine (CPA) method, which has been used to compute Lyapunov functions for nonlinear dynamical systems given by an autonomous ODE [8, 11, 14, 15, 18, 20]. Here, the Lyapunov function is constructed as a continuous function, which is affine on each simplex of a fixed triangulation of a given set S and thus determined by the values at the vertices of the triangulation. The CPA method determines suitable values of the Lyapunov function at the vertices by solving a linear optimization problem, where the conditions of the Lyapunov function are transformed into linear constraints. The method includes a verification that the CPA function is indeed a Lyapunov function. In

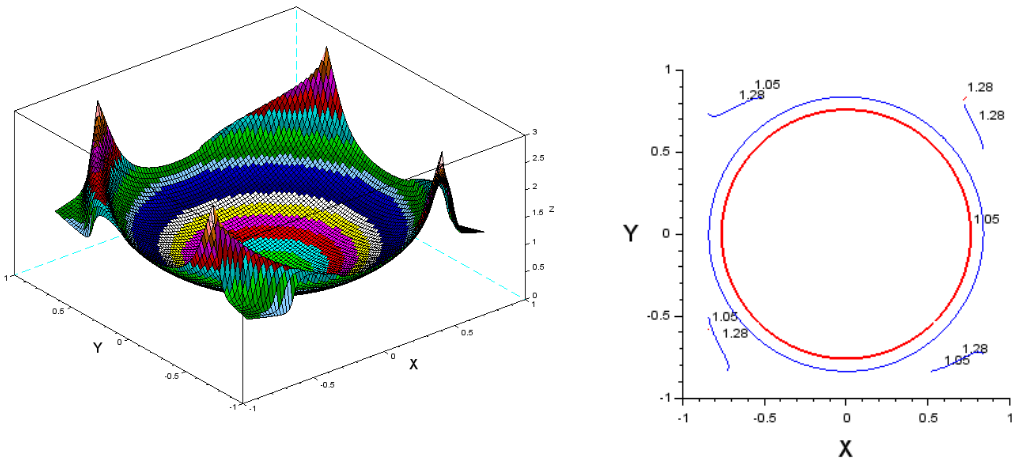

Figure 1. *Left: CPA Lyapunov function for the system* $\dot{x} = -y + x(1 - x^2 - y^2)$, $\dot{y} = x + y(1 - x^2 - y^2)$. *Right: The contour lines for the values* 1.05 *(red) and* 1.28 *(blue). The largest lower bound on the basin of attraction of the equilibrium at the origin is not obtained by considering the minimum of the function at the boundary (red), but by considering a local minimum such that the connected component containing the equilibrium extends to the boundary (blue).*

particular, the constraints include an error estimate which proves that if the problem is feasible, i.e., the constraints are satisfied, then the CPA Lyapunov function is strictly decreasing along solutions. It has been shown that a CPA Lyapunov function exists if the set S is a subset of the basin of attraction and the triangulation is sufficiently fine; see [8]. Hence, the method always succeeds in finding a Lyapunov function. This is an advantage over, e.g., the SOS method, which is not guaranteed to find a Lyapunov function due to the fact that a positive definite function is not necessarily SOS. While the SOS method is suited to prove global stability, the CPA method is not able to show global stability, but rather determine a bounded subset of the basin of attraction. The determination of a sublevel set of the CPA Lyapunov, which determines a subset of the basin of attraction being as large as possible, however, has so far not been solved satisfactorily.

Previously, the largest sublevel set was found in an ad hoc way. The problems are that we want to determine the connected component of the sublevel set that includes the equilibrium and it is thus a challenge to find both the connected component as well as the largest possible value for the level. Taking the minimal value at the boundary leads in general to a suboptimal value for the sublevel set, and the point, where it is attained, is in general not in the connected component including the equilibrium. Consider, for example, the system $\dot{x} = -y + x(1 - x^2 - y^2)$, $\dot{y} = x + y(1 - x^2 - y^2)$. This system has an asymptotically stable equilibrium at the origin and a periodic orbit at the unit circle. The equilibrium's basin of attraction is easily seen to be the open unit circular disc. In Figure 1 a CPA Lyapunov function with domain $[-0.85, 0.85]^2$ is depicted. We see that the minima at the boundary of this Lyapunov function are attained close to the points $(\pm 0.85, \pm 0.85)$, but this Lyapunov function delivers a larger

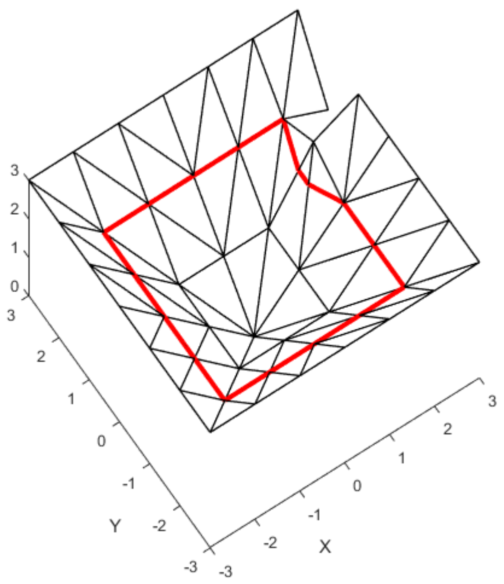

Figure 2. *CPA Lyapunov function for the system $\dot{x} = -x$, $\dot{y} = -y$. The largest sublevel set (red) affirmed by the Lyapunov function to be within the basin of attraction does not intersect the boundary of the domain of the Lyapunov function in a local minimum.*

lower bound on the basin of attraction if one uses a local minimum attained close to $(0, \pm 0.85)$ or $(\pm 0.85, 0)$.

Note that the level of the optimal sublevel set touching the boundary is not even necessarily a local minimum at the boundary as the following example shows: We consider the system $\dot{x} = -x$, $\dot{y} = -y$, which has a globally asymptotically stable equilibrium at the origin. We define $S = [-3, 3]^2 \setminus (2, 3] \times (1, 2)$ and use the standard triangulation $\mathcal{T}$ of this set with vertices $\mathrm{ve}(\mathcal{T}) = \{(k, l) \mid k, l \in \{-3, -2, \dots, 2, 3\}\}$. A CPA Lyapunov function is determined by its values at the vertices, and we fix $V(x, y) = \|(x, y)\|_\infty = \max(|x|, |y|)$ for all $(x, y) \in \mathrm{ve}(\mathcal{T}) \setminus \{(2, 1), (3, 2)\}$ and $V(2, 1) = 2.5$, $V(3, 2) = 1.5$. Then the largest sublevel set is obtained with the value 2 (see Figure 2), and it is found with the algorithm described in this paper. The vertex $(2, 2) \in \partial S$ is the vertex which terminates the algorithm, but note that it is not a local minimum at the boundary as the two adjacent vertices $(2, 1)$ and $(3, 2)$ have higher and lower V-value, respectively.

In this paper we propose an algorithm to find the optimal sublevel set and automatically determine the connected component which contains the equilibrium. Given a triangulation of the set S and the values of a Lyapunov function at the vertices, the algorithm considers a level m and gives every vertex a color: Green vertices are within the connected component of the sublevel set $\{x \in S \mid V(x) < m\}$, which contains the equilibrium, red vertices are outside, and yellow vertices provide a layer between the green and red ones. The algorithm increases the level m and recolors vertices at certain discrete levels—one can think of filling the graph of the

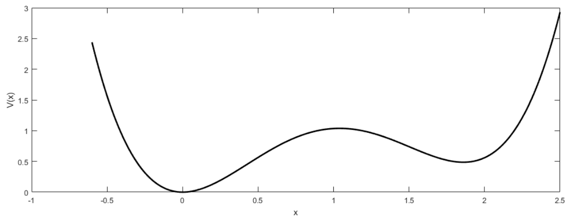

Figure 3. *When raising the level from the global minimum at* 0*, water would flow over the local maximum at* 1 *when reaching a certain level. This situation cannot arise if the function* $V(x)$ *is a Lyapunov function.*

function with water poured in at the equilibrium and the green vertices are the ones covered by water. The algorithm ends with the optimal level when the water reaches the boundary of S. The general idea of the algorithm is also applicable to finding sublevel sets of Lyapunov functions which are generated by other methods—note that if a specific class of functions is chosen, then there might be easier ways of checking the assumptions. The algorithm is also applicable to finding sublevel sets of a general function, starting at the global minimum. However, note that for a general function, the water could flow over a barrier; see Figure 3. This case cannot occur in the case of a Lyapunov function, as we will show in this paper. In the future, this algorithm could be combined with the search for a CPA Lyapunov function by maximizing the volume of the sublevel set to obtain a large subset of the basin of attraction as an optimization criterion.

Let us give an overview of the contents: In section 2 we introduce triangulations and the CPA method in more detail and recall the definition of $\mathcal{L}_V^{\sup}$, the largest subset of the basin of attraction which can be obtained with the Lyapunov function V. Section 3 introduces the algorithm and proves that the coloring of the vertices at each level m characterizes the connected component of the sublevel set at level m, which contains the equilibrium. Moreover, the coloring at the final level m_N, when the algorithm terminates, gives $\mathcal{L}_V^{\sup}$. Section 4 applies the algorithm to several examples. The appendix contains several proofs.

2. Triangulation and CPA method.

In this section we summarize the CPA method; for more details see [8]. To define a CPA function we must first fix a suitable simplicial complex and the corresponding triangulation.

Definition 2.1 (simplicial complex and adjacent vertices). *For $p+1$ vectors $v_0, v_1, \ldots, v_p \in \mathbb{R}^n$ their convex hull is defined as*

$$\mathrm{co}\{v_0, v_1, \ldots, v_p\} := \left\{ \sum_{i=0}^{p} \lambda_i x_i \mid \sum_{i=0}^{p} \lambda_i = 1, \lambda_i \geq 0 \textit{ for all } i \right\}.$$

A p-simplex is a set

$$T = \mathrm{co}\{v_0, v_1, \ldots, v_p\},$$

$p \geq 0$, where $v_0, \ldots, v_p$ are affinely independent, i.e., $v_1 - v_0, v_2 - v_0, \ldots, v_p - v_0$ are linearly independent. We denote by $\mathrm{ve}(T) = \{v_0, \ldots, v_p\}$ the set of vertices of T. A subsimplex of a simplex T is a simplex T' such that $\emptyset \neq \mathrm{ve}(T') \subseteq \mathrm{ve}(T)$.

A set of simplices $\mathcal{K}$ is called a simplicial complex of $S \subseteq \mathbb{R}^n$ if the following hold:

- $\bigcup_{T\in\mathcal{K}} T = S$;
- *if* $T \in \mathcal{K}$, *then also all its subsimplices are in* $\mathcal{K}$;
- *if* $T_1, T_2 \in \mathcal{K}$, *then* $T_1 \cap T_2$ *is either empty or a subsimplex of both* T_1 *and* T_2.

We denote $\mathrm{ve}(\mathcal{K}) = \bigcup_{T\in\mathcal{K}} \mathrm{ve}(T)$ *and call two vertices* $v_1, v_2 \in \mathrm{ve}(\mathcal{K})$ *adjacent if there exists a simplex* $T \in \mathcal{K}$, *such that* $v_1, v_2 \in \mathrm{ve}(T)$.

For a vertex $v \in \mathrm{ve}(\mathcal{K})$ *we denote by* $\mathrm{ad}(v) = \{w \in \mathrm{ve}(\mathcal{K}) \setminus \{v\} \mid w \text{ is adjacent to } v\}$ *the set of adjacent vertices and for* $X \subseteq \mathrm{ve}(\mathcal{K})$

$$\mathrm{ad}(X) = \bigcup_{x\in X} \mathrm{ad}(x).$$

Note that $\mathrm{ad}(v)$ does not include the vertex v itself, while $\mathrm{ad}(X)$ could include a vertex v of the set X if X contains an adjacent vertex to v.

Instead of a simplicial complex, containing p-simplices with $0 \le p \le n$, we can just consider the n-simplices. This is called a triangulation and there is a one-to-one correspondence between a simplicial complex and a triangulation.

Note that we will often refer to p-simplices to stress the fact that we consider any $p \in \{0, \dots, n\}$ in contrast to n-simplices.

Definition 2.2 (triangulation). *A triangulation* $\mathcal{T}$ *of* $S = \cup_{T\in\mathcal{T}} T$ *is a set of* n-*simplices such that the intersection of two different simplices is either empty or a* p-*simplex,* $0 \le p < n$, *and its vertices are the common vertices of the two different* n-*simplices.*

Given a simplicial complex $\mathcal{K}$ of S, the corresponding triangulation $\mathcal{T}$ of S is given by the collection of all n-simplices in $\mathcal{K}$. Conversely, given a triangulation $\mathcal{T}$ of S, the corresponding simplicial complex $\mathcal{K}$ of S is given by the collection of all subsimplices of simplices in $\mathcal{T}$; see [1].

One example of a triangulation $\mathcal{T}$ of $\mathbb{R}^n$ is the standard triangulation defined below, where S_n denotes the set of all permutations of the numbers $1, 2, \dots, n$, $\mathcal{X}_{\mathcal{J}(i)}$ denotes the characteristic function equal to one if $i \in \mathcal{J}$ and equal to zero if $i \notin \mathcal{J}$, and $e_1, e_2, \dots, e_n$ denotes the standard orthonormal basis of $\mathbb{R}^n$. Further, we use functions $R^{\mathcal{J}} : \mathbb{R}^n \to \mathbb{R}^n$, defined for every $\mathcal{J} \subseteq \{1, 2, \dots, n\}$ by

$$R^{\mathcal{J}}(x) := \sum_{i=1}^{n} (-1)^{\mathcal{X}_{\mathcal{J}(i)}} x_i e_i.$$

Thus $R^{\mathcal{J}}$ puts a minus in front of the coordinate of x_i of x if $i \in \mathcal{J}$.

Definition 2.3 (standard triangulation). *The standard triangulation* $\mathcal{T}$ *consists of the simplices*

$$T_{z,n,\sigma} = \mathrm{co}\left\{ R^{\mathcal{J}}\left(z + \sum_{i=1}^{j} e_{\sigma(i)} \right) \mid j = 0, 1, 2, \dots, n \right\}$$

for all $z \in \mathbb{N}_0^n$, *all* $\mathcal{J} \subseteq \{1, 2, \dots, n\}$, *and all* $\sigma \in S_n$. *The corresponding simplicial complex consists of all elements of* $\mathcal{T}$ *and all their subsimplices.*

Before we define a CPA function on a triangulation, we show that the boundary of S consists of simplices with all vertices contained in the boundary. For the proof of Lemma 2.4, see Appendix A.

Lemma 2.4. *Let $\mathcal{T}$ be a triangulation of $S = \bigcup_{T\in\mathcal{T}} T \subseteq \mathbb{R}^n$, let $\mathcal{K}$ be the corresponding simplicial complex, and assume that $\mathcal{T}$ is locally finite, i.e., for every compact set $C \subseteq \mathbb{R}^n$ the cardinality of the set $\{T \in \mathcal{T} \mid T \cap C \neq \emptyset\}$ is finite.*

Then the boundary ∂S consists of p-simplices $T \in \mathcal{K}$ such that for all vertices $v \in \mathrm{ve}(T)$ we have $v \in \partial S$.

Now we can define a CPA function on a general triangulation $\mathcal{T}$; see [8] for more details.

Definition 2.5 (CPA function). *Let $\mathcal{T}$ be a finite triangulation of $S \subseteq \mathbb{R}^n$. Then a function $V : S \to \mathbb{R}$ is said to be a CPA function on the triangulation $\mathcal{T}$, written $V \in \mathrm{CPA}[\mathcal{T}]$, if V is a continuous function, which is affine on each simplex $T \in \mathcal{T}$. Since V is affine on each $T \in \mathcal{T}$, there are a vector $n_T \in \mathbb{R}^n$ and a number $a_T \in \mathbb{R}$ such that*

$$V(x) = n_T \cdot x + a_T \text{ for all } x \in T.$$

Furthermore, the function V is uniquely defined by the values $V(v)$ for all vertices $v \in \mathrm{ve}(\mathcal{T})$.

Consider a simplex $T \in \mathcal{T}$ with $T = \mathrm{co}\{v_0, \dots, v_n\}$. Then $\nabla V\big|_T := n_T = X_T^{-1} v_T$, where the matrix $X_T \in \mathbb{R}^{n\times n}$ is defined by writing the components of the vector $(v_i - v_0)^T$ in its ith row and the ith element of the column vector v_T is defined by $V(v_i) - V(v_0)$.

Consider the autonomous ODE $\dot{x} = f(x)$ with $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and denote the solution $x(t)$ with initial value $x(0) = \xi$ by $S_t\xi := x(t)$ for all $t \geq 0$ for which it exists. Furthermore, assume without loss of generality that the equilibrium under consideration is at 0.

Note that the property that a function V is strictly decreasing along solutions can be expressed by the orbital derivative if V is sufficiently smooth. This does not hold for a CPA function, which is not differentiable, but, using the Dini derivative, one can define a weaker notion of the orbital derivative. To show that the Dini orbital derivative is negative, it is sufficient to show that the usual orbital derivative, defined on each simplex T by $\dot{V}(x) = \nabla V\big|_T \cdot f(x)$, is negative. Note that $\nabla V\big|_T$, as defined above, is constant on each simplex. By Taylor-type estimates, using a bound on the second derivatives of f, the requirement that $\dot{V}(x)$ is negative for all $x \in T$ and all simplices $T \in \mathcal{T}$ can be written as linear constraints on the values $V(v)$ at the vertices $v \in \mathrm{ve}(\mathcal{T})$.

If values $V(v)$ can be found such that all the constraints are fulfilled, then V is a strict CPA Lyapunov function. On the other hand, these linear constraints are feasible if the equilibrium at the origin is exponentially stable and the triangulation is fine enough [8].

A CPA Lyapunov function can either be determined by solving a suitable linear programming problem, or by computing the values at the vertices by other methods and then checking that the constraints hold.

We will in the following assume that we are given a strict CPA Lyapunov function $V \in \mathrm{CPA}[\mathcal{T}]$ in the following sense.

Definition 2.6 (strict CPA Lyapunov function). *Consider the autonomous ODE $\dot{x} = f(x)$, where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, with equilibrium at 0.*

The CPA function $V \in CPA[\mathcal{T}]$ (see Definition 2.5) is called a strict CPA Lyapunov function if $V(0) = 0$, $V(x) > 0$ holds for all $x \in S \setminus \{0\}$ and, moreover, for each simplex $T \in \mathcal{T}$ we have

$$\dot{V}(x) = \nabla V\big|_T \cdot f(x) < 0 \text{ for all } x \in T \setminus \{0\}.$$

In particular, a strict CPA Lyapunov function satisfies that if $x \in S \setminus \{0\}$ and if $t > 0$ is such that $S_\tau x \in S$ for all $\tau \in [0,t]$, then $V(S_t x) < V(x)$. In other words, V is strictly decreasing along positive orbits in $S \setminus \{0\}$.

We will now define sublevel sets of V which will be subsets of the basin of attraction of 0; see Theorem 2.8. Further, such a sublevel set is forward invariant. Note that S° denotes the interior of the set S.

Definition 2.7 (level sets). *Consider the autonomous ODE $\dot{x} = f(x)$ with $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and denote the solution $x(t)$ with initial value $x(0) = \xi$ by $S_t\xi := x(t)$. Furthermore, assume without loss of generality that the equilibrium under consideration is at 0.*

Assume that $\mathcal{T}$ is a finite triangulation of the compact set $S \subseteq \mathbb{R}^n$, S° is a neighborhood of the origin, $\overline{S^\circ} = S$, and we have for all simplices $T \in \mathcal{T}$ that $0 \in T$ implies $0 \in \mathrm{ve}(T)$.

Let $V \in \mathrm{CPA}[\mathcal{T}]$ be a strict CPA Lyapunov function (see Definition 2.6), and let $m \in \mathbb{R}$ be a constant. Define the set

$$\mathcal{O}_{V,m} := \{0\} \cup \{x \in S \mid V(x) < m\} \subseteq S.$$

Denote by $\mathcal{O}_{V,m,0}$ the connected component of $\mathcal{O}_{V,m}$ satisfying $0 \in \mathcal{O}_{V,m,0} \subseteq \mathcal{O}_{V,m}$. If $\{0\} \subseteq \mathcal{O}^\circ_{V,m,0} \subseteq \overline{\mathcal{O}_{V,m,0}} \subseteq S^\circ$, then we define the sublevel set $\mathcal{L}_{V,m} := \mathcal{O}_{V,m,0}$. If no such $\mathcal{O}_{V,m,0}$ exists, then we define $\mathcal{L}_{V,m} := \emptyset$. We can further define

$$\mathcal{L}_V^{\sup} := \bigcup_{m \in \mathbb{R}} \mathcal{L}_{V,m}.$$

Theorem 2.8. *Let S and V be as in Definition 2.7. Then $\mathcal{L}_V^{\sup}$ is a subset of the basin of attraction of 0. Note that with $b := \sup\{m \in \mathbb{R} \mid \mathcal{L}_{V,m} \neq \emptyset\}$ we have that $\mathcal{L}_V^{\sup}$ is the connected component of $\{x \in S \colon V(x) < b\}$ that contains the origin.*

For a proof of this theorem (cf. [3, Thm. 2.5] and [9, Thm. 2.6]), note that the proof works also in the case that S° is not simply connected.

The goal of this paper is, given a strict CPA Lyapunov function V, to algorithmically determine the set $\mathcal{L}_V^{\sup}$ and thus a subset of the basin of attraction of 0.

3. Algorithm.

In this section we will present an algorithm to determine the set $\mathcal{L}_V^{\sup}$ in Definition 2.7 for a strict CPA Lyapunov function V.

After introducing the algorithm, which is based on coloring the vertices of the triangulation, we will prove the relation of the different colored simplices with $\mathcal{O}_{V,m,0}$, the connected component of the sublevel set of level m, which contains the origin, in Theorem 3.10. Then we will prove the relation between the colored simplices and $\mathcal{L}_V^{\sup}$ in Theorem 3.13.

In the following we will assume that 0 is an equilibrium, and $S \subseteq \mathbb{R}^n$ is a compact set such that S° is a neighborhood of 0 and $\overline{S^\circ} = S$. Furthermore, $\mathcal{T}$ is a finite triangulation of S such that if 0 is in a simplex, then 0 is a vertex, and $V \in \mathrm{CPA}[\mathcal{T}]$ is a strict CPA Lyapunov function; see Definition 2.7.

3.1. The algorithm.

Let us first give some definitions.

Definition 3.1. *Denote by $\mathrm{col}_k(v) \in \{g, y, r\}$, the color of vertex $v \in \mathrm{ve}(\mathcal{T})$ in step $k \in \mathbb{N}_0$. Denote by $G_k = \{v \in \mathrm{ve}(\mathcal{T}) \mid \mathrm{col}_k(v) = g\}$, $Y_k = \{v \in \mathrm{ve}(\mathcal{T}) \mid \mathrm{col}_k(v) = y\}$, and $R_k = \{v \in \mathrm{ve}(\mathcal{T}) \mid \mathrm{col}_k(v) = r\}$ the set of green, yellow, and red vertices in step k, respectively.*

The algorithm will define colorings of all vertices $\mathrm{col}_k(v)$, a set of vertices $X_k \subseteq \mathrm{ve}(\mathcal{T})$, marked to be turned green in step $k+1$, as well their V-value m_k in each step $k = 0, 1, \ldots$.

To start the algorithm in step 0, set $X_0 = \{0\}$, $\mathrm{col}_0(0) = y$ and $\mathrm{col}_0(v) = r$ for all $v \in \mathrm{ve}(\mathcal{T}) \setminus \{0\}$ as well as $m_0 = V(0) = 0$.

In step k, where $k = 1, 2, \ldots$,

(i) set $\mathrm{col}_k(v) = \mathrm{col}_{k-1}(v)$ for all $v \in \mathrm{ve}(\mathcal{T})$;
set $\mathrm{col}_k(x) = g$ for all $x \in X_{k-1}$;
set $\mathrm{col}_k(v) = y$ for all $v \in \mathrm{ad}(X_{k-1}) \cap R_{k-1}$;

(ii) set $m_k := \min_{v \in Y_k} V(v)$ as well as $X_k := \{v \in Y_k \mid V(v) = m_k\}$.
Then we have two options.
  (a) If $B := X_k \cap \partial S \neq \emptyset$, then set $M := k$ and terminate the algorithm.
  (b) Otherwise, set k to $k+1$ and repeat.

Note that in (i) we keep the coloring from the previous step and turn all vertices in X_{k-1} (from yellow to) green. Furthermore, we turn all red vertices adjacent to any vertex in X_{k-1} yellow. (If adjacent vertices are already green or yellow, then they remain unchanged.)

In (ii) we determine the minimal V-value m_k among all yellow vertices, and collect the yellow vertices, where the minimal V-value is attained, in the set X_k, i.e., these yellow vertices are marked to be turned green in the next step. Finally, we terminate the algorithm if at least one of the vertices in X_k is at the boundary of S; otherwise, we go to the next step.

For (ii) we need to show that $Y_k \neq \emptyset$. Note that $0 \in G_k$ for all $k \geq 1$, as 0 is turned green in step 1 (i) and then stays green throughout the algorithm. Hence, the alternatives to having a yellow vertex are that (A) all vertices are green and (B) all vertices are green or red (and there are both green and red vertices). Case (A) is not possible since by the termination criterion (iia) boundary points cannot become green. Case (B) is not possible since by Lemma 3.8 (see later) if there are green and red vertices, then there also must be yellow vertices.

Note that the algorithm will always terminate in a finite number of steps, since the set of vertices $\mathrm{ve}(\mathcal{T})$ is finite and in each step of the algorithm vertices change color from red to yellow and at least one from yellow to green, never in the opposite direction.

In Lemma 3.5 we want to show that the sequence m_k is strictly increasing. For the proof we need to show that $V(v) > m_k$ for all $v \in \mathrm{ad}(X_k) \cap R_k$. We will show this later in Lemma C.4, but for the moment we will define N as the first step, where this is violated; see the following definition.

Definition 3.2. *If there exists a step $k \in \{0, \ldots, M\}$ such that there exists $v \in \mathrm{ad}(X_k) \cap R_k$ with $V(v) \leq m_k$, then define $N \in \{0, \ldots, M\}$ to be the minimal step $N = k$ with this property. If no such step exists, then define $N := M$.*

Note that $N \neq 0$ since $V(x) > 0 = m_0$ for all $x \in S \setminus \{0\}$, so in particular for all $x \in R_0$.

From now on we will only consider steps k up to N, but we will later show that $N = M$ in Lemma C.4. We begin by introducing the following notation to label simplices according to the colors of their vertices.

Definition 3.3. *Consider step $k \in \{1, \ldots, M\}$. We denote by $\mathcal{S}_{n_g n_y n_r}$ the set of simplices $T \in \mathcal{K}$ such that T has exactly n_g green, n_y yellow, and n_r red vertices, where $n_g, n_y, n_r \in \mathbb{N}_0$. Furthermore, we denote $\mathcal{S}_{gy0} = \bigcup_{n_g, n_y \geq 1} \mathcal{S}_{n_g n_y 0}$ etc. If we want to note the step k to which* *the color corresponds, then we write* $\mathcal{S}^{(k)}_{n_g n_y n_r}$ *and* $\mathcal{S}^{(k)}_{gy0}$.

For example, $\mathcal{S}_{120}$ represents the set of 2-simplices with exactly one green and two yellow vertices, and $\mathcal{S}_{gy0}$ denotes all simplices with at least one green and one yellow vertex, but no red vertex.

We start with the observation that every yellow vertex has an adjacent green vertex.

Lemma 3.4 (yellow vertices have adjacent green vertex). *Consider step* $k \in \{1, \ldots, M\}$. *If* $v \in Y_k$, *then* $\mathrm{ad}(v) \cap G_k \neq \emptyset$.

Proof. Consider step $k \in \{1, \ldots, M\}$ and let $1 \leq l \leq k$ be the step when the yellow vertex was turned from red to yellow (note that 0 is green from step 1 onwards). In step l (i) there is an adjacent green vertex to the yellow vertex, and the green vertex stays green until step k. ∎

Lemma 3.5 (m_k is strictly increasing). *We have* $m_k > m_{k-1}$ *for all* $k \in \{1, \ldots, N\}$. *In particular, if we consider a level* $m \in (0, m_N]$, *then there is a unique* $k \in \{1, \ldots, N\}$ *such that* $m_{k-1} < m \leq m_k$.

Proof. If $k = 1$, then $m_0 = V(0) = 0$ and $V(x) > 0$ for all other vertices, so $m_1 = \min_{x \in Y_1} V(x) > m_0$, as there are finitely many yellow vertices.

Now consider step $k \geq 2$. By (i) of the algorithm we have $Y_k = (Y_{k-1} \setminus X_{k-1}) \cup (\mathrm{ad}(X_{k-1}) \cap R_{k-1})$. If $y \in Y_{k-1} \setminus X_{k-1}$, then $V(y) > m_{k-1} = \min_{\tilde{y} \in Y_{k-1}} V(\tilde{y})$ by definition of X_{k-1}. If $y \in \mathrm{ad}(X_{k-1}) \cap R_{k-1}$, then $V(y) > m_{k-1}$ since $k - 1 < N$ (see Definition 3.2 of N). Hence, taking the minimum over the finitely many elements in Y_k we obtain $m_k = \min_{y \in Y_k} V(y) > m_{k-1}$. ∎

Lemma 3.6 (yellow simplices have higher V-value than m_k). *Let* $y \in T$ *with* $T \in \mathcal{S}^{(k)}_{0y0}$ *at step* $k \in \{0, \ldots, M\}$. *Then* $V(y) \geq m_k$.

Proof. For $k = 0$ the statement is true as the only point in $\mathcal{S}^{(0)}_{0y0}$ is 0 and $V(0) = 0 = m_0$. Now let $k \geq 1$ and first let $y \in \mathcal{S}^{(k)}_{010} = Y_k$ be a (yellow) vertex of T. Then we have $m_k = \min_{\tilde{y} \in \mathcal{S}^{(k)}_{010}} V(\tilde{y}) \leq V(y)$.

Now let $y \in T = \mathrm{co}\{y_0, \ldots, y_p\}$ with $T \in \mathcal{S}^{(k)}_{0y0}$. Since V is affine on T and $V(y_i) \geq m_k$ for all yellow vertices as shown above, we have $V(y) = \sum_{i=0}^{p} \lambda_i V(y_i) \geq \sum_{i=0}^{p} \lambda_i m_k = m_k$, where $y = \sum_{i=0}^{p} \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=0}^{p} \lambda_i = 1$. ∎

In the next lemma we will show that we can never have a green and a red vertex adjacent to each other.

Lemma 3.7 (green and red vertices are not adjacent). *At every step* $k \in \{0, \ldots, M\}$, *if* $T \in \mathcal{K}$ *is a* p*-simplex, then there cannot be two vertices* $v_i, v_j \in \mathrm{ve}(T)$ *such that* v_i *is green and* v_j *is red.*

Proof. First note that the algorithm only allows for a vertex to change color from yellow to green or from red to yellow.

Note that in step 0, there is no green vertex. Assume in contrast to the statement that at step $k \in \{1, \ldots, M\}$, $\mathrm{col}_k(v) = g$, $\mathrm{col}_k(w) = r$, and $w \in \mathrm{ad}(v)$. As neither a green nor a yellow vertex can change to red in the algorithm, we conclude that

$$\mathrm{col}_e(w) = r \text{ for all } 0 \leq e \leq k.$$

Since the algorithm starts with no green vertex, and they can only be introduced by turning yellow ones to green, there exists an $i \in \{1, 2, \ldots, k\}$ such that $\mathrm{col}_{i-1}(v) = y$, $\mathrm{col}_i(v) = g$. At step i we would have turned all adjacent vertices yellow if they were red (see (i)), i.e., $\mathrm{col}_i(w) = y$. This is a contradiction, so we cannot have a simplex with both a green and a red vertex. ∎

Now we show that in a given step k any continuous path between a point of an entirely red or a red and yellow simplex and a point of an entirely green simplex must cross an entirely yellow simplex. In particular, if there are green and red vertices, then there must also be a yellow vertex.

Lemma 3.8 (any continuous path from a point in $\mathcal{S}_{g00}$ to $\mathcal{S}_{00r}$ or $\mathcal{S}_{0yr}$ must cross $\mathcal{S}_{0y0}$). *Consider step $k \in \{1, \ldots, M\}$. Let $l\colon [0,1] \to S$ be a continuous function with $l(0) \in T_0$ and $l(1) \in T_e$, where $T_0 \in \mathcal{S}_{g00}$ and $T_e \in \mathcal{S}_{00r} \cup \mathcal{S}_{0yr}$.*

Then there exists $\theta^* \in (0,1)$ such that $l(\theta^*) \in T'$ with $T' \in \mathcal{S}_{0y0}$.

Proof. Assume there exists a continuous path $l : [0,1] \to S$ as above and denote $l(0) =: x$ and $l(1) =: y$.

Using Lemma 3.7 it is immediate that the nonempty sets of colored simplices are

$$\mathcal{S}_{g00}, \mathcal{S}_{gy0}, \mathcal{S}_{0y0}, \mathcal{S}_{0yr}, \mathcal{S}_{00r}$$

since all other sets contain simplices with both green and red vertices.

Consider the finite sequence $(T_i)_{i=0}^m$ of p_i-simplices traversed by l, where each p_i is minimal. This means that if for instance $l(\theta) = v$, where v is a vertex, then we take the 0-simplex equal to v as an element of the sequence rather than any of the j-simplices, $1 \leq j \leq n$ with vertex v.

For the continuous path to go from one simplex to another, the simplices T_i and T_{i+1} must share at least one vertex. This means in particular they must have at least one vertex of the same color. Because of this the options for traversing simplices are

$$\begin{tikzcd} \mathcal{S}_{g00} \arrow[r, leftrightarrow] & \mathcal{S}_{gy0} \arrow[r, leftrightarrow] \arrow[rr, leftrightarrow, bend right] & \mathcal{S}_{0y0} \arrow[r, leftrightarrow] & \mathcal{S}_{0yr} \arrow[r, leftrightarrow] & \mathcal{S}_{00r}. \end{tikzcd}$$

Let us assume that there is no simplex in $\mathcal{S}_{0y0}$ in the sequence; then, as $x = l(0) \in T_0$ and $y = l(1) \in T_e$ with $T_0 \in \mathcal{S}_{g00}$ and $T_e \in \mathcal{S}_{0yr} \cup \mathcal{S}_{00r}$, at some point l traverses from $\mathcal{S}_{gy0}$ to $\mathcal{S}_{0yr}$. Denote these simplices by T and T'. In particular, there is a point $s := l(\theta^*) \in T \cap T'$. Note that $T \cap T'$ is again a simplex and $\mathrm{ve}(T \cap T') = \mathrm{ve}(T) \cap \mathrm{ve}(T')$. Since any vertex of $T \cap T'$ can by Lemma 3.7 be neither green, since $T' \in \mathcal{S}_{0yr}$, nor red, since $T \in \mathcal{S}_{gy0}$, all vertices of $T \cap T'$ are yellow, i.e., $T \cap T' \in \mathcal{S}_{0y0}$ in contraction to the assumption. ∎

The next lemma considers the situation described in Definition 3.2 and is used in Lemma C.4.

Lemma 3.9. *Let $x \in S$ and T be the p-simplex with $x \in T$ and minimal p. Let $k \in \{1,\ldots,N\}$, let $T \in \mathcal{S}_{0yr}^{(k)} \cup \mathcal{S}_{00r}^{(k)}$, and assume $V(x) \leq m_k$. Then either $x \notin A(0)$ or the positive orbit through x leaves S.*

Proof. Let x be as in the lemma. In contrast to the statement we assume that x is in the basin of attraction of 0 and the positive orbit through x does not leave S. In particular, $S_t x$ is defined for all $t \geq 0$.

Since x is in the basin of attraction of 0 there exists continuous path $\alpha : [0,1] \to \mathbb{R}^n$ defined by $\alpha(s) = S_{\exp(\frac{s}{1-s})-1}x$ for $s \in [0,1)$ and $\alpha(1) = \lim_{s\to 1} S_{\exp(\frac{s}{1-s})-1}x = \lim_{t\to\infty} S_t x = 0$, which follows the positive orbit. We have $\alpha(0) = x$, $\alpha(1) = 0$, and $\alpha([0,1]) \subseteq S$ by assumption. Since $x \neq 0$ because 0 is green, the function $s \mapsto V(\alpha(s))$, $s \in [0,1)$, is strictly decreasing as s increases, because V is a strict CPA Lyapunov function. Further, $\{0\}$ is in $\mathcal{S}_{g00}$. By Lemma 3.8 there exists a $\theta^* \in (0,1)$ with $x^* := \alpha(\theta^*) \in T$ with $T \in \mathcal{S}_{0y0}$.

By Lemma 3.6 we have

$$V(\alpha(\theta^*)) = V(x^*) \geq m_k \geq V(x) = V(\alpha(0)).$$

This is a contradiction to the fact that $V(\alpha(\cdot))$ is strictly decreasing. ■

3.2. Colored simplices and sublevel sets. We now introduce the main theorem of this section, which characterizes the set $\mathcal{O}_{V,m,0}$ (see Definition 2.7) by the colored simplices. More precisely, all simplices in $\mathcal{S}_{g00}$ lie inside $\mathcal{O}_{V,m,0}$ as well as parts of simplices in $\mathcal{S}_{gy0}$, while all other simplices are disjoint from $\mathcal{O}_{V,m,0}$.

Theorem 3.10 (colored simplices and sublevel sets). *In step $k \in \{1,\ldots,N\}$ in the algorithm we have with $m_{k-1} < m \leq m_k$ and $\mathcal{O}_{V,m,0}$, as in Definition 2.7,*

1. $\mathcal{S}_{00r}^{(k)} \cap \mathcal{O}_{V,m,0} = \emptyset$,
2. $\mathcal{S}_{0y0}^{(k)} \cap \mathcal{O}_{V,m,0} = \emptyset$,
3. $\mathcal{S}_{0yr}^{(k)} \cap \mathcal{O}_{V,m,0} = \emptyset$,
4. $\mathcal{S}_{g00}^{(k)} \subseteq \mathcal{O}_{V,m,0}$,
5. *If $T \in \mathcal{S}_{gy0}^{(k)}$ is an n-simplex, then there exist $y_T \in T$ and $n_T \in \mathbb{R}^n$ such that*

$$T \cap \mathcal{O}_{V,m,0} = T \cap \{x \in \mathbb{R}^n \mid (x - y_T)^T n_T < 0\} = T \cap \{x \in \mathbb{R}^n \mid V(x) < m\}.$$

Furthermore, $T \cap \mathcal{O}_{V,m,0} \neq \emptyset$ and $T \setminus \mathcal{O}_{V,m,0} \neq \emptyset$.

The proof of the theorem is given in Appendix B. Let us now state a consequence of Theorem 3.10 in the following corollary, linking P_m (referring to the coloring, defined below) to $\mathcal{O}_{V,m,0}$.

Definition 3.11. *For $0 < m \leq m_N$ there is a unique $k \in \{1,\ldots,N\}$ such that $m_{k-1} < m \leq m_k$ (see Lemma 3.5). We define*

$$P_m := \bigcup_{T \in \mathcal{S}_{g00}^{(k)},\ T \text{ is } n\text{-simplex}} T \cup \bigcup_{T \in \mathcal{S}_{gy0}^{(k)},\ T \text{ is } n\text{-simplex}} \{x \in T \mid V(x) < m\}.$$

Corollary 3.12. *Let $m \in (0, m_N]$. Then $P_m = \mathcal{O}_{V,m,0}$. Let $k \in \{1,\ldots,N\}$. Then $X_k \subseteq \overline{\mathcal{O}_{V,m_k,0}}$.*

Proof. Let $x \in S$; then there is an n-simplex $T \in \mathcal{T}$ such that $x \in T$. We will show $x \in P_m$ if and only if $x \in \mathcal{O}_{V,m,0}$ and distinguish between the different colorings of T.

If $T \in \mathcal{S}^{(k)}_{0y0} \cup \mathcal{S}^{(k)}_{0yr} \cup \mathcal{S}^{(k)}_{00r}$, then $x \notin P_m$ by definition and $x \notin \mathcal{O}_{V,m,0}$ by Theorem 3.10.

If $T \in \mathcal{S}^{(k)}_{g00}$, then $x \in P_m$ by definition and $x \in \mathcal{O}_{V,m,0}$ by Theorem 3.10.

If $T \in \mathcal{S}^{(k)}_{gy0}$, then we have by definition and Theorem 3.10 for $x \in T$:

$$x \in P_m \iff V(x) < m \iff x \in \mathcal{O}_{V,m,0}.$$

For the last statement, fix $k \in \{1, \dots, N\}$ and $x \in X_k$. Note that $x \in Y_k$. By Lemma 3.4 there is an adjacent green vertex $g \in X_l$ with $l < k$, and hence $m_l < m_k$. The sequence $y_p = \frac{1}{p}g + \left(1 - \frac{1}{p}\right)x \in T$ for all $p \in \mathbb{N}$ satisfies $y_p \to x$ as $p \to \infty$ and

$$V(y_p) = \frac{1}{p}V(g) + \left(1 - \frac{1}{p}\right)V(x) < m_k,$$

and $y_p \in \mathcal{O}_{V,m_k,0}$ by Theorem 3.10. Thus, $x \in \overline{\mathcal{O}_{V,m_k,0}}$. ■

3.3. Colored simplices and maximal sublevel set. We now proceed to the main theorem of this paper, linking the colored simplices to the maximal sublevel set $\mathcal{L}_V^{\sup}$.

Theorem 3.13 ($P_{m_M} = \mathcal{L}_V^{\sup}$). *Suppose that we have a finite triangulation $\mathcal{T}$ defined as before, that is, $S = \bigcup_{T\in\mathcal{T}} T \subseteq \mathbb{R}^n$ is compact and connected, $0 \in S^\circ$, and $0 \in T$ implies $0 \in \mathrm{ve}(T)$. Suppose further that V is a strict CPA Lyapunov function.*

Then for all $m_1 \leq m < m_N$,

$$P_m = \mathcal{L}_{V,m} \neq \emptyset; \tag{3.1}$$

see Definitions 2.7 and 3.11, i.e., P_m is the connected component of $\mathcal{O}_{V,m}$ which includes 0 and $0 \in P_m^\circ \subseteq \overline{P_m} \subseteq S^\circ$.

Moreover, we have $N = M$ and

$$P_{m_M} = \mathcal{L}_V^{\sup}.$$

Note that we require $N \geq 2$. If $N = 1$, then we need to define a new, finer triangulation and start the process again. The proof of the theorem is in Appendix C.

4. Examples. In this section we give three examples of our algorithm in action. The first two systems are two-dimensional and the third one is three-dimensional.

4.1. Example 1. The first example is the two-dimensional system

$$\left\{ \begin{array}{rcl} \dot{x} &=& -x(x+0.5), \\ \dot{y} &=& x - y(x+0.5). \end{array} \right. \tag{4.1}$$

We chose this system because the CPA Lyapunov function, which was computed using the linear optimization from [8], delivers a Lyapunov function with nonconvex stretched level sets because of the separatrix at $x = -0.5$. We used the standard triangulation (cf., e.g., [9]) and then mapped the vertices in $\{(z_x, z_y) \in \mathbb{Z}^2 \mid -22 \leq z_x \leq 65,\ -300 \leq z_y \leq 65\}$ using $\mathbf{F}(x, y) = (a_x x, a_y y)^T$ with $a_x = a_y = 0.025$. The domain of the CPA Lyapunov

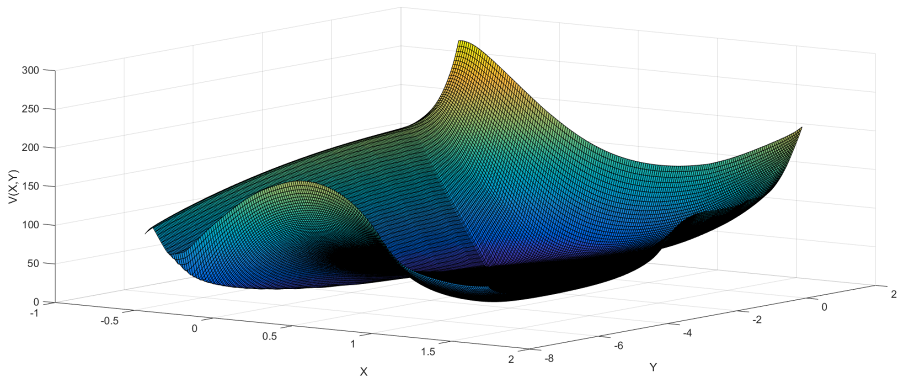

Figure 4. *CPA Lyapunov function computed for the system* (4.1) *using linear optimization.*

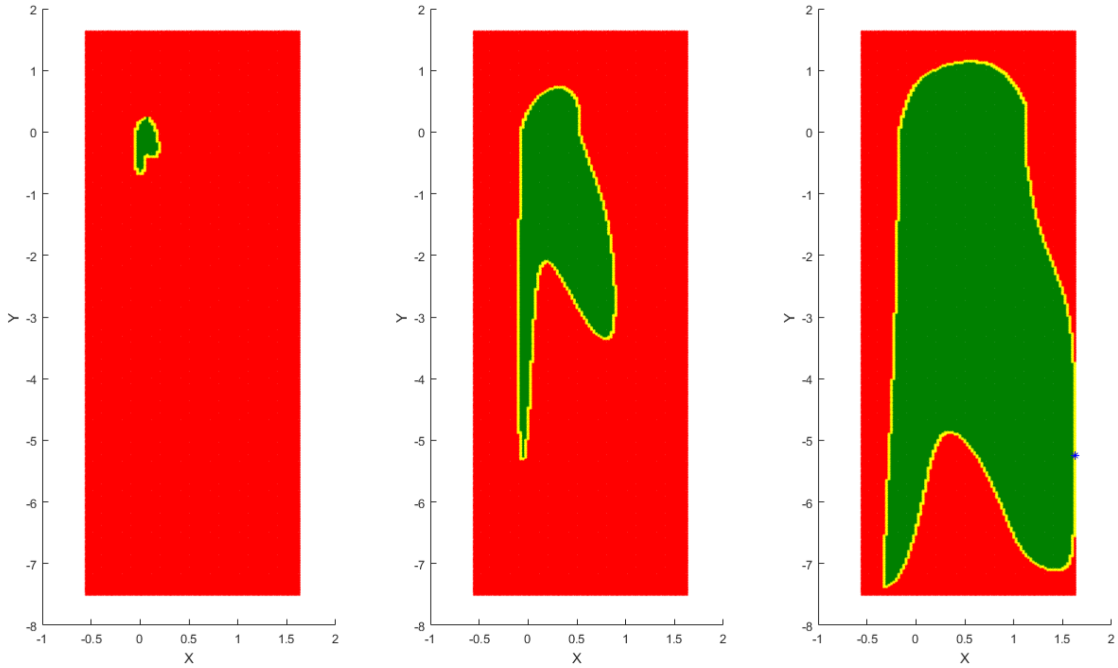

Figure 5. *Level sets for the CPA Lyapunov function computed for system* (4.1)*. The green, yellow, and red areas are as described in the algorithm, and the blue point is the point at the boundary that terminates the algorithm in the last figure. In particular, the green area is the connected component containing the equilibrium at the origin and is a lower bound on its basin of attraction.*

function is thus $[-0.55, 1.625]\times[-7.5, 1.625] = \mathbf{F}([-22, 65]\times[-300, 65])$. Further, in the linear programming problem we set the constants $B_\nu = 2$ for all simplices T_ν in the triangulation $\mathcal{T}$ (cf. [8]), and we minimized $\max\{|(\nabla V_\nu)_i| \mid T_\nu \in \mathcal{T}$ and $i = 1, \ldots, n\}$. The CPA Lyapunov function computed is depicted in Figure 4 and the level sets computed for the function with our algorithm are shown in Figure 5 and movie M126252_01.mp4 [local/web 904KB].

4.2. Example 2. The second example is a two-dimensional system taken from [6, Ex. 6], for which a CPA Lyapunov function that guarantees a much larger domain of attraction than previous approaches was computed in [12, Ex. 1]. The dynamics of the system are given by the ODE

$$\begin{cases} \dot{x} &= -x + y, \\ \dot{y} &= 0.1x - 2y - x^2 - 0.1x^3. \end{cases} \tag{4.2}$$

This example differs from Example 1 in two important aspects. First, its CPA Lyapunov function is computed by numerically integrating

$$V(\xi) = \int_0^{20} \frac{\|S_\tau \xi\|_2^2}{0.6 + \|S_\tau \xi\|_2^{1.2}} d\tau \tag{4.3}$$

for the vertices ξ of the triangulation with a subsequent verification of the linear constraints of a feasibility problem. The constraints will fail on a subset of the domain where we generate the CPA Lyapunov function, and this set is not considered to be in its domain in our algorithm. In particular, the domain of the CPA Lyapunov function is not a regular square as in Example 1. Second, the simplicial complex is much larger than in Example 1 and has $2001 \times 2001 = 4,004,001$ vertices, compared to $(22+65+1)\times(300+65+1) = 32,208$ in Example 1. For more detailed information on how the CPA Lyapunov function is computed we refer to [12, Ex. 1], where it is explained in detail. We used the standard triangulation as in Example 1 and then mapped the vertices in $\{(z_x, z_y) \in \mathbb{Z}^2 \mid -1333 \le z_x \le 667, \ -1000 \le z_y \le 1000\}$ using $\mathbf{F}(x, y) = (a_x x, a_y y)^T$ with $a_x = 0.015$ and $a_y = 0.04$. The domain of the CPA Lyapunov function is thus $\mathbf{F}([-1333, 667] \times [-1000, 1000]) \approx [-20, 10] \times [-40, 40]$. Further, we set the constants $B_{i,j}^\nu$ as in [12, Ex. 1]. The CPA Lyapunov function computed is depicted in Figure 6, and the level sets computed for the function with our algorithm are shown in Figure 7 and in the movie M126252_02.mp4 [local/web 451KB].

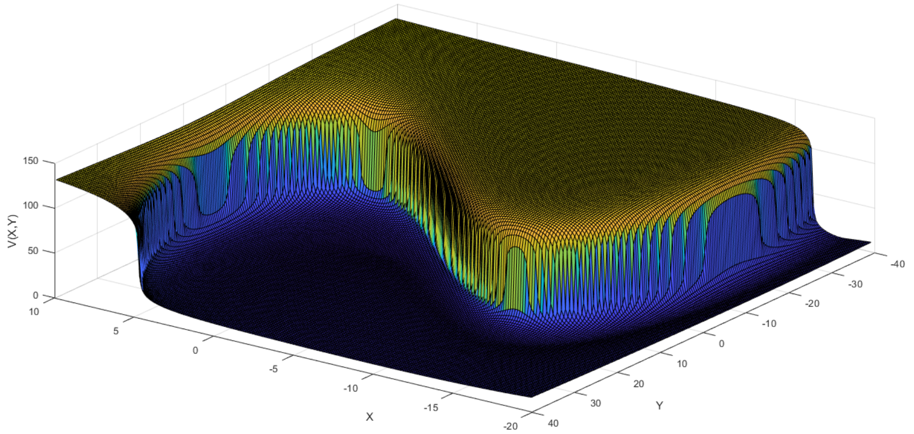

Figure 6. *CPA Lyapunov function computed for the system* (4.2) *using formula* (4.3).

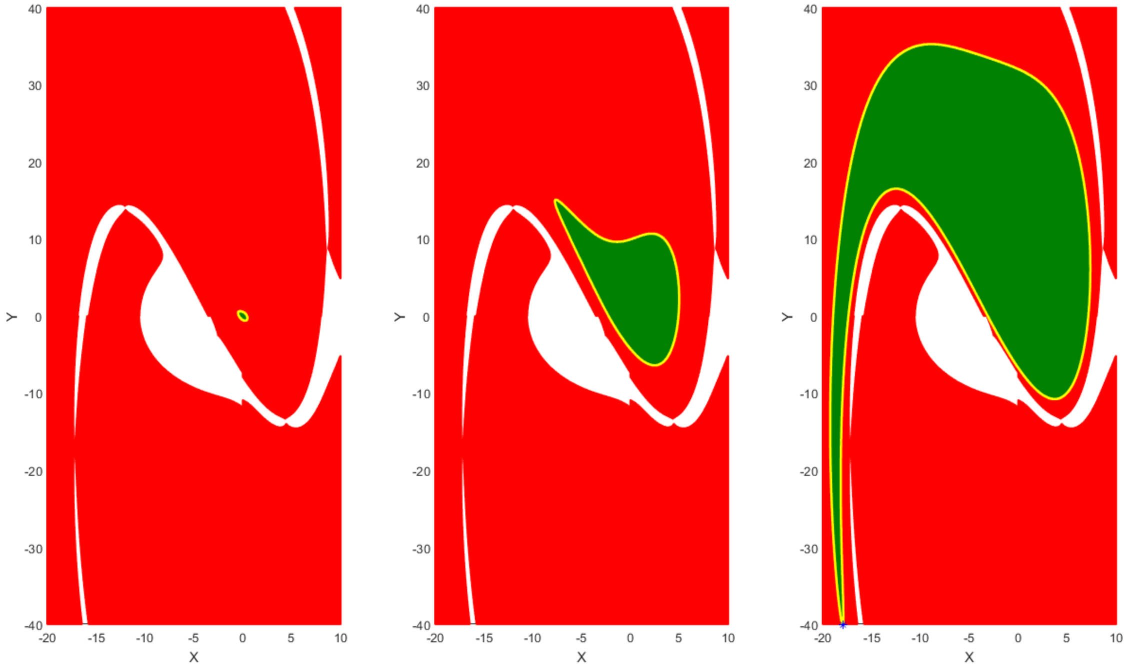

Figure 7. *Level sets for the CPA Lyapunov function computed for system (4.2). The green, yellow, and red areas are as described in the algorithm, and the blue point is the point at the boundary that terminates the algorithm in the last figure. In particular, the green area is the connected component containing the equilibrium at the origin and is a lower bound on its basin of attraction. Note that the CPA Lyapunov function does not fulfill the decrease condition in the white area and we do not consider it to be defined there in our algorithm.*

4.3. Example 3. The third and last example is the three-dimensional system given by the ODE

$$\begin{cases} \dot{x} &=& x - x^3 + y^2 + 0.5z, \\ \dot{y} &=& -x^2 - y - y^3 + 0.5z^2, \\ \dot{z} &=& x + x^2 + 2y - y^2 - z^3. \end{cases} \tag{4.4}$$

This system has two equilibria, which we computed numerically at

$$q_1 = (1.1097993202745274, -0.5236146236173852, -1.0625246308420705)^T \quad \text{and}$$
$$q_2 = (1.3621076986428951, -0.8290987269673093, 0.9553138697139557)^T.$$

We computed a CPA Lyapunov function for the system similarly as in Example 2, but for the equilibrium q_1 and using the formula

$$V(\xi) = \int_0^{20} \frac{\|S_\tau \xi - q_1\|_2^2}{0.2 + \|S_\tau \xi - q_1\|_2^{0.6}} d\tau. \tag{4.5}$$

This system is included because it is very difficult to determine by trial-and-error a subset of the domain of attraction from the level sets of the computed CPA Lyapunov function. Our algorithm, however, does it with ease. As before we use the standard triangulation and map the vertices $\{(z_x, z_y, z_z) \in \mathbb{Z}^3 \mid -80 \leq z_x \leq 10,\ -54 \leq z_y \leq 36,\ -80 \leq z_z \leq 10\}$ using $\mathbf{F}(x,y,z) = (a_x x, a_y y, a_z z)^T$ with $a_x = 0.1$ and $a_y = a_z = 1/9$. As in Example 2 the validity of the conditions for a Lyapunov function is checked by considering a feasibility problem and these conditions fail on a subset of the domain where we generated the problem. Again, this set is not considered to be in the domain of the CPA Lyapunov function in our algorithm. The domain of the generated function is thus $\mathbf{F}([-80,10] \times [-54,36] \times [-80,10]) \approx [-8,1] \times [-6,4] \times [-8.89, 0.111]$ and the domain of the CPA Lyapunov function is this cube minus the area where the conditions for a Lyapunov function are not fulfilled. In the verification we set the constants $B^{\nu}_{i,j}$, $i,j = 1,2,3$, as described in [12], equal to

$$B^{\nu}_{1,1} = \max_{(x,y,z)\in T_\nu} \max\{6x, 2\}, \quad B^{\nu}_{2,2} = \max_{(x,y,z)\in T_\nu} \max\{6y, 2\}, \quad B^{\nu}_{3,3} = \max_{(x,y,z)\in T_\nu} \max\{6z, 1\},$$

and zero otherwise. The level sets computed for the function with our algorithm are depicted in Figure 8 and the movie M126252_03.mp4 [local/web 7.70MB]. In the movie and the first two figures of Figure 8 we draw the green vertices computed by the algorithm. Then, to emphasize the form of the sublevel set computed, we draw it as a three-dimensional object and rotate it; see also the rightmost plot in Figure 8. The area where the Lyapunov function fails the decrease condition is shown in purple; the algorithm does not consider the Lyapunov function to be defined in this area.

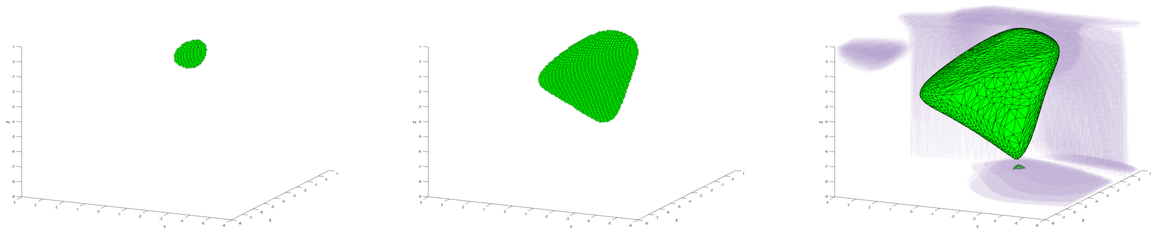

Figure 8. *Level sets for the CPA Lyapunov function computed for system (4.4). The green area is the connected component containing the equilibrium at the origin and is a lower bound on its basin of attraction. Note that the CPA Lyapunov function does not fulfill the decrease condition in the purple area in the rightmost figure and we do not consider it to be defined there in our algorithm.*

Appendix A. Proof of Lemma 2.4.

In this section we give the proof of Lemma 2.4.

Proof of Lemma 2.4. Since $\partial S \subseteq S$, for a point $x \in \partial S$ there exists by [1, Lemma 2.5] a unique p-simplex $T \in \mathcal{K}$ such that x is an inner point of T, i.e., $\mathrm{ve}(T) = \{v_0, \ldots, v_p\}$,

$$x = \sum_{i=0}^{p} \lambda_i v_i \quad \text{with} \quad \sum_{i=0}^{p} \lambda_i = 1, \quad \text{and} \quad \lambda_i > 0 \quad \text{for } i = 0, \ldots, p.$$

Let us assume in contrast to the statement that one of the vertices, say v_0, is in the interior of S, i.e., $v_0 \in S^\circ$. Note that $\lambda_0 \neq 1$, since otherwise $x = v_0$, which cannot hold since $x \in \partial S$

while $v_0 \in S^\circ$. Since $x \in \partial S$ there is a sequence of points $x_j \notin S$ with $\lim_{j\to\infty} x_j = x$. Let $d := 2\sup_{j\in\mathbb{N}} \|x_j - v_0\| < \infty$, where $\|\cdot\| = \|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^n$.

Since $v_0 \in S^\circ$, we have $B_\epsilon(v_0) \subseteq S$ for $\epsilon > 0$ which can be chosen such that

$$\epsilon < \frac{d}{1-\lambda_0}. \tag{A.1}$$

For each $j \in \mathbb{N}$ we now define the point y_j which lies on the straight line between v_0 and x_j by

$$y_j := v_0 + \frac{\epsilon}{d}(x_j - v_0) \in B_\epsilon(v_0) \subseteq S,$$

since by definition of d

$$\|y_j - v_0\| = \frac{\epsilon}{d}\|x_j - v_0\| \ < \ \epsilon.$$

Since there are finitely many simplices in $\mathcal{T}$ that intersect $B_\epsilon(v_0)$, there exists an n-simplex $T' \in \mathcal{T}$ such that a convergent subsequence (which we still call y_j) of the y_j lies in T'. For its limit we have

$$y := \lim_{j\to\infty} y_j \ = \ v_0 + \frac{\epsilon}{d}(x - v_0)$$

and we have $y \in T'$ since T' is closed.

Let us show that $y \in T$. Indeed, we can write

$$y = v_0\left(1 - \frac{\epsilon}{d}\right) + \frac{\epsilon}{d}x \ = \ v_0 \underbrace{\left(1 + (\lambda_0 - 1)\frac{\epsilon}{d}\right)}_{=:\mu_0} + \sum_{i=1}^{p} \underbrace{\frac{\epsilon}{d}\lambda_i}_{=:\mu_i} v_i$$

with $\sum_{i=0}^p \mu_i = 1$ (since $\sum_{i=0}^p \lambda_i = 1$) and $\mu_i > 0$ for all $i \geq 0$. Indeed, for $i \geq 1$ we have $\mu_i > 0$ since $\lambda_i > 0$. For $i = 0$ we have with (A.1) and $1 - \lambda_0 > 0$ that

$$\mu_0 = 1 + (\lambda_0 - 1)\frac{\epsilon}{d} \ > \ 0.$$

Since $\mathcal{T}$ is a triangulation, there is a unique way of writing y in this form (see [1, Lemma 2.5]), and thus $\mathrm{ve}(T) \subseteq \mathrm{ve}(T')$; i.e., we have $\mathrm{ve}(T') = \{v_0, \dots, v_p, v_{p+1} \dots, v_n\}$.

Now we show that $x_j \in T'$ if j is sufficiently large. We can write

$$y_j = \sum_{i=0}^{n} \mu_i^{(j)} v_i$$

with $\sum_{i=0}^n \mu_i^{(j)} = 1$ and $\mu_i^{(j)} \geq 0$. Note that $\lim_{j\to\infty} \mu_i^{(j)} = \mu_i$ for each fixed i, since $\lim_{j\to\infty} y_j = y$; here, we set $\mu_i = 0$ for $i > p$. We have

$$\begin{aligned} x_j &= \frac{d}{\epsilon}(y_j - v_0) + v_0 \\ &= v_0\left(1 - \frac{d}{\epsilon}\right) + \frac{d}{\epsilon}y_j \\ &= v_0 \underbrace{\left(1 + (\mu_0^{(j)} - 1)\frac{d}{\epsilon}\right)}_{=:\nu_0^{(j)}} + \sum_{i=1}^{n} \underbrace{\frac{d}{\epsilon}\mu_i^{(j)}}_{=:\nu_i^{(j)}} v_i. \end{aligned}$$

It is easy to see that $\sum_{i=0}^n \nu_i^{(j)} = 1$ since $\sum_{i=0}^n \mu_i^{(j)} = 1$. We have $\nu_i^{(j)} \geq 0$ for $i \geq 1$ as $\mu_i^{(j)} \geq 0$. Since $\lim_{j\to\infty} \mu_0^{(j)} = \mu_0$ there is a $J \in \mathbb{N}$ such that

$$|\mu_0^{(j)} - \mu_0| < \lambda_0 \frac{\epsilon}{d} \text{ for all } j \geq J.$$

Hence, we have, using $\mu_0 = 1 + \frac{\epsilon}{d}(\lambda_0 - 1)$,

$$\begin{aligned}
\nu_0^{(j)} &= 1 + (\mu_0^{(j)} - 1)\frac{d}{\epsilon} \\
&\geq 1 - \frac{d}{\epsilon} + \frac{d}{\epsilon}\left(\mu_0 - |\mu_0^{(j)} - \mu_0|\right) \\
&> 1 - \frac{d}{\epsilon} + \frac{d}{\epsilon}\left(1 + \frac{\epsilon}{d}(\lambda_0 - 1) - \lambda_0 \frac{\epsilon}{d}\right) \\
&\geq 1 - \frac{d}{\epsilon} + \frac{d}{\epsilon} - 1 \\
&= 0 \text{ for } j \geq J.
\end{aligned}$$

Thus, $x_j \in T' \subseteq S$ for all $j \geq J$, which is a contradiction to $x_j \notin S$. This proves the lemma. ∎

Appendix B. Proof of Theorem 3.10. We will break the proof of Theorem 3.10 into several parts, proving first statement 4 in Lemma B.1, statement 2 in Lemma B.2, and statements 1 and 3 in Lemma B.3. Finally, Lemma B.4 implies statement 5.

Let us begin by looking at entirely green simplices.

Lemma B.1 ($\mathcal{S}_{g00}$ inside). *At step $k \in \{1, \dots, N\}$ in the algorithm $\mathcal{S}_{g00}^{(k)} \subseteq \mathcal{O}_{V,m,0}$ holds for all $m_{k-1} < m \leq m_k$.*

Proof. Select $T \in \mathcal{S}_{g00}^{(k)}$ and $s \in T$. To show that s is an element of $\mathcal{O}_{V,m,0}$, we show that there exists a continuous path $l : [0,1] \to S$, $l(0) = 0$, and $l(1) = s$ with $V(l(\theta)) < m$ for all $\theta \in [0,1]$ by induction with respect to n_g, the number of green vertices of T. We start by looking at $\mathcal{S}_{200}$.

Let $g \in X_i$ be a green vertex that has been turned green in step $i+1$, so $i \in \{0, \dots, k-1\}$. Then we have with Lemma 3.5 that $V(g) = m_i \leq m_{k-1} < m$ for all $0 \leq i \leq k-1$.

Now we show that for every point $\tilde{g} \in \mathcal{S}_{200}$ we have $V(\tilde{g}) < m$. Select $\tilde{g} \in \mathcal{S}_{200}$; then there exist green vertices $v_i, v_j \in \mathcal{S}_{100}$, $i, j \in \{0, 1, 2, \dots, k-1\}$ adjacent to one another such that $\tilde{g} \in \{\theta v_i + (1-\theta)v_j | \theta \in [0,1]\}$. Then, as V is affine on all simplices, this means that $V(\tilde{g}) \leq \max(V(v_i), V(v_j)) < m$. So $T \in \mathcal{S}_{200}$ implies $T \subseteq \mathcal{O}_{V,m}$.

To show that $\mathcal{S}_{200} \subseteq \mathcal{O}_{V,m,0}$ it remains to show that there is a continuous path connecting any $\tilde{g} \in \mathcal{S}_{200}$ to 0, which lies in $\mathcal{O}_{V,m}$. If $\tilde{g} \in T$ with $T \in \mathcal{S}_{200}$, then there is a finite sequence of green vertices $x_{q(i)} \in X_{q(i)}$, $i = 0, \dots, l$ with $q(\cdot)$ strictly increasing, $q(0) = 0$, and $x_{q(l)} \in \operatorname{ve}(T)$; furthermore, $x_{q(i)}$ and $x_{q(i+1)}$ are adjacent. Define the straight lines $\beta_i : [0,1] \to S$ for each $i = 0, \dots, l-1$ connecting $x_{q(i)}$ with $x_{q(i+1)}$. By concatenating these paths and finally with the straight line connecting $x_{q(l)}$ with $\tilde{g}$, we have constructed a path connecting 0 with $\tilde{g}$, which lies completely in elements of $\mathcal{S}_{200}$ and thus, as we have shown above, in $\mathcal{O}_{V,m}$. Hence, $\tilde{g} \in \mathcal{O}_{V,m,0}$.

After having established that $\mathcal{S}_{200} \subseteq \mathcal{O}_{V,m,0}$, we will now use this as the inductive step to show that $\mathcal{S}_{n_g 00} \subseteq \mathcal{O}_{V,m,0}$ for all $n_g \leq n+1$. Assume that $\mathcal{S}_{n_g-1,00} \subseteq \mathcal{O}_{V,m,0}$ for $3 \leq n_g \leq n+1$. Select an element $s \in T$ with $T \in \mathcal{S}_{n_g 00}$. Then s is either in the boundary of T, in which case $s \in \tilde{T}$ with $\tilde{T} \in \mathcal{S}_{n_g' 00} \subseteq \mathcal{O}_{V,m,0}$ with $n_g' < n_g$ by inductive hypothesis, or s is in the interior of $T = \mathrm{co}\{v_0, \dots, v_{n_g}\}$. Denote by $\bar{v}$ a vertex with $V(\bar{v}) = \max_{j=0,\dots,n_g} V(v_j)$. As $\bar{v} \in \mathcal{S}_{100} \subseteq \mathcal{S}_{200} \subseteq \mathcal{O}_{V,m,0}$ as shown above, there is a path connecting 0 with $\bar{v}$. By concatenating this path with the straight line from $\bar{v}$ to s we have constructed a path $l\colon [0,1] \to S$ from 0 to s such that $V(l(\theta)) < m$ holds for all $\theta \in [0,1]$—this is clear for the path from 0 to $\bar{v}$ since $\bar{v} \in \mathcal{O}_{V,m,0}$, and follows from the fact that V is affine on T and thus $V(l(\theta)) \leq V(\bar{v}) = \max_{j=0,\dots,n_g} V(v_j)$ for the part such that $l(\theta) \in T$. Since $\{\bar{v}\} \in \mathcal{S}_{100} \subseteq \mathcal{S}_{200} \subseteq \mathcal{O}_{V,m,0}$ we have $V(\bar{v}) < m$. This concludes the proof. ■

We now consider the yellow simplices and show that they are outside $\mathcal{O}_{V,m,0}$.

Lemma B.2 ($\mathcal{S}_{0y0}$ outside). *At step $k \in \{1, \dots, N\}$ in the algorithm $\mathcal{S}_{0y0}^{(k)} \cap \mathcal{O}_{V,m,0} = \emptyset$ for all $m_{k-1} < m \leq m_k$.*

Proof. Select $T \in \mathcal{S}_{0y0}$ and $y \in T$. To show that $y \notin \mathcal{O}_{V,m,0}$ it is sufficient to establish $V(y) \geq m$. Indeed, this is true since by Lemma 3.6 we have $V(y) \geq m_k \geq m$. ■

We now look at $\mathcal{S}_{00r}$ and $\mathcal{S}_{0yr}$ and show that they are also outside $\mathcal{O}_{V,m,0}$.

Lemma B.3 ($\mathcal{S}_{00r}$ and $\mathcal{S}_{0yr}$ outside). *At step $k \in \{1, \dots, N\}$ in the algorithm $\mathcal{S}_{00r}^{(k)} \cap \mathcal{O}_{V,m,0} = \emptyset$ as well as $\mathcal{S}_{0yr}^{(k)} \cap \mathcal{O}_{V,m,0} = \emptyset$ for all $m_{k-1} < m \leq m_k$.*

Proof. Assume that $x \in T \cap \mathcal{O}_{V,m,0}$ with $T \in \mathcal{S}_{00r}$ or $T \in \mathcal{S}_{0yr}$. Then there is a path connecting 0 to x, i.e., a continuous function $l : [0,1] \to S$, with $l(0) = 0$, $l(1) = x$, and $l(\theta) \subseteq \mathcal{O}_{V,m,0}$ for all $\theta \in [0,1]$. By Lemma 3.8 there exists $\theta^* \in [0,1]$ with $l(\theta^*) \in T'$ and $T' \in \mathcal{S}_{0y0}$. However, by Lemma B.2 we have $l(\theta^*) \notin \mathcal{O}_{V,m,0}$, which is a contradiction. ■

We will now look at $\mathcal{S}_{gy0}$, showing that the level set goes through those simplices. In particular, the level set intersects each of the n-simplices in an $(n-1)$-dimensional hypersurface.

Lemma B.4 (level set goes through $\mathcal{S}_{gy0}$). *Consider step $k \in \{1, \dots, N\}$ and let $m_{k-1} < m \leq m_k$.*

Let $T' \in \mathcal{S}_{gy0}^{(k)}$; then there exists an n-simplex $T \in \mathcal{S}_{gy0}^{(k)}$, such that T' is a subsimplex of T, as well as $y_T \in T'$ and $n_T \in \mathbb{R}^n$ such that

$$T' \cap \mathcal{O}_{V,m,0} = T' \cap \{x \in \mathbb{R}^n \mid (x - y_T)^T n_T < 0\} = T' \cap \{x \in \mathbb{R}^n \mid V(x) < m\}.$$

Furthermore, $T' \cap \mathcal{O}_{V,m,0} \neq \emptyset$ and $T' \setminus \mathcal{O}_{V,m,0} \neq \emptyset$.

Proof. Consider a p-simplex $T' \in \mathcal{S}_{gy0}$ with $0 \leq p \leq n$. Then there is an n-simplex $T \in \mathcal{S}_{gy0}$ such that T' is a subsimplex of T. Indeed, we can add $n - p$ (adjacent) vertices to the vertices $\mathrm{ve}(T')$ to obtain an n-simplex T, and since the vertices in T cannot be red by Lemma 3.7, $T \in \mathcal{S}_{gy0}$.

We order the vertices of T, $(v_i)_{i=0}^n$, such that all green vertices are $v_0, \dots, v_{g-1}$, and all yellow vertices are $v_g, \dots, v_n$. We can assume that $v_0, v_n \in \mathrm{ve}(T')$ and we have $1 \leq g \leq n$.

For a point $x \in T' \subseteq T$ there are $\lambda_i \geq 0$ with $\sum_{i=0}^n \lambda_i = 1$ such that $x = \sum_{i=0}^n \lambda_i v_i$. We have $V(x) = m$ if and only if

$$\sum_{i=0}^{n} \lambda_i m = m \;=\; V\left(\sum_{i=0}^{n} \lambda_i v_i\right) \;=\; \sum_{i=0}^{n} \lambda_i V(v_i) \;=\; \sum_{i=0}^{g-1} \lambda_i V(v_i) + \sum_{i=g}^{n} \lambda_i V(v_i). \tag{B.1}$$

From this we have

$$\sum_{i=0}^{g-1} \lambda_i (V(v_i) - m) + \sum_{i=g}^{n} \lambda_i (V(v_i) - m) = 0. \tag{B.2}$$

Remember that all green vertices satisfy $V(v_i) \leq m_{k-1} < m$ for all $0 \leq i \leq g-1$ and we have for the yellow vertices $V(v_i) \geq m_k \geq m$ for all $i \geq g$ by Lemma 3.6. This means $(V(v_i) - m) < 0$ for $0 \leq i \leq g-1$ and $(V(v_i) - m) \geq 0$ for $g \leq i \leq n$. In particular, $V(v_0) - m < 0$ and $V(v_n) - m \geq 0$. Setting $\lambda_0^* = \frac{V(v_n)-m}{V(v_n)-V(v_0)} \geq 0$, $\lambda_n^* = \frac{m-V(v_0)}{V(v_n)-V(v_0)} > 0$, and $\lambda_i^* = 0$ for all other i, we have $\sum_{i=0}^{n} \lambda_i^* = 1$. Setting $y_T = \sum_{i=0}^{n} \lambda_i^* v_i \in T'$ we have $V(y_T) = m$ by (B.2).

We will show that for all $x \in T$ we have

$$(x - y_T)^T n_T = V(x) - m$$

with $n_T = \nabla V\big|_T$.

For a fixed point $x \in T$ we can write $x = \sum_{i=0}^{n} \lambda_i v_i$, with $\sum_{i=0}^{n} \lambda_i = 1$, $\lambda_i \geq 0$. By Definition 2.5 we have $n_T = \nabla V\big|_T = X_T^{-1} v_T$, where the matrix X_T is defined by writing the components of the vector $(v_i - v_0)^T$ in its ith row, $i = 1, 2, \ldots, n$, and the column vector v_T is defined by setting $V(v_i) - V(v_0)$ as its ith component for $i = 1, 2, \ldots, n$.

Now we have

$$\begin{aligned} x - y_T &= (x - v_0) - (y_T - v_0) \\ &= \sum_{i=0}^{n} \lambda_i (v_i - v_0) - \sum_{i=0}^{n} \lambda_i^* (v_i - v_0) \\ &= \sum_{i=0}^{n} (\lambda_i - \lambda_i^*)(v_i - v_0) \\ &= \sum_{i=1}^{n} (\lambda_i - \lambda_i^*)(v_i - v_0) \\ &= X_T^T \begin{pmatrix} \lambda_1 - \lambda_1^* \\ \lambda_2 - \lambda_2^* \\ \vdots \\ \lambda_n - \lambda_n^* \end{pmatrix}, \end{aligned}$$

$$
\begin{aligned}
(x-y_T)^T n_T &= (\lambda_1-\lambda_1^*, \lambda_2-\lambda_2^*, \dots, \lambda_n-\lambda_n^*)\, X_T X_T^{-1} v_T \\
&= \sum_{i=1}^{n} (\lambda_i-\lambda_i^*)(V(v_i)-V(v_0)) \\
&= \sum_{i=0}^{n} (\lambda_i-\lambda_i^*)(V(v_i)-V(v_0)) \\
&= \sum_{i=0}^{n} \lambda_i V(v_i) - V(v_0) - \sum_{i=0}^{n} \lambda_i^* V(v_i) + V(v_0) \\
&= V(x) - V(y_T) \\
&= V(x) - m. \qquad \text{(B.3)}
\end{aligned}
$$

Note that for $x \in T$ we have $V(x) < m$ if and only if $(x-y_T)^T n_T < 0$. Since all green vertices are in $\mathcal{O}_{V,m,0}$ (see Lemma B.1), i.e., in particular path-connected to 0, and since V is affine on each simplex, also the points $x \in T$ with $T \in \mathcal{S}_{gy0}$ and $V(x) < m$ satisfy $x \in \mathcal{O}_{V,m,0}$. We have $v_0 \in T' \cap \mathcal{O}_{V,m,0}$ and $v_n \in T' \setminus \mathcal{O}_{V,m,0}$. This completes the proof. ∎

Consider step $k = 1$. Using similar arguments as in the proof of the previous lemma, we will show that $\|x\|$ for all x with $V(x) = m_1$ is bounded below away from 0.

Corollary B.5. *Consider step $k = 1$ and let $T \in \mathcal{S}_{gy0}^{(1)}$ be an n-simplex, where $\mathrm{ve}(T) = \{v_0, \dots, v_n\}$, $v_0 = 0$ and $0 = V(v_0) < V(v_1) \le \cdots \le V(v_n)$.*

Then $V(v_n) \ge m_1$ and

$$\min_{x \in T, V(x)=m_1} \|x\|^2 = \frac{m_1^2}{\|X_T^{-1} v_T\|^2} > 0.$$

Proof. We have $V(v_n) \ge m_1$ by Lemma 3.6, since v_n is yellow in step 1, as it is adjacent to 0. We can follow the proof of the previous lemma with $k = 1$ and $m = m_1$. To compute the minimum of $h(x) = x^T x$ under the condition $g(x) = (x-y_T)^T n_T = 0$, which is equivalent to $V(x) = m_1$ (see (B.3)), we can conclude with $\nabla h(x) = 2x$ and $\nabla g(x) = n_T$, that there is a Lagrange multiplier $\lambda \in \mathbb{R}$ with $x = \lambda n_T$. Since $(x-y_T)^T n_T = 0$, we have $\lambda = \frac{y_T^T n_T}{\|n_T\|^2}$, i.e., $x^T x = \frac{(y_T^T n_T)^2}{\|n_T\|^2}$.

Using $v_0 = 0$ we obtain (see proof of Lemma B.4)

$$
\begin{aligned}
y_T &= \lambda_0^* v_0 + \lambda_n^* v_n \;=\; \lambda_n^* v_n, \\
\lambda_n^* &= \frac{m_1}{V(v_n)} \in (0,1], \\
y_T^T n_T &= \lambda_n^* (v_n - v_0)^T X_T^{-1} v_T \;=\; \lambda_n^* (V(v_n) - V(v_0)) \;=\; m_1, \\
x^T x &= \frac{m_1^2}{\|n_T\|^2} = \frac{m_1^2}{\|X_T^{-1} v_T\|^2}.
\end{aligned}
$$

This shows the corollary. ∎

Appendix C. Proof of Theorem 3.13. The proof will be divided into several parts. Initially we prove (3.1), before we show $N = M$ and $P_{m_M} = \mathcal{L}_V^{\sup}$.

Let $m \in [m_1, m_N)$. To establish $P_m = \mathcal{L}_{V,m}$, recall that by Corollary 3.12 $P_m = \mathcal{O}_{V,m,0}$. Hence, it suffices to show that

1. $0 \in P_m^\circ$,
2. $\overline{P_m} \subseteq S^\circ$.

We will look at each of these points individually in Lemmas C.3 and C.2.

Lemma C.1. *Let $A, B \subseteq \mathbb{R}^n$ both be nonempty sets such that A is path-connected, and assume that $A \cap B^\circ \neq \emptyset$ and $A \cap \partial B = \emptyset$ hold. Then $A \subseteq B^\circ$.*

Proof. Assume in contrast to the statement that there exists $x \in A$ such that $x \notin B^\circ$. Then, as $A \cap B^\circ \neq \emptyset$, pick $y \in A \cap B^\circ$. As A is path-connected there exists a continuous path $l : [0,1] \to A$ with $l(0) = x$, $l(1) = y$. But $x \notin B^\circ$ so there exists an $\epsilon \in [0,1]$ such that $l(\epsilon) \in \partial B$. A contradiction, hence $A \subseteq B^\circ$. ■

Lemma C.2. $\overline{P_m} \subseteq S^\circ$ *for all* $0 < m < m_N$.

Proof. Note that it is sufficient to show the statement for $m_{N-1} < m < m_N$ as $\overline{P_m} \subseteq \overline{P_{m'}}$ for $m \leq m'$ by the characterization as $P_m = \mathcal{O}_{V,m,0}$.

Hence, let us now assume $m_{N-1} < m < m_N$. We use Lemma C.1 with $A = \overline{P_m} = \overline{\mathcal{O}_{V,m,0}}$, which is path-connected by definition, and $B = S$. We have $0 \in \overline{P_m} \cap S^\circ \neq \emptyset$. Hence, we need to show that $\overline{P_m} \cap \partial S = \emptyset$; then by Lemma C.1 this will imply that $\overline{P_m} \subseteq S^\circ$.

Now assume that $x \in \overline{P_m} \cap \partial S \neq \emptyset$. There is a p-simplex T with $x \in T$ and we can choose p minimal with this property. By Lemma 2.4 we have for all vertices $\mathrm{ve}(T) \subseteq \partial S$ since $x \in \partial S$. As the algorithm terminates, if it reaches the boundary of S, there will be no green vertices in ∂S. This means that $T \in \mathcal{S}_{0y0} \cup \mathcal{S}_{00r} \cup \mathcal{S}_{0yr}$.

P_m, on the other hand, only contains green vertices, and parts of green and yellow simplices, but no entirely yellow simplices, and we will show below that also all points in $\overline{P_m} = \overline{\mathcal{O}_{V,m,0}}$ lie in a simplex in $\mathcal{S}_{g00} \cup \mathcal{S}_{gy0}$. Hence T lies in both $\mathcal{S}_{0y0} \cup \mathcal{S}_{00r} \cup \mathcal{S}_{0yr}$ and $\mathcal{S}_{g00} \cup \mathcal{S}_{gy0}$, which is impossible.

We will now show that $x \in \overline{P_m}$ implies that $x \in T''$, where T'' is a p-simplex with minimal p and $T'' \in \mathcal{S}_{g00} \cup \mathcal{S}_{gy0}$. Assume that $y_i \in \mathcal{O}_{V,m,0} = P_m$ is a sequence with limit $\lim_{i\to\infty} y_i = x$. Since there are finitely many simplices, there exists a simplex $T' \in \mathcal{S}_{g00} \cup \mathcal{S}_{gy0}$ such that a subsequence, which we still call y_i, lies in T'. We have $x \in T'$ as T' is closed, and we have $V(x) \leq m$ as $V(y_i) < m$.

Either $T' \in \mathcal{S}_{g00}$ or $T' \in \mathcal{S}_{gy0}$. x cannot lie in a subsimplex $T'' \subseteq T'$ with $T'' \in \mathcal{S}_{0y0}$ since by Lemma 3.6 in that case $V(x) \geq m_N$, which is a contradiction to $V(x) \leq m < m_N$. Altogether, this shows $T'' \in \mathcal{S}_{g00} \cup \mathcal{S}_{gy0}$. ■

Lemma C.3. $0 \in P_m^\circ$ *for all* $m \in [m_1, m_N]$.

Proof. We will show that $B_\epsilon(0) \subseteq P_{m_1}$ with $\epsilon > 0$ defined below. This will imply $0 \in P_m^\circ$ since $P_{m_1}^\circ \subseteq P_m^\circ$ for all $m \geq m_1$.

Consider step $k = 1$ and define $\mathcal{T}_0 := \bigcup_{T \in \mathcal{T}, 0 \in \mathrm{ve}(T)} T$, i.e., the union of all n-simplices which have 0 as a vertex. Note that $\mathcal{T}_0 = \mathcal{S}_{gy0}$, and in particular we have for $T \in \mathcal{T}_0$ that $\mathrm{ve}(T) = \{0, v_1, \ldots, v_n\}$, where v_i are yellow vertices, since 0 is the only green vertex in step 1 and the v_i are adjacent to 0. For each $T \in \mathcal{T}_0$ let us define with Corollary B.5

$$\epsilon_T := \min_{x \in T, V(x)=m_1} \|x\| > 0$$

and

$$\epsilon := \min_{T \in \mathcal{T}_0} \epsilon_T > 0$$

as there are finitely many simplices in $\mathcal{T}_0$.

Assume in contradiction to the statement that there exists $x \in B_\epsilon(0) \setminus P_{m_1} = B_\epsilon(0) \setminus \mathcal{O}_{V,m_1,0}$; recall that $\mathcal{O}_{V,m_1,0} = P_{m_1}$ by Corollary 3.12. Consider the straight line $l\colon [0,1] \to S$, $l(\theta) = \theta x$. There is a simplex $T \in \mathcal{T}_0$ and $\theta' > 0$ such that $\theta x \in T$ for all $\theta \in [0, \theta']$ and $\theta x \notin T$ for all $\theta > \theta'$. Denote $\mathrm{ve}(T) = \{v_0 = 0, v_1, \dots, v_n\}$, where v_i are yellow vertices; see above. We can write $x = \sum_{i=1}^n \lambda_i (v_i - v_0)$ with $\lambda_i \geq 0$. Then for $\theta \geq 0$ we have $\theta x = \sum_{i=1}^n \theta \lambda_i v_i = \sum_{i=0}^n \mu_i v_i$ with $\mu_i = \theta \lambda_i \geq 0$ for $i \geq 1$ and $\mu_0 = 1 - \sum_{i=1}^n \mu_i$. We have $\sum_{i=0}^n \mu_i = 1$, so $\theta x \in T$, if and only if $\mu_0 \geq 0$, i.e., $\theta \leq (\sum_{i=1}^n \lambda_i)^{-1}$. This shows that $\theta' = (\sum_{i=1}^n \lambda_i)^{-1}$. In particular,

$$\theta' x = \sum_{i=1}^n \mu_i v_i$$

with $\mu_0 = 0$, so $\theta' x \in \partial T$ and is a convex combination of yellow vertices. Thus, $V(\theta' x) \geq m_1$ by Lemma 3.6.

Since $0 \in \mathcal{O}_{V,m_1,0}$ and $x \notin \mathcal{O}_{V,m_1,0}$ there is $\theta^* \in (0,1]$ such that $l(\theta) \in \mathcal{O}_{V,m_1,0}$ for all $\theta \in [0, \theta^*)$ and $x^* = l(\theta^*) \in \partial \mathcal{O}_{V,m_1,0}$; in particular, $V(x^*) = m_1$. Since $V(\theta x) = \theta V(x)$ for $0 \leq \theta \leq \theta'$ and $V(\theta' x) \geq m_1$, we have $\theta^* \leq \theta'$, so in particular, $\theta^* x \in T$. Note that $x^* = \theta^* x \in B_\epsilon(0)$ since $\theta^* \in (0,1]$. Then we have

$$\epsilon \leq \epsilon_T = \min_{x \in T, V(x)=m_1} \|x\| \leq \|x^*\| < \epsilon,$$

which is a contradiction. ■

Note that Lemmas C.2 and C.3 together show (3.1). The next lemma implies that the case mentioned in Definition 3.2 never occurs and thus $N = M$.

Lemma C.4. *Consider step $k \in \{1, \dots, N\}$. Assume that $X_k \cap \partial S = \emptyset$ holds. If $r \in \mathrm{ad}(X_k) \cap R_k$, then $V(r) > m_k$. In particular, $N = M$.*

Proof. Assume in contradication to the statement that $X_k \cap \partial S = \emptyset$ holds and there is a red vertex $\bar{x}_k \in \mathrm{ad}(X_k) \cap R_k$, adjacent to an $x_k \in X_k$, with $V(\bar{x}_k) \leq m_k = V(x_k)$.

Step 1: *$\overline{\mathcal{O}_{V,m_k,0}} \subseteq S^\circ$ and $\mathcal{L}_{V,m_k} \neq \emptyset$.* We will first establish

$$\overline{\mathcal{O}_{V,m_k,0}} \cap \partial S = \emptyset.$$

Indeed, assume that in contrast to the statement above there was a sequence of points $y_i \in \mathcal{O}_{V,m_k,0} = P_{m_k}$ with limit $x := \lim_{i\to\infty} y_i \in \partial S$. Since $V(y_i) < m_k$, we have $V(x) \leq m_k$.

Since there are finitely many simplices, there exists a simplex $T' \in \mathcal{S}_{g00} \cup \mathcal{S}_{gy0}$ such that a subsequence, which we still call y_i, lies in T'. We have $x \in T'$ as T' is closed.

Since $x \in \partial S$, there is a p-simplex T with $x \in T$ and we can choose p minimal with this property. By Lemma 2.4 we have for all vertices $\mathrm{ve}(T) \subseteq \partial S$. By the definition of the algorithm, there are no green vertices in ∂S. Since $x \in T' \cap T$, T' does not have any red vertices and T does not have any green vertices, $T \cap T' \in \mathcal{S}_{0y0}$ and thus $V(x) \geq m_k$ by Lemma 3.6. This shows $V(x) = m_k$ and, in particular, that there is a vertex $v \in X_k$ with $v \in \partial S$. This is a contradiction to $X_k \cap \partial S = \emptyset$.

By Lemma C.1 with $A = \overline{\mathcal{O}_{V,m_k,0}}$ and $B = S$ we have $\overline{\mathcal{O}_{V,m_k,0}} \subseteq S^\circ$ since $\overline{\mathcal{O}_{V,m_k,0}}$ is path connected, $0 \in \overline{\mathcal{O}_{V,m_k,0}} \cap S^\circ \neq \emptyset$, and, as just shown, $\overline{\mathcal{O}_{V,m_k,0}} \cap \partial S = \emptyset$. Hence, we have $\overline{\mathcal{O}_{V,m_k,0}} \subseteq S^\circ$ and $\mathcal{L}_{V,m_k} = \mathcal{O}_{V,m_k,0} = P_{m_k} \neq \emptyset$, since $0 \in P^\circ_{m_k}$ by Lemma C.3.

Step 2: $x_k \in A(0)$. We have $x_k \in \overline{\mathcal{O}_{V,m_k,0}}$ by Corollary 3.12. Since V is a strict Lyapunov function in S, $\overline{\mathcal{O}_{V,m_k,0}}$ is positively invariant and compact, so $\emptyset \neq \omega(x_k) \subseteq \overline{\mathcal{O}_{V,m_k,0}} \subseteq S^\circ$ by Step 1. By LaSalle's principle $\dot{V}(w) = 0$ for all $w \in \omega(x_k)$, i.e., $\omega(x_k) = \{0\}$, and, since 0 is an asymptotically stable equilibrium as V is a strict Lyapunov function, we have $x_k \in A(0)$.

Step 3: *Contradiction*. Consider the straight line between x_k and $\bar{x}_k$, namely $L = \{\theta \bar{x}_k + (1-\theta)x_k \mid \theta \in (0,1)\}$. For all $x \in L$ the minimal p-simplex T with $x \in T$ is $T = \mathrm{co}\{x_k, \bar{x}_k\} \in \mathcal{S}_{0yr}$, since x_k is yellow and $\bar{x}_k$ is red. Moreover, $V(x) \leq m_k$ for all $x \in L$, since $V(x_k), V(\bar{x}_k) \leq m_k$ by assumption. By Lemma 3.9 we have for any $x \in L$ that either $x \notin A(0)$ or the positive orbit through x leaves S.

Since $0 \in S^\circ$, there exists $\epsilon > 0$ such that $B_\epsilon(0) \subseteq S^\circ$. Since 0 is stable, there exists $\delta > 0$ such that $S_t B_\delta(0) \subseteq B_\epsilon(0)$ holds for all $t \geq 0$. Since $x_k \in A(0)$, which is open, there exists $\nu > 0$ such that $B_\nu(x_k) \subseteq A(0)$. Moreover, there exists $T_0 > 0$ such that $S_{T_0} x_k \in B_{\delta/2}(0)$ since $x_k \in A(0)$. We have

$$\min_{t \in [0,T_0]} \mathrm{dist}(S_t x_k, \partial S) =: c > 0$$

since $x_k \in \overline{\mathcal{O}_{V,m_k,0}} \subseteq S^\circ$, $\overline{\mathcal{O}_{V,m_k,0}}$ is positively invariant, and the function is continuous on a compact interval. Also, S_t is defined on a neighborhood of x_k and is uniformly continuous for $[0, T_0]$, so there is $\eta > 0$ such that

$$x \in B_\eta(x_k) \Rightarrow \|S_t x - S_t x_k\| < \frac{1}{2} \min(\delta, c) \text{ for all } t \in [0, T_0].$$

Choose

$$x \in L \cap B_\eta(x_k) \cap B_\nu(x_k) \neq \emptyset.$$

Then we have

$$S_t x \in S^\circ \text{ for all } t \geq 0. \tag{C.1}$$

Indeed, for $t \in [0, T_0]$ we have $S_t x \subseteq S^\circ$ due to Lemma C.1 with $A = \bigcup_{t \in [0,T_0]} S_t x$ and $B = S$ and the fact that $S_t x \notin \partial S$ since

$$\mathrm{dist}(S_t x, \partial S) \geq \mathrm{dist}(S_t x_k, \partial S) - \|S_t x - S_t x_k\| > c/2 > 0 \text{ for } t \in [0, T_0].$$

For $t \geq T_0$ we have

$$\|S_{T_0} x\| \leq \|S_{T_0} x_k\| + \|S_{T_0} x - S_{T_0} x_k\| < \delta/2 + \delta/2$$

and thus $S_{T_0+\theta}x \in B_\epsilon(0) \subseteq S^\circ$ for all $\theta \geq 0$.

We have $x \in A(0)$ since $x \in B_\nu(x_k)$ and the positive orbit through x stays in S by (C.1). This is a contradiction to the fact that either $x \notin A(0)$ or the positive orbit through x leaves S. ∎

We are now in a position to show that P_{m_N} is in fact equal to $\mathcal{L}_V^{\sup}$ as defined in Definition 2.7. This, together with Lemma C.4 concludes the proof of Theorem 3.13.

Lemma C.5. *We have $P_{m_N} = \mathcal{L}_V^{\sup}$.*

Proof. To show that $P_{m_N} \subseteq \mathcal{L}_V^{\sup}$, we first show

$$P_{m_N} \subseteq \bigcup_{m_{N-1}<m<m_N} P_m.$$

We consider the coloring in step N. Note that if $x \in P_{m_N}$, then either $x \in T$ with $T \in \mathcal{S}_{g00}$ and then $x \in \bigcup_{m_{N-1}<m<m_N} P_m$ or $x \in T$ with $T \in \mathcal{S}_{gy0}$ and $V(x) < m_N$. In this case we also have $V(x) < \frac{m_N+V(x)}{2} =: m < m_N$, which shows $x \in \bigcup_{m_{N-1}<m<m_N} P_m$.

Now pick $x \in P_{m_N}$; then there exists m with $m_{N-1} < m < m_N$ such that $x \in P_m$, as shown above. By (3.1) $\mathcal{L}_{V,m} = P_m \neq \emptyset$, which implies that $x \in \bigcup_{m\in\mathbb{R}} \mathcal{L}_{V,m} = \mathcal{L}_V^{\sup}$. So $P_{m_N} \subseteq \mathcal{L}_V^{\sup}$.

It remains to show that $\mathcal{L}_V^{\sup} \subseteq P_{m_N}$. From Definition 2.7 remember that $\mathcal{L}_V^{\sup} := \bigcup_{m\in\mathbb{R}} \mathcal{L}_{V,m}$. Hence we look at different values $m \in \mathbb{R}$ and check to see when $\mathcal{L}_{V,m} \subseteq P_{m_N}$. We can split this into three cases. In each case either $\mathcal{L}_{V,m} = \mathcal{O}_{V,m,0}$, if a suitable $\mathcal{O}_{V,m,0}$ exists from Definition 2.7, or $\mathcal{L}_{V,m} = \emptyset$. Remember that a suitable $\mathcal{O}_{V,m,0}$ means that there exists $\mathcal{O}_{V,m,0}$ such that $\{0\} \subseteq \mathcal{O}_{V,m,0}^\circ \subseteq \overline{\mathcal{O}_{V,m,0}} \subseteq S^\circ$. If $\mathcal{L}_{V,m} = \emptyset$, then clearly $\mathcal{L}_{V,m} \subseteq P_{m_N}$.

Case A: $0 \geq m$. As $V(x) \geq 0$ for all $x \in S$, we have that $\mathcal{O}_{V,m,0} = \{0\}$ and thus $\mathcal{L}_{V,m} = \emptyset$ by Definition 2.7.

Case B: $0 < m < m_N$. Then there exists $\tilde{m} \geq m$ with $m_1 \leq \tilde{m} < m_N$. By (3.1) we have $\mathcal{L}_{V,\tilde{m}} = \mathcal{O}_{V,\tilde{m},0} = P_{\tilde{m}} \neq \emptyset$. And since $\mathcal{L}_{V,m} \subseteq \mathcal{L}_{V,\tilde{m}}$ as well as $P_{\tilde{m}} \subseteq P_{m_N}$, we have $\mathcal{L}_{V,m} \subseteq P_{\tilde{m}} \subseteq P_{m_N}$.

Case C: $m \geq m_N$. Since we have shown in Lemma C.4 that $N = M$, there exists $y \in X_N \cap \partial S$. We will conclude that $\mathcal{L}_{V,m} = \emptyset$ for all $m \geq m_N$.

We will show that $y \in \partial S \cap \overline{\mathcal{O}_{V,m_N,0}}$. y is yellow and by Lemma 3.4 there is an adjacent green vertex $x \in X_k$ with $k \in \{0, \ldots, N-1\}$ and thus $V(x) < V(y)$ by Lemma 3.5. Consider the 1-simplex $L = \mathrm{co}\{x, y\} \in \mathcal{S}_{gy0}$ given by the straight line $l(\theta) = x + \theta(y - x)$, $\theta \in [0, 1]$. Then

$$V(l(\theta)) = V(x) + \theta(V(y) - V(x)) < V(y) = m_N \text{ for } 0 \leq \theta < 1.$$

Thus, the sequence $l(1 - 1/p)$, $p \in \mathbb{N}$, satisfies $l(1 - 1/p) \in \mathcal{O}_{V,m_N,0}$ by Lemma B.4 as well as $l(1 - 1/p) \to l(1) = y$ as $p \to \infty$. Hence, $y \in \partial S \cap \overline{\mathcal{O}_{V,m_N,0}}$. Thus, $\mathcal{L}_{V,m} = \emptyset$ for all $m \geq m_N$. ∎

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