Computation of Lyapunov Functions for Nonautonomous Systems on Finite Time-Intervals by Linear Programming

Sigurdur F. Hafstein[a,*], Huijuan Li[b]

[a]School of Science and Engineering, Reykjavik University, Menntavegur 1, IS-101 Reykjavik, Iceland
[b]School of Mathematics and Physics, China University of Geosciences (Wuhan), 430074, Wuhan, China

Abstract

We present an algorithm for numerically computing Lyapunov functions for nonautonomous systems on finite time-intervals. The algorithm relies on a linear optimization problem and delivers a continuous and piecewise affine function on a compact set. The level-sets of such a Lyapunov function give concrete bounds on the time-evolution of the system on the time-interval and for time-periodic systems they deliver an ultimate bound on solutions. Four examples of computed finite-time Lyapunov functions are given.

Keywords: Lyapunov function, Finite-time Lyapunov function, Periodic-time system, Linear programming

1. Introduction

In the usual setting for a nonautonomous and continuous-time system, given by the differential equation $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$, one studies the stability properties of the zero solution. Traditionally concepts such as uniform asymptotic stability or uniform ultimate boundedness, also known as practical stability, are studied in classical textbooks [15, 17, 24, 23]. Both of these, together with numerous other stability concepts, can be characterized by the existence of so-called Lyapunov functions. Lyapunov functions are real-valued functions from the state-space that are nonincreasing along the system's trajectories. They are fundamental tools when studying the qualitative behaviour of dynamical systems. Note that for nonlinear systems it is generally a very hard problem to identify or compute Lyapunov functions and for nonautonomous nonlinear systems even the linear case, i.e. $\dot{\mathbf{x}} = A(t)\mathbf{x}$, is quite involved.

Recently in [12] the concept of a finite-time Lyapunov function for nonautonomous systems on finite time intervals was introduced. The existence of such a function was shown to be equivalent to solutions being attracted to the zero solution on the interval in an appropriate norm. A different and much less strict approach was followed earlier in [13]

*Corresponding author
Email addresses: `sigurdurh@ru.is` (Sigurdur F. Hafstein), `huijuanhongyan@gmail.com` (Huijuan Li)
[1]Li is partially supported by the Fundamental Research Funds for the Central Universities (CUG160603), China University of Geosciences (Wuhan).

where an algorithm to compute functions decreasing along solution trajectories on finite time-intervals was developed. One must however be very careful when drawing conclusions about the behaviour of solution trajectories from such functions, cf. [13, Theorem 6.11].

In this paper we will define finite-time Lyapunov functions, or short FT Lyapunov functions, in Definition 1 that are in some sense between these two approaches above. An FT Lyapunov function in our sense does not imply attractiveness of the zero solution as in [12], indeed the system does not even have to possess the zero solution, but in contrast to [13] its existence implies certain boundedness properties of the solution trajectories. We then proceed to derive a linear programming (LP) problem in LP Problem 5, of which every feasible solution can be used to parameterize a continuous and piecewise affine (CPA) FT Lyapunov function in our sense. Finally, we give four worked out examples where we generate FT Lyapunov functions for nonlinear systems and make some conclusions.

Note that there have been several methods proposed of how to numerically compute Lyapunov functions, for example using semidefinite programming to parameterize polynomial Lyapunov functions [21, 4, 20, 1], solving a Zubov type partial differential equation [18, 3, 6], or using LP to parameterize CPA Lyapunov functions [16, 19, 13]; to name a few. We refer the interested reader to the recent review [11] on numerical methods for the computation of Lyapunov functions for more references on many different methods. However, most of these computational approaches only work for autonomous systems.

Notations: We write a column vector $\mathbf{x} \in \mathbb{R}^n$ in boldface and denote its transpose by $\mathbf{x}^\top$. For a $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$ we often write $(t, \mathbf{x})$ for the $(n+1)$-dimensional column vector with t as its first element. By $\|\cdot\|$ we denote an arbitrary norm on $\mathbb{R}^n$ and $\|\cdot\|_p$ denotes the norm $\|\mathbf{x}\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ if $1 \leq p < \infty$ and $\|\mathbf{x}\|_\infty := \max\{|x_i| \,|\, i = 1, 2, \ldots, n\}$. We denote the open ball $\{\mathbf{x} \in \mathbb{R}^n \,\big|\, \|\mathbf{x}\|_2 < r\}$ by $\mathcal{B}_r^n$. For a set $U \subset \mathbb{R} \times \mathbb{R}^n$ denote the t-fibre of U as $U(t) := \{\mathbf{x} \in \mathbb{R}^n \,|\, (t, \mathbf{x}) \in U\}$. The closure, interior, complement, and boundary of a set $U \subset \mathbb{R}^n$ is denoted by $\mathrm{cl}(U)$, $\mathrm{int}(U)$, $\mathrm{comp}(U)$, and $\mathrm{bdy}(U)$ respectively. We frequently use the relative topology of a set $X := [0, T] \times \mathbb{R}^n$ where $T > 0$. Recall that the open sets in the relative topology of X are sets that are the intersection of open sets in $\mathbb{R} \times \mathbb{R}^n$ and X. The closure, interior, complement, and boundary of a set $U \subset X$ in the relative topology are denoted by $\mathrm{cl}_X(U)$, $\mathrm{int}_X(U)$, $\mathrm{comp}_X(U)$, and $\mathrm{bdy}_X(U)$ respectively. For integers $a, b \in \mathbb{Z}$ we write $[a, b]_{\mathbb{Z}}$ for the set $\{a, a+1, \ldots, b\}$. We denote the set of the nonnegative integers by $\mathbb{N}_0$.

2. Finite-time Lyapunov functions

We consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \tag{1}$$

where $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz in its second argument, i.e. for every compact $C \subset \mathbb{R} \times \mathbb{R}^n$ there exists a constant $L_C > 0$ such that

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\| \leq L_C \|\mathbf{x} - \mathbf{y}\| \;\text{ for all } (t, \mathbf{x}), (t, \mathbf{y}) \in C.$$

We denote by $\boldsymbol{\phi}(t, s, \mathbf{x})$ the solution to (1), i.e.

$$\frac{d}{dt}\boldsymbol{\phi}(t, s, \mathbf{x}) = \mathbf{f}(t, \boldsymbol{\phi}(t, s, \mathbf{x})) \;\text{ and }\; \boldsymbol{\phi}(s, s, \mathbf{x}) = \mathbf{x}.$$

It is well known that under these assumptions the solution $\boldsymbol{\phi}(t,s,\mathbf{x})$ is unique and for each $(s,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^n$ there exists a maximum open interval (a,b), $-\infty \le a < s < b \le +\infty$, such that $t \mapsto \boldsymbol{\phi}(t,s,\mathbf{x})$ is well defined, c.f. e.g. [25, III.§10.VI].

To simplify notation and the discussion we will consider the system (1) on sets that are neighbourhoods of $[0,T]\times\{\mathbf{0}\}$ in the relative topology of $X := [0,T]\times\mathbb{R}^n$. The ideas can, however, be applied to more general finite-time situations by the use of coordinate transforms. In this paper we are interested in deriving bounds on solution trajectories in a vicinity of $[0,T]\times\{\mathbf{0}\}$ and to this end we will use finite-time Lyapunov functions. Note that we are not discussing the stability of the zero solution and we do not even assume that $t \mapsto \boldsymbol{\phi}(t,s,\mathbf{0}) = \mathbf{0}$.

Definition 1. Consider the system (1), let $T > 0$, $X := [0,T]\times\mathbb{R}^n$, and let $M, I \subset X$ be bounded, connected sets that are open neighbourhoods of $[0,T]\times\{\mathbf{0}\}$ in the relative topology of X. Assume further that $\mathrm{cl}_X(I) \subset M$ and set $D_V := \mathrm{cl}_X(M)\setminus I$. A continuous function $V : D_V \to \mathbb{R}$, such that $V(t,\mathbf{x})$ is Lipschitz continuous in its second argument and such that:

i)
$$M_V := \min_{(t,\mathbf{x})\in\mathrm{bdy}_X(M)} V(t,\mathbf{x}) > \max_{(t,\mathbf{x})\in\mathrm{bdy}_X(I)} V(t,\mathbf{x}) =: m_V$$

ii) and for all $(t,\mathbf{x}) \in \mathrm{int}(D_V)$
$$D^+_{\mathbf{f}}V(t,\mathbf{x}) := \limsup_{h\to 0+} \frac{V(t+h,\mathbf{x}+h\mathbf{f}(t,\mathbf{x})) - V(t,\mathbf{x})}{h} \le 0$$

is called a finite-time Lyapunov function (FTLF) for (1) on D_V.

Note that the term *finite-time Lyapunov function* has been used in the literature for different concepts, cf. e.g. [12, 7] where it is used for functions that fulfill considerably stricter conditions than our FT Lyapunov function in Definition 1, or [5] where it is used for functions that decrease along solution trajectories on time-intervals of certain length, but not necessarily monotonically.
An FT Lyapunov function for system (1) is nonincreasing along its solution's trajectories as long as they stay in D_V.

Lemma 1. *Assume V is an FT Lyapunov function for (1) on D_V. Then, for $0 \le a < b \le T$ such that $(a,\boldsymbol{\phi}(a,s,\mathbf{x})), (b,\boldsymbol{\phi}(b,s,\mathbf{x})) \in D_V$ and $(t,\boldsymbol{\phi}(t,s,\mathbf{x})) \in \mathrm{int}_X(D_V)$ for all $a < t < b$, we have $V(a,\boldsymbol{\phi}(a,s,\mathbf{x})) \ge V(b,\boldsymbol{\phi}(b,s,\mathbf{x}))$.*

In other words: V is nonincreasing along solution trajectories or, more exactly, the mapping $t \mapsto V(t,\boldsymbol{\phi}(t,s,\mathbf{x}))$ is nonincreasing on $\mathrm{int}_X(D_V)$.

Proof. Fix $t \in (a,b)$ and set $\mathbf{y} = \boldsymbol{\phi}(t,s,\mathbf{x})$. Let $L > 0$ be a Lipschitz constant for V on

D_V, then

$$\lim_{h\to 0}\left|\frac{V(t+h,\boldsymbol{\phi}(t+h,t,\mathbf{y}))-V(t+h,\mathbf{y}+h\mathbf{f}(t,\mathbf{y}))}{h}\right|$$
$$\leq \lim_{h\to 0} L\left|\frac{\boldsymbol{\phi}(t+h,t,\mathbf{y})-\mathbf{y}-h\mathbf{f}(t,\mathbf{y})}{h}\right| = L\left|\lim_{h\to 0}\frac{\boldsymbol{\phi}(t+h,t,\mathbf{y})-\mathbf{y}}{h}-\mathbf{f}(t,\mathbf{y})\right|$$
$$= L|\mathbf{f}(t,\mathbf{y})-\mathbf{f}(t,\mathbf{y})| = 0.$$

Hence, the equality

$$D^+_{\mathbf{f}}V(t,\mathbf{y}) = \limsup_{h\to 0+}\frac{V(t+h,\boldsymbol{\phi}(t+h,t,\mathbf{y}))-V(t,\mathbf{y})}{h}$$

follows and by condition ii) in Definition 1 and by [26, §12.24], the continuous function $t \mapsto V(t,\boldsymbol{\phi}(t,\mathbf{y})) = V(t,\boldsymbol{\phi}(t,s,\mathbf{x}))$ is nonincreasing on $[a,b]$. Especially, $V(a,\boldsymbol{\phi}(a,s,\mathbf{x}) \geq V(b,\boldsymbol{\phi}(b,s,\mathbf{x}))$.

Lemma 1 implies that certain sub-level sets of an FT Lyapunov function are forward invariant within the set M. This is proved in Theorem 2 below.

Definition 2. For an FT Lyapunov function V as in Definition 1 and a constant C, $m_V < C < M_V$, denote by $\mathcal{L}_{V,C}$ the connected component of $\{(t,\mathbf{x}) \in M \mid (t,\mathbf{x}) \in I \text{ or } V(t,\mathbf{x}) \leq C\}$ containing I.

Definition 3. We call a set $U \subset M$ forward-M invariant iff $(s,\mathbf{x}) \in U$ implies $\boldsymbol{\phi}(t,s,\mathbf{x}) \in U$ for all $s \leq t \leq T$.

Clearly, if U is forward-M invariant, then so is $U \cap \{(t,\mathbf{x}) \in M \mid t \geq s\}$ for all $s \in [0,T]$.

We have

Theorem 2. *For every $m_V < C < M_V$ the set $\mathcal{L}_{V,C}$ is forward-M invariant. Additionally, the sets*

$$I_V := \bigcap_{m_V<C<M_V}\mathcal{L}_{V,C} \quad \textit{and} \quad O_V := \bigcup_{m_V<C<M_V}\mathcal{L}_{V,C}$$

are forward-M invariant.

PROOF. Let $(s,\mathbf{x}) \in \mathcal{L}_{V,C}$ and assume there is a t, $s < t \leq T$, such that $(t,\boldsymbol{\phi}(t,s,\mathbf{x})) \notin \mathcal{L}_{V,C}$. Since $\{(\tau,\boldsymbol{\phi}(\tau,s,\mathbf{x})) \in X \mid s \leq \tau \leq t\}$ is connected and $\mathcal{L}_{V,C}$ is a closed subset of the open set M (both in the topology of X), there exists a t^*, $s < t^* \leq t$, such that $\{(\tau,\boldsymbol{\phi}(\tau,s,\mathbf{x})) \in X \mid s \leq \tau \leq t^*\} \subset M$, $(t^*,\boldsymbol{\phi}(t^*,s,\mathbf{x})) \in M \setminus \mathcal{L}_{V,C}$, and $V(t^*,\boldsymbol{\phi}(t^*,s,\mathbf{x})) > C$. Set

$$\mathcal{A} := \{\tau \in [s,t^*] \mid (\tau,\boldsymbol{\phi}(\tau,s,\mathbf{x})) \in \mathrm{cl}_X(I))\}$$

and define

$$s^* := \begin{cases} s, & \text{if } \mathcal{A} = \emptyset, \\ \sup\mathcal{A}, & \text{if } \mathcal{A} \neq \emptyset. \end{cases}$$

We have that $s \leq s^* < t^*$, and $(\tau,\boldsymbol{\phi}(\tau,s,\mathbf{x})) \in \mathcal{L}_{V,C} \setminus \mathrm{cl}_X(I)$ for all $s^* < \tau < t^*$. Thus

$$V(s^*,\boldsymbol{\phi}(s^*,s,\mathbf{x})) \leq C < V(t^*,\boldsymbol{\phi}(t^*,s,\mathbf{x})),$$

contradictory to Lemma 1.

Theorem 3. *Consider the FT Lyapunov function from Definition 1 and assume there are open and connected sets $S, R \subset \mathbb{R}^n$ and a number C, $m_V < C < M_V$, such that*

$$\mathrm{cl}(I(0)) \subset S \subset \mathrm{cl}(S) \subset M(0), \tag{2}$$
$$V(0,\mathbf{x}) \le C \text{ for all } \mathbf{x} \in \mathrm{cl}(S) \setminus I(0), \tag{3}$$

and

$$\mathrm{cl}(I(T)) \subset E \subset \mathrm{cl}(E) \subset \mathrm{int}(M(T)), \tag{4}$$
$$V(0,\mathbf{x}) > C \text{ for all } \mathbf{x} \in M(T) \setminus \mathrm{int}(E). \tag{5}$$

Then $\mathcal{L}_{V,C}$ is an M-forward invariant set such that $S \subset \mathcal{L}_{V,C}(0)$ and $\mathcal{L}_{V,C}(T) \subset E$. Especially, the Poincare mapping $\Phi(\boldsymbol{\xi}) := \boldsymbol{\phi}(T, 0, \boldsymbol{\xi})$ fulfills $\Phi(S) \subset E$.

Proof. That $\mathcal{L}_{V,C}$ is M-forward invariant follows directly from Theorem 2. Further, $S \subset \mathcal{L}_{V,C}(0)$ follows from (3) and $\mathcal{L}_{V,C}(T) \subset E$ follows from (5) because S and E are connected. The proposition on the Poincare mapping is a direct consequence.

The system (1) is said to P-periodic for a $P > 0$, iff $\mathbf{f}(t+P,\mathbf{x}) = \mathbf{f}(t,\mathbf{x})$ for all $(t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$. If (1) is P-periodic, then clearly whenever $\boldsymbol{\phi}(t,s,\mathbf{x})$ is defined for some $s, t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then so is $\boldsymbol{\phi}(t+P, s+P, \mathbf{x})$ and $\boldsymbol{\phi}(t+P, s+P, \mathbf{x}) = \boldsymbol{\phi}(t,s,\mathbf{x})$. An FT Lyapunov function for a time-periodic system can deliver more information on the system trajectories than Theorem 3. This is the subject of the next corollary. Recall that a set $U \subset \mathbb{R} \times \mathbb{R}^n$ is called forward invariant for the system (1) iff $(s,\mathbf{x}) \in U$ implies $\boldsymbol{\phi}(t,s,\mathbf{x}) \in U$ for all $t \ge s$. For a number $P \in \mathbb{R}$ we define the following equivalence relation on $\mathbb{R}$:

$$s \equiv_P t \ \text{ iff } \ s = t + kP \ \text{ for some } k \in \mathbb{Z}.$$

A direct consequence of Theorem 3 and the properties of the solution trajectories of periodic systems is:

Corollary 4. *Assume that the system in Theorem 3 is T-periodic and that the Lyapunov function additionally fulfills $\mathcal{L}_{V,C}(T) \subset \mathcal{L}_{V,C}(0)$. Then*

$$\mathcal{L}_{V,C}^{\equiv_T} := \{(t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n \,|\, (s,\mathbf{x}) \in \mathcal{L}_{V,C} \ \text{ for some } s \equiv_T t\}$$

is forward invariant.

3. Algorithm

The algorithm to compute an FT Lyapunov function for system (1) first generates a system specific LP problem using a triangulation of the domain of the FT Lyapunov functions to be computed. A solution to this LP problem is then used to parameterize an FT Lyapunov function for the system that is continuous and affine on each simplex of the triangulation (CPA).

Let us first recall the definition of an m-simplex, before we define triangulations and CPA functions suited for our needs. Let $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m)$ be an ordered $(m+1)-$tuple of vectors in $\mathbb{R}^n$. The set of all **convex combinations** of these vectors is denoted by

$\mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m) := \{\sum_{i=0}^m \lambda_i \mathbf{x}_i : 0 \leq \lambda_i \leq 1, \sum_{i=0}^m \lambda_i = 1\}$. The vectors $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m)$ are called **affinely independent** if the vectors $\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_m - \mathbf{x}_0$ are linearly independent. If $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m)$ are affinely independent, then the set $\mathfrak{S} := \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m)$ is called an m-**simplex** and the vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m$ are said to be its **vertices**. Note that the m-dimensional measure of an m-simplex is a finite positive number. We consider two simplices $\mathfrak{S}_1$ and $\mathfrak{S}_2$ to be equal if they are equal as sets, although we represent them as the convex combination of an ordered tuple of its vertices. A **face** of an m-simplex $\mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m)$ is a k-simplex $\mathrm{co}(\mathbf{x}_{i_0}, \mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_k})$, where $0 \leq k < m$ and the $0 \leq i_0, i_1, \ldots, i_k \leq m$ are pairwise different integers.

An m-dimensional triangulation $\mathcal{T}$ in $\mathbb{R}^n$, $m \leq n$, is a set of countably many m-simplices $\mathfrak{S}_\nu \subset \mathbb{R}^n$. To simplify notations we often write $\mathcal{T} = \{\mathfrak{S}_\nu\}$, where it is to be understood that $\nu \in [1, N]_\mathbb{Z}$ if $\mathcal{T}$ has a finite number N of (different) simplices, or $\nu \in \mathbb{N}$, if $\mathcal{T}$ is infinite. We will briefly describe triangulations suited for our needs; for more details, cf. [8, 9].

Definition 4 (Triangulation). Let $\mathcal{T}$ be a set of m-simplices $\mathfrak{S}_\nu$ in $\mathbb{R}^n$. $\mathcal{T}$ is called an m-dimensional triangulation if for every $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$, $\nu \neq \mu$, either $\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \emptyset$ or $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ intersect in a common face.

For a triangulation $\mathcal{T}$ we define

$$\mathcal{V}_\mathcal{T} := \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{x} \text{ is a vertex of a simplex in } \mathcal{T}\}$$

and

$$\mathcal{D}_\mathcal{T} := \bigcup_{\mathfrak{S}_\nu \in \mathcal{T}} \mathfrak{S}_\nu.$$

We call $\mathcal{V}_\mathcal{T}$ the **vertex set** of the triangulation $\mathcal{T}$ and we say that $\mathcal{T}$ is a triangulation of the set $\mathcal{D}_\mathcal{T}$.

Definition 5 (CPA function). Let $\mathcal{T} = \{\mathfrak{S}_\nu\}$ be an n-dimensional triangulation of a set $\mathcal{D}_\mathcal{T} \subset \mathbb{R}^n$. A continuous and piecewise affine (CPA) function $P : \mathcal{D}_\mathcal{T} \to \mathbb{R}$ can be defined by fixing its value at every vertex in the vertex set $\mathcal{V}_\mathcal{T}$.

More exactly, assume that for every $\mathbf{x} \in \mathcal{V}_\mathcal{T}$ we are given a number $P_\mathbf{x} \in \mathbb{R}$. Then we can uniquely define a continuous function $P : \mathcal{D}_\mathcal{T} \to \mathbb{R}$ through:

i. $P(\mathbf{x}) := P_\mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}_\mathcal{T}$,
ii. P is affine on every simplex $\mathfrak{S}_\nu \in \mathcal{T}$, i.e. there is a vector $\mathbf{a}_\nu \in \mathbb{R}^n$ and a number $b_\nu \in \mathbb{R}$, such that $P(\mathbf{x}) = \mathbf{a}_\nu^\top \mathbf{x} + b_\nu$ for all $\mathbf{x} \in \mathfrak{S}_\nu$.

The set of all such continuous and piecewise affine functions $\mathcal{D}_\mathcal{T} \to \mathbb{R}$ fulfilling (i) and (ii) is denoted by $\mathrm{CPA}[\mathcal{T}]$ or $\mathrm{CPA}[\{\mathfrak{S}_\nu\}]$.

For $P \in \mathrm{CPA}[\mathcal{T}]$ and $\mathfrak{S}_\nu \in \mathcal{T}$ we define $\nabla P_\nu = \nabla P|_{\mathfrak{S}_\nu} := \mathbf{a}_\nu$, where $\mathbf{a}_\nu \in \mathbb{R}^n$ is as in (ii). Note that $\nabla P_\nu = \mathbf{a}_\nu$ is a constant for every simplex $\mathfrak{S}_\nu$.

Remark 6. If $\mathbf{x} \in \mathfrak{S}_\nu = \mathrm{co}\,(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_m^\nu) \in \mathcal{T}$, then $\mathbf{x}$ can be written uniquely as a convex combination $\mathbf{x} = \displaystyle\sum_{i=0}^m \lambda_i \mathbf{x}_i^\nu$, $0 \leq \lambda_i \leq 1$ for all $i = [0, m]_\mathbb{Z}$, and $\sum_{i=0}^m \lambda_i = 1$, of the vertices of $\mathfrak{S}_\nu$ and

$$P(\mathbf{x}) = P\left(\sum_{i=0}^m \lambda_i \mathbf{x}_i^\nu\right) = \sum_{i=0}^m \lambda_i P(\mathbf{x}_i^\nu) = \sum_{i=0}^m \lambda_i P_{\mathbf{x}_i^\nu}.$$

For a simplex $\mathfrak{S}_\nu = \text{co}\,(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu) \in \mathcal{T}$ we define its **shape matrix** $X_\nu \in \mathbb{R}^{n\times n}$ through

$$X_\nu := (\mathbf{x}_1^\nu - \mathbf{x}_0^\nu, \mathbf{x}_2^\nu - \mathbf{x}_0^\nu, \ldots, \mathbf{x}_n^\nu - \mathbf{x}_0^\nu)^\top .$$

Thus, the matrix X_ν is defined by writing the entities of the vector $\mathbf{x}_i^\nu - \mathbf{x}_0^\nu$ in the i-th row of X_ν for $i \in [1, n]_\mathbb{Z}$.

It is not difficult to derive a formula for ∇P_ν in terms of the shape matrix X_ν of $\mathfrak{S}_\nu$ and the values of the affine functions P at the vertices of $\mathfrak{S}_\nu$, cf. [10, Remark 9]. If $\mathcal{T}$ is a triangulation, $P \in \text{CPA}[\mathcal{T}]$, and $\mathfrak{S}_\nu = \text{co}\,(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu) \in \mathcal{T}$. Then $\nabla P_\nu = X_\nu^{-1}\mathbf{p}$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)^\top$ is a column vector with $p_i := P_{\mathbf{x}_i^\nu} - P_{\mathbf{x}_0^\nu}$ for $i \in [1, n]_\mathbb{Z}$.

Before we can define our system specific LP problem, of which every feasible solution parameterizes an FT Lyapunov function for the system, we need some preparation:

Consider the system (1) and define

$$\widetilde{\mathbf{f}} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n, \;\; \widetilde{\mathbf{f}}(t, \mathbf{x}) = (1, \mathbf{f}(t, \mathbf{x})).$$

Let $T > 0$ and let $\mathcal{T}_M$ and $\mathcal{T}_I$ be $(n+1)$-dimensional triangulations, $\mathcal{T}_I \subset \mathcal{T}_M$, and of which the simplices are in $X := [0, T] \times \mathbb{R}^n$. Assume further that the sets

$$M := \text{int}_X(\bigcup_{\mathfrak{S}\in\mathcal{T}_M} \mathfrak{S}) \;\; \text{and} \;\; I := \text{int}_X(\bigcup_{\mathfrak{S}\in\mathcal{T}_I} \mathfrak{S})$$

are as in Definition 1. That is the sets M and I are bounded connected sets that are open neighbourhoods of $[0, T] \times \{\mathbf{0}\}$ in the relative topology of X and $\text{cl}_X(I) \subset M$. Define the triangulation $\mathcal{T}_{D_V} := \mathcal{T}_M \setminus \mathcal{T}_I$. Then

$$D_V := \text{cl}_X(M) \setminus I = \bigcup_{\mathfrak{S}\in\mathcal{T}_{D_V}} \mathfrak{S}$$

is as in Definition 1 too.

The $(n + 1)$-dimensional simplices in $\mathcal{T}_M$ now generate n-dimensional triangulations $\mathcal{T}_{\text{bdy}_X(M)}$, $\mathcal{T}_{\text{bdy}_X(I)}$, $\mathcal{T}_{M(0)}$, and $\mathcal{T}_{M(T)}$ of $\text{bdy}_X(M)$, $\text{bdy}_X(I)$, $M(0)$, and $M(T)$ respectively in the obvious sense by taking intersection of the simplices in $\mathcal{T}_M$ with the corresponding sets. For example for $\text{bdy}_X(M)$ and $\mathcal{T}_{\text{bdy}_X(M)}$ we have $\text{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{T}_{\text{bdy}_X(M)}$ iff there is a permutation σ of $[0, n+1]_\mathbb{Z}$ such that $\text{co}\left(\mathbf{x}_{\sigma(0)}, \mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(n+1)}\right) \in \mathcal{T}_M$. Assume there are triangulations $\mathcal{T}_S \subset \mathcal{T}_{M(0)}$ and $\mathcal{T}_E \subset \mathcal{T}_{M(T)}$ such that the sets $S := \text{int}(\bigcup_{\mathfrak{S}\in\mathcal{T}_S} \mathfrak{S})$ and $E := \text{int}(\bigcup_{\mathfrak{S}\in\mathcal{T}_E} \mathfrak{S})$ are connected and fulfill the conditions (2) and (4) in Theorem 3 respectively. Let $\varepsilon > 0$ be an arbitrary (small) constant.

We now have everything in place to define our system specific LP problem:

LP Problem 5. *The variables of our LP problem are A, D, and $V_{(t,\mathbf{x})}$ for every $(t, \mathbf{x}) \in \mathcal{V}_{\mathcal{T}_{D_V}}$.*

The linear constraints of the LP problem are as follows: For every vertex $(t, \mathbf{x}) \in \mathcal{V}_{\mathcal{T}_{D_V}}$ we add the constraint(s) to the LP problem

1. $V_{(t,\mathbf{x})} \geq D + \varepsilon$ *if* $(t, \mathbf{x}) \in \text{bdy}_X(M)$ *or* $t = T$ *and* $\mathbf{x} \in M(T) \setminus E$
2. $V_{(t,\mathbf{x})} \leq A - \varepsilon$ *if* $(t, \mathbf{x}) \in \text{bdy}_X(I)$
3. $V_{(t,\mathbf{x})} \leq D$ *if* $t = 0$ *and* $\mathbf{x} \in \text{cl}(S)$

4. $V_{(t,\mathbf{x})} \geq A$ *if* $t = 0$ *and* $\mathbf{x} \in \mathrm{bdy}(S)$

For every simplex $\mathfrak{S}_\nu = \mathrm{co}\left((t_0, \mathbf{x}_0), (t_1, \mathbf{x}_1), \ldots, (t_{n+1}, \mathbf{x}_{n+1})\right) \in \mathcal{T}_{D_V}$ *and every vertex* $(t_i, \mathbf{x}_i)$, $i = [0, n+1]_{\mathbb{Z}}$, *of* $\mathfrak{S}_\nu$, *we add the constraint*

$$\mathbf{w}_\nu \cdot \widetilde{\mathbf{f}}(t_i, \mathbf{x}_i) + E_{\nu,i} \|\mathbf{w}_\nu\|_1 \leq 0 \tag{6}$$

to the problem. Here

$$\mathbf{w}_\nu := X_\nu^{-1} \left[V_{(t_1,\mathbf{x}_1)} - V_{(t_0,\mathbf{x}_0)},\ V_{(t_2,\mathbf{x}_2)} - V_{(t_0,\mathbf{x}_0)}, \ldots, V_{(t_{n+1},\mathbf{x}_{n+1})} - V_{(t_0,\mathbf{x}_0)}\right]^\top,$$

where X_ν *is the shape-matrix of* $\mathfrak{S}_\nu$,

$$X_\nu = \left[(t_1, \mathbf{x}_1) - (t_0, \mathbf{x}_0), (t_2, \mathbf{x}_2) - (t_0, \mathbf{x}_0), \ldots, (t_{n+1}, \mathbf{x}_{n+1}) - (t_0, \mathbf{x}_0)\right]^\top,$$

and

$$E_{\nu,i} := \frac{(n+1)B_\nu}{2} \|(t_i, \mathbf{x}_i) - (t_0, \mathbf{x}_0)\|_2 \left(h_\nu^0 + \|(t_i, \mathbf{x}_i) - (t_0, \mathbf{x}_0)\|_2\right), \tag{7}$$

$$\textit{where}\quad h_\nu^0 := \max_{j \in [0,n+1]_{\mathbb{Z}}} \|(t_j, \mathbf{x}_j) - (t_0, \mathbf{x}_0)\|_2 \quad \textit{and} \quad B_\nu \geq \max_{\substack{m,r,s \in [1,n+1]_{\mathbb{Z}} \\ \mathbf{z} \in \mathfrak{S}_\nu}} \left| \frac{\partial^2 \widetilde{f}_m}{\partial x_r \partial x_s}(\mathbf{z}) \right|.$$

Remark 7. The constraints (6) can be implemented as linear constraints using a standard trick. For each simplex $\mathfrak{S}_\nu$ one introduces the auxiliary variables C_j^ν, $j \in [1, n+1]_{\mathbb{Z}}$, and the constraints

$$-C_j^\nu \leq (\mathbf{w}_\nu)_j \leq C_j^\nu,\ j \in [1, n+1]_{\mathbb{Z}},$$

where $(\mathbf{w}_\nu)_j$ denotes the j-th component of the vector $\mathbf{w}_\nu$. Then the constraints (6) are replaced by the linear constraints

$$\mathbf{w}_\nu \cdot \widetilde{\mathbf{f}}(t_i, \mathbf{x}_i) + E_{\nu,i} \sum_{j=1}^{n+1} C_j^\nu \leq 0$$

for all $i \in [0, n+1]_{\mathbb{Z}}$.

Remark 8. Note that B_ν is any upper bound on the second-derivatives. Tight bounds are preferable but not necessary. Also note that it is obviously possible to use the simpler formula

$$E_{\nu,i} := (n+1)B_\nu h_\nu^2 \ \text{ with } h_\nu := \max_{\mathbf{x},\mathbf{y} \in \mathfrak{S}_\nu} \|\mathbf{x} - \mathbf{y}\|_2$$

for the $E_{\nu,i}$s. These $E_{\nu,i}$s are however larger than necessary.

A feasible solution to the LP Problem 5 parameterizes a CPA$[\mathcal{T}_{D_V}]$ FT Lyapunov function for the system (1). Before we prove this we recall a fact about approximations of functions by CPA functions.

Lemma 6. *Let $\mathfrak{S} := \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m)$ be an m-simplex in $\mathbb{R}^m$ and $g \in C^2(U)$, where $U \subset \mathbb{R}^m$ is an open set and $\mathfrak{S} \subset U$. Then we have the estimate*

$$\left| g(\mathbf{x}) - \sum_{i=0}^{m} \lambda_i g(\mathbf{x}_i) \right| \leq \frac{mB}{2} \sum_{i=0}^{m} \lambda_i \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left(\max_{j \in [0,m]_{\mathbb{Z}}} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2 \right),$$

where the λ_is are the barycentric coordinates of $\mathbf{x} = \sum_{i=0}^{m} \lambda_i \mathbf{x}_i$ and

$$B \geq \max_{\substack{r,s=\in[1,m]_{\mathbb{Z}} \\ \mathbf{z}\in\mathfrak{S}}} \left| \frac{\partial^2 g}{\partial x_r \partial x_s}(\mathbf{z}) \right|.$$

Proof. Follows directly from Proposition 4.1 and Lemma 4.2 in [2].

Theorem 7. *Assume the LP problem 5 has a feasible solution. Then the function $V \in \mathrm{CPA}[\mathcal{T}_{D_V}]$ defined by $V(t,\mathbf{x}) = V_{(t,\mathbf{x})}$ for all $(t,\mathbf{x}) \in \mathcal{V}_{D_\mathcal{T}}$ is an FT Lyapunov function for the system (1). Further, $\mathcal{L}_{D,V}$ is forward-M invariant (D is the value of the variable D in the LP problem), $S \subset \mathcal{L}_{D,V}(0)$, and $E \subset \mathcal{L}_{D,V}(T)$.*

Proof. We only have to establish that V is an FT Lyapunov function for the system, the other propositions follow directly from Theorem 3. The condition i) in Definition 1 of an FT Lyapunov function follows from the following estimates: $V(t,\mathbf{x}) > D$ for all $(t,\mathbf{x}) \in \mathrm{bdy}_X(M)$. This is enforced by the constraints 1. in the LP problem for all vertices $(t,\mathbf{x})$ in $\mathrm{bdy}_X(M)$ and for every $(t,\mathbf{x}) \in \mathrm{bdy}_X(M)$ the value $V(t,\mathbf{x})$ is the convex combination of such vertex values. Similarly $V(t,\mathbf{x}) < A$ for all $(t,\mathbf{x}) \in \mathrm{bdy}_X(I)$. The constraints 3. and 4. of the LP problem imply together that $A \leq V(0,\mathbf{x}) \leq D$ for all vertices $(0,\mathbf{x})$ such that $\mathbf{x} \in \mathrm{bdy}(S)$. Especially $A \leq D$. Thus

$$M_V := \min_{(t,\mathbf{x})\in\mathrm{bdy}_X(M)} V(t,\mathbf{x}) > D \geq A > \max_{(t,\mathbf{x})\in\mathrm{bdy}_X(I)} V(t,\mathbf{x}) =: m_V.$$

To prove the condition ii) in Definition 1 consider an arbitrary $(t,\mathbf{x}) \in \mathrm{int}(D_V)$. By virtue of the triangulation $\mathcal{T}_{D_V}$ there exists a $\rho > 0$ and a $\mathfrak{S}_\nu \in \mathcal{T}_{D_V}$ such that $(t+h, \mathbf{x}+h\mathbf{f}(t,\mathbf{x})) \in \mathfrak{S}_\nu$ for all $0 \leq h \leq \rho$. For every such $h > 0$ we have

$$\frac{V(t+h, \mathbf{x}+h\mathbf{f}(t,\mathbf{x})) - V(t,\mathbf{x})}{h} = \mathbf{w}_\nu \cdot \widetilde{\mathbf{f}}(t,\mathbf{x}).$$

Now $(t,\mathbf{x})$ can be written as a convex combination of the vertices of

$$\mathfrak{S}_\nu = \mathrm{co}\left((t_0^\nu, \mathbf{x}_0^\nu), (t_1^\nu, \mathbf{x}_1^\nu), \dots, (t_{n+1}^\nu, \mathbf{x}_{n+1}^\nu)\right), \quad \text{i.e. } (t,\mathbf{x}) = \sum_{i=0}^{n+1} \lambda_i (t_i^\nu, \mathbf{x}_i^\nu),$$

and by Lemma 6 and the definition of the $E_{\nu,i}$ we have

$$\left\| \widetilde{\mathbf{f}}(t,\mathbf{x}) - \sum_{i=0}^{n+1} \lambda_i \widetilde{\mathbf{f}}(t_0^\nu, \mathbf{x}_i^\nu) \right\|_\infty \leq \sum_{i=0}^{n+1} \lambda_i E_{\nu,i}.$$

Thus, by Hölder's inequality

$$
\begin{aligned}
\mathbf{w}_\nu \cdot \widetilde{\mathbf{f}}(t,\mathbf{x}) &\leq \sum_{i=0}^{n+1} \lambda_i \mathbf{w}_\nu \cdot \widetilde{\mathbf{f}}(t_i^\nu, \mathbf{x}_i^\nu) + \|\mathbf{w}_\nu\|_1 \left\| \widetilde{\mathbf{f}}(t,\mathbf{x}) - \sum_{i=0}^{n+1} \lambda_i \widetilde{\mathbf{f}}(t_0^\nu, \mathbf{x}_i^\nu) \right\|_\infty \\
&\leq \sum_{i=0}^{n+1} \lambda_i \underbrace{\left(\mathbf{w}_\nu \cdot \widetilde{\mathbf{f}}(t_i^\nu, \mathbf{x}_i^\nu) + \|\mathbf{w}_\nu\|_1 E_{\nu,i} \right)}_{\leq 0 \quad \text{by (6)}} \leq 0
\end{aligned}
$$

and it follows that

$$
D_{\mathbf{f}}^+ V(t,\mathbf{x}) = \limsup_{h \to 0+} \frac{V(t+h, \mathbf{x} + h\mathbf{f}(t,\mathbf{x})) - V(t,\mathbf{x})}{h} \leq 0.
$$

4. Examples

Our starting point is a regular triangulation $\mathcal{T}^{\text{std}}$ of $\mathbb{R}^{n+1}$, defined in Definition 10, whose vertices are the set $\mathbb{Z}^{n+1}$. For some illustrations of the simplices in $\mathcal{T}^{\text{std}}$ together with a discussion on how they can be efficiently generated, see e.g. [14, 9]. The only difference here is that we are triangulating $\mathbb{R} \times \mathbb{R}^n$ (time and space) instead of $\mathbb{R}^n$.

Remark 9. For the construction of our triangulations we use the set S_{n+1} of all permutations of the set $[1, n+1]_{\mathbb{Z}}$, the characteristic function $\chi_{\mathcal{J}}(i)$ equal to one if $i \in \mathcal{J}$ and equal to zero if $i \notin \mathcal{J}$, and the standard orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1}$ of $\mathbb{R} \times \mathbb{R}^n$. Further, we use the functions $\mathbf{R}^{\mathcal{J}} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, defined for every $\mathcal{J} \subset [1, n+1]_{\mathbb{Z}}$ by

$$
\mathbf{R}^{\mathcal{J}}(\mathbf{x}) := \sum_{i=1}^{n+1} (-1)^{\chi_{\mathcal{J}}(i)} x_i \mathbf{e}_i.
$$

$\mathbf{R}^{\mathcal{J}}(\mathbf{x})$ puts a minus in front of the coordinate x_i of $\mathbf{x}$ whenever $i \in \mathcal{J}$.

Definition 10 (Basic triangulation $\mathcal{T}^{\text{std}}$). The triangulation $\mathcal{T}^{\text{std}}$ consists of the $(n+1)$-dimensional simplices

$$
\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \operatorname{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_{n+1}^{\mathbf{z}\mathcal{J}\sigma} \right)
$$

for all $\mathbf{z} \in \mathbb{N}_0^{n+1}$, all $\mathcal{J} \subset [1, n+1]_{\mathbb{Z}}$, and all $\sigma \in S_{n+1}$ (cf. Remark 9 for notations), where

$$
\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} := \mathbf{R}^{\mathcal{J}} \left(\mathbf{z} + \sum_{j=1}^{i} \mathbf{e}_{\sigma(j)} \right) \quad \text{for } i \in [0, n+1]_{\mathbb{Z}}. \tag{8}
$$

For constructing triangulations of I and M in the examples below we start with grids of the form $\mathbb{G} := [0, T^*]_{\mathbb{Z}} \times [-X^*, X^*]_{\mathbb{Z}} \subset \mathbb{Z} \times \mathbb{Z}$ or $\mathbb{G} := [0, T^*]_{\mathbb{Z}} \times [-X^*, X^*]_{\mathbb{Z}}^2 \subset \mathbb{Z} \times \mathbb{Z}^2$. Here $T^* > 0$ and $X^* > 0$ are some given integers. These grids are then mapped to $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}^2$ respectively by using a mapping $\mathbf{F} : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{R} \times \mathbb{R}^n$, $n = 1, 2$, dependent on two parameters $c_t > 0$ and $c_{\mathbf{x}} > 0$. The formula for $\mathbf{F}$ is

$$
\mathbf{F}(i_t, \mathbf{j}_{\mathbf{x}}) := (c_t i_t, c_x \mathbf{F}_{\mathbf{x}}(\mathbf{j}_{\mathbf{x}})), \tag{9}
$$

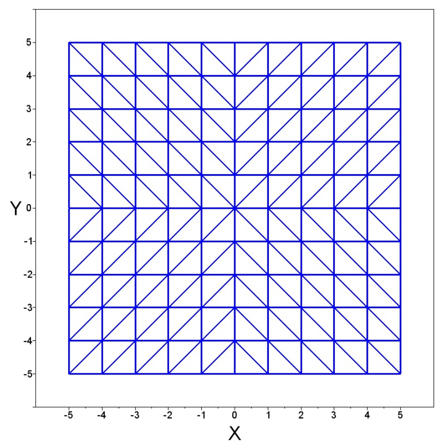
where

$$\mathbf{F_x}(\mathbf{0}) := \mathbf{0} \text{ and } \mathbf{F_x}(\mathbf{x}) = \frac{\|\mathbf{x}\|_\infty}{\|\mathbf{x}\|_2}\mathbf{x} \text{ if } \mathbf{x} \neq \mathbf{0}.$$

A triangulation $\mathcal{T}$ using such a grid $\mathbb{G}$ and a mapping $\mathbf{F}$ is now constructed in the following way:

For every simplex $\mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n+1}) \in \mathcal{T}^{\mathrm{std}}$ such that $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\} \subset \mathbb{G}$ we add the simplex $\mathrm{co}\,(\mathbf{F}(\mathbf{x}_0), \mathbf{F}(\mathbf{x}_1), \ldots, \mathbf{F}(\mathbf{x}_{n+1}))$ to the triangulation $\mathcal{T}$.

By using this procedure we create triangulations that are approximately cylinders with the line-segment $\{(t, \mathbf{0}) \in \mathbb{R} \times \mathbb{R}^n \,|\, 0 \le t \le c_t T^*\}$ as axis and with radius $c_\mathbf{x} X^*$. In $\mathbb{R} \times \mathbb{R}$ this is obvious and in $\mathbb{R} \times \mathbb{R}^2$ this is a consequence of the fact that $\mathbf{F}$ maps squares to circles, cf. Figure 1.



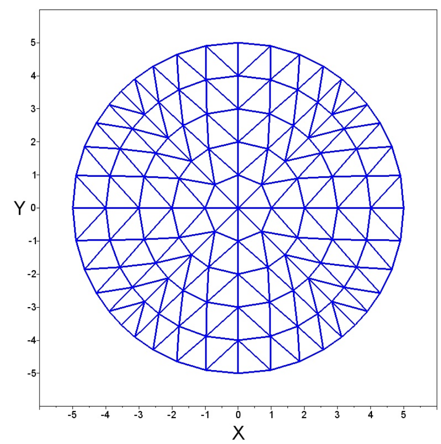


Figure 1: On the left are the simplices $\mathrm{co}\,(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) \in \mathcal{T}^{\mathrm{std}}$ such that $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\} \subset [-5, 5]^2$. On the right we depict the simplices $\mathrm{co}\,(\mathbf{F}(\mathbf{x}_0), \mathbf{F}(\mathbf{x}_1), \mathbf{F}(\mathbf{x}_2))$. Note that the square $[-5, 5]^2$ becomes approximately a circle with radius 5.

As described in LP Problem 5 the simplices in the triangulation $\mathcal{T}_M$, of which the simplices are subsets of $\mathbb{R} \times \mathbb{R}^n$, generate the n-dimensional triangulations $\mathcal{T}_{M(0)}$ and $\mathcal{T}_{M(T)}$. The triangulation $\mathcal{T}_S \subset \mathcal{T}_{M(0)}$ in the examples is created by specifying a grid $\mathbb{G}_S \subset \mathbb{Z}^n$. A simplex in $\mathcal{T}_S$ is an n-face of a simplex $\mathfrak{S} = \mathrm{co}\,(\mathbf{F}(\mathbf{x}_0), \mathbf{F}(\mathbf{x}_1), \ldots, \mathbf{F}(\mathbf{x}_{n+1})) \in \mathcal{T}_M$ such that all but one vertex $\mathbf{F}(\mathbf{x}_j)$ of $\mathfrak{S}$ lie in the hyperplane $\{0\} \times \mathbb{R}^n$ and such that $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\} \setminus \{\mathbf{x}_j\} \subset \{0\} \times \mathbb{G}_S$. In an identical manner the simplices in $\mathcal{T}_E \subset \mathcal{T}_{M(T)}$ are specified by a grid $\mathbb{G}_E \subset \mathbb{Z}^n$.

In [7] a numerical method was developed to compute Lyapunov functions on finite-time intervals in the sense of [12]. The Lyapunov functions considered there are, however, somewhat different than our FT Lyapunov functions and must fulfill considerably stricter criteria. More exactly, they must fulfill $V(0, \mathbf{x}) = V(T, \mathbf{x}) = \|\mathbf{x}\|^2$ and decrease linearly with respect to time along solutions. Especially, solution trajectories must fulfill $\|\boldsymbol{\phi}(T, 0, \mathbf{x})\| < \|\mathbf{x}\|$ for all $\mathbf{x}$ in a neighbourhood of the origin if such a Lyapunov function exists. The numerical method in [7] uses radial basis functions and meshless collocation to solve the PDE $V''(t,\mathbf{x})=0$ with appropriate boundary values, where V'' denotes the second orbital derivative (orbital derivative of the orbital derivative) along the system trajectories. In Examples 2 and 3 below we compute FT Lyapunov functions in the sense of Definition 1 for systems studied in [7] and discuss the differences.

The examples were programmed in C++ and using the Armadillo linear algebra library [22] and the LP problems generated were solved using the state of the art Gurobi Optimizer. The plots for Examples 1 and 2 were drawn with Scilab and the plots for Examples 3 and 4 with Matlab. We set $\varepsilon = 10^{-3}$ in all the examples. The computer we used has a i4790K@4600MHz CPU and 32GB RAM.

4.1. Example 1

The first example is the system

$$\dot{x} = xt\cos t. \tag{10}$$

By separation of variables it is easily seen that its solution is

$$\phi(t,s,x) = x\cdot\frac{e^{\cos t + t\sin t}}{e^{\cos s + s\sin s}}.$$

Especially, the null solution is not stable. Simple analysis shows that $t\mapsto|\phi(t,0,x)| = |x|e^{\cos t+t\sin t-1}$ has a local maximum at $t=\pi/2$ and a local minimum at $t=3\pi/2$. Further, $|\phi(\pi/2,0,x)/x| = e^{\pi/2-1}\approx 1.77$ and $|\phi(3\pi/2,0,x)/x| = e^{-3\pi/2-1}\approx 0.0033$. From $t=3\pi/2$ on $t\mapsto|\phi(t,0,x)/x|$ grows fast; for example $|\phi(5\pi/2,0,x)/\phi(3\pi/2,0,x)|\approx 286751$.

We computed an FT Lyapunov function for the system on $M:=[0,1.5\pi]\times[-2.355,2.355]$ with $I:=[0,1.5\pi]\times[-0.2355,0.2355]$, $S=\{0\}\times[-1.1775,1.1775]$, and $E:=\{1.5\pi\}\times[-0.35325,0.35325]$. To create the triangulations $\mathcal{T}_M$ and $\mathcal{T}_I$ we used the procedure described above using the mapping

$$\mathbf{F}(i_t,j_x) := \left(\frac{1.5\pi}{100}i_t,\ \frac{2.355}{40}j_x\right)$$

and the grid $[0,100]_{\mathbb{Z}}\times[-40,40]_{\mathbb{Z}}$ for $\mathcal{T}_M$, $[0,100]_{\mathbb{Z}}\times[-4,4]_{\mathbb{Z}}$ for $\mathcal{T}_I$, $[-20,20]_{\mathbb{Z}}$ for $\mathcal{T}_S$, and $[-6,6]_{\mathbb{Z}}$ for $\mathcal{T}_E$. We set $B_\nu := \max_{(t,x)\in\mathfrak{S}_\nu}\max(t+1,(2+t)|x|)$. Setting up the LP problem with 40,184 variables and 139,723 constraints took 0.26 sec. Solving the LP problem took 2.29 sec. using the Gurobi barrier method. In Figure 2 the computed FT Lyapunov function is depicted and in Figure 3 the M-forward set $\mathcal{L}_{V,D}$ delivered by the Lyapunov function, where $D:=0.0291863$ is the value of the corresponding variable D in the LP problem, is depicted.

4.2. Example 2

The second example is

$$\dot{x} = x\left(0.25-(1-t)^2\right)+x^3. \tag{11}$$

It is taken from [7], where it is shown that with

$$\alpha_t := 2\int_0^t e^{-2\tau/3+2\tau^2-3\tau/2}d\tau$$

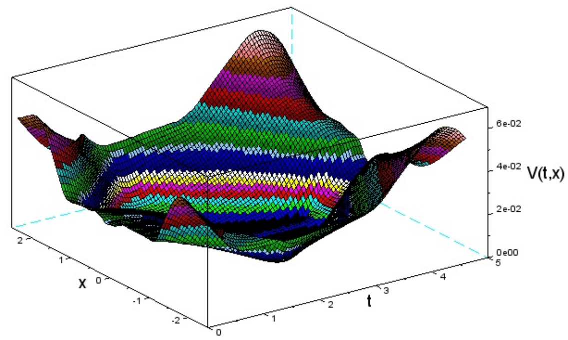


Figure 2: FT Lyapunov function $V(t,x)$ delivered by the algorithm for system (10) on $[0, 1.5\pi] \times [-2.355, 2.355]$.

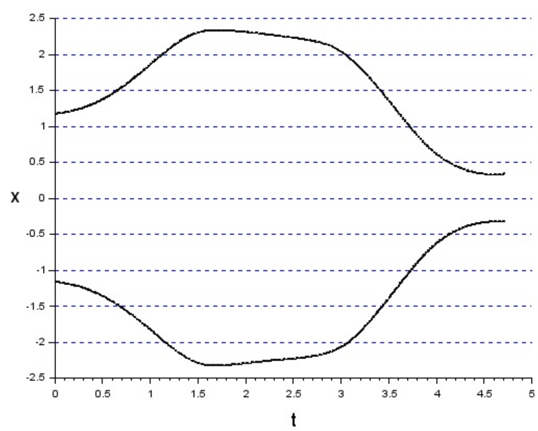


Figure 3: The M-forward invariant set $\mathcal{L}_{V,D}$ for the FT Lyapunov function in Figure 2 for system (10).

its solution is given with

$$\phi(t,0,x) = xe^{-t^3/3+t^2-3t/4}\left(1 - x^2\alpha_t\right)^{-\frac{1}{2}}.$$

For a fixed $x \neq 0$ the solution is defined on the interval $(-\infty, s_x)$, where s_x is such that $\alpha_{s_x} = x^{-2}$. Especially, for every initial value $x \neq 0$ the solution $t \mapsto \phi(t,0,x)$ diverges at s_x.

We computed two FT Lyapunov functions for the system. In both cases we set $B_\nu := \max_{(t,x)\in\mathfrak{S}_\nu} \max(6|x|, 2|t-1|)$. In the first computation we set $M := [0,2] \times [-0.5, 0.5]$, $I := [0,2] \times [-0.125, 0.125]$, $S = \{0\} \times [-0.25, 0.25]$, and $E = \{2\} \times [-0.25, 0.25]$. We used the mapping

$$\mathbf{F}(i_t, j_x) := \left(\frac{2}{100}i_t,\ \frac{0.5}{60}j_x\right)$$

and the grid $[0,100]_{\mathbb{Z}} \times [-60,60]_{\mathbb{Z}}$ for creating $\mathcal{T}_M$, $[0,100]_{\mathbb{Z}} \times [-15,15]_{\mathbb{Z}}$ for $\mathcal{T}_I$, and $[-30,30]_{\mathbb{Z}}$ for $\mathcal{T}_S$ and $\mathcal{T}_E$.

Setting up the LP problem with 60,224 variables and 198,535 constraints took 0.36 sec. Solving the LP problem took 9.73 sec. using the Gurobi barrier method. In Figure 4 the computed FT Lyapunov function is depicted and in Figure 5 the M-forward set $\mathcal{L}_{V,D}$ delivered by the Lyapunov function, where $D := 0.0928254$ is the value of the corresponding variable D in the LP problem, is depicted.

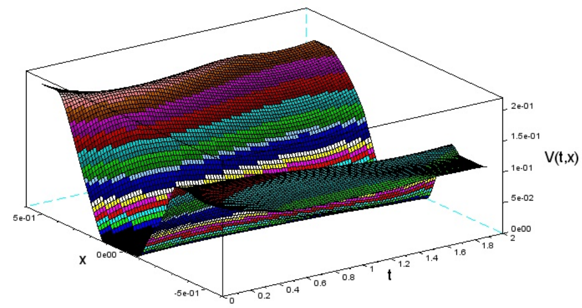


Figure 4: FT Lyapunov function $V(t,x)$ delivered by the algorithm for system (11) on $[0,2] \times [-0.5,0.5]$.

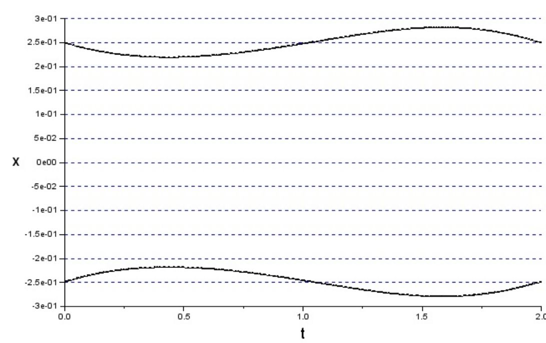


Figure 5: The M-forward invariant set $\mathcal{L}_{V,D}$ for the FT Lyapunov function in Figure 4 for system (11).

Then we did another computation with $M := [0,3]\times[-0.5,0.5]$, $I := [0,3]\times[-0.025,0.025]$, $S = \{0\} \times [-0.3,0.3]$, and $E = \{3\} \times [0.05,0.05]$. We used the mapping

$$\mathbf{F}(i_t, j_x) := \left(\frac{3}{100} i_t,\ \frac{0.5}{60} j_x \right)$$

and the grid $[0,100]_{\mathbb{Z}} \times [-60,60]_{\mathbb{Z}}$ for creating $\mathcal{T}_M$, $[0,100]_{\mathbb{Z}} \times [-3,3]_{\mathbb{Z}}$ for $\mathcal{T}_I$, $[-36,36]_{\mathbb{Z}}$ for $\mathcal{T}_S$, and $[-6,6]_{\mathbb{Z}}$ for $\mathcal{T}_E$. Setting up the LP problem with 60,224 variables and 212,995 constraints took 0.40 sec. Solving the LP problem took 8.76 sec. using the Gurobi barrier method. In Figure 6 the computed FT Lyapunov function is depicted and in Figure 7 the M-forward set $\mathcal{L}_{V,D}$ delivered by the Lyapunov function, where $D := 0.0356067$ is the value of the corresponding variable D in the LP problem, is depicted. In Figure 8 we draw

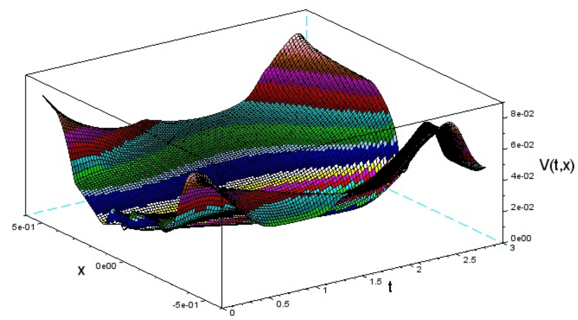


Figure 6: FT Lyapunov function $V(t,x)$ delivered by the algorithm for system (11) on $[0,3] \times [-0.5, 0.5]$.

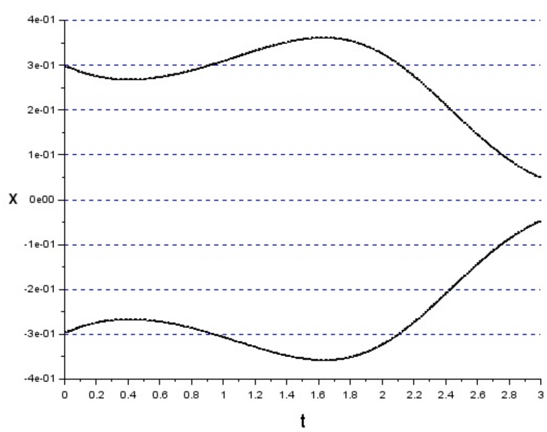


Figure 7: The M-forward invariant set $\mathcal{L}_{V,D}$ for the FT Lyapunov function in Figure 6 for system (11).

some additional level-sets $\mathcal{L}_{V,C}$ together with $\mathcal{L}_{V,D}$ and I. Note that if I is not completely enclosed in $\mathcal{L}_{V,C}$, then $\mathcal{L}_{V,C}$ it is not necessarily M-forward invariant. Indeed, the value D in LP Problem 5 is optimized in such a way that $\mathcal{L}_{V,D}$ is the largest M-forward invariant set delivered by the FT Lyapunov function computed.

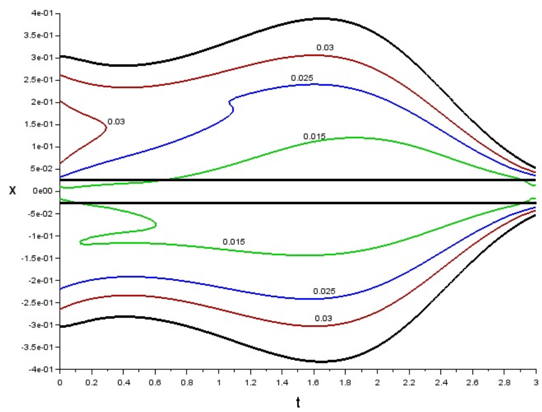


Figure 8: The M-forward invariant sets $\mathcal{L}_{V,0.025}$ (blue) and $\mathcal{L}_{V,0.03}$ (red) for the FT Lyapunov function in Figure 6 for system (11). The sets $\mathcal{L}_{V,D}$ and I are drawn in black. Note that $\mathcal{L}_{V,0.015}$ (green) is not necessarily M-forward invariant because it does not enclose I completely.

4.3. Example 3

The third example is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(0.25 - (t-1)^2) + x^2 y \\ y(-1 + (t-1)^2) + xy^2 \end{pmatrix}. \tag{12}$$

Like Example 2 it is taken from [7].

We computed an FT Lyapunov function for the system using the mapping

$$\mathbf{F}(i_t, \mathbf{i_x}) = \left(\frac{2.5}{120} i_t, \frac{0.7}{8} \mathbf{F_x}(\mathbf{i_x}) \right)$$

and the grid $[0,120]_\mathbb{Z} \times [-8,8]^2_\mathbb{Z}$ for $\mathcal{T}_M$, $[0,120]_\mathbb{Z} \times [-3,3]^2_\mathbb{Z}$ for $\mathcal{T}_I$, and $[-6,6]^2_\mathbb{Z}$ for $\mathcal{T}_S$ and $\mathcal{T}_E$. Thus M is approximately the cylinder $[0,2.5] \times \mathcal{B}^2_{0.7}$, I approximately the cylinder $[0,2.5] \times \mathcal{B}^2_{0.2625}$, S approximately the circular disc $\{0\} \times \mathcal{B}^2_{0.525}$, and E approximately the circular disc $\{2.5\} \times \mathcal{B}^2_{0.525}$. We set $B_\nu := 2\max_{(t,x,y)\in\mathfrak{S}_\nu} \max(|x|,|y|,|x(t-1)|,|y(t-1)|)$.

Setting up the LP problem with 587,932 variables and 2,303,769 constraints took 33.39 sec. Solving the LP problem took 91.37 sec. using the Gurobi barrier method. In Figure 9 the M-forward set $\mathcal{L}_{V,D}$ delivered by the computed FT Lyapunov function is depicted, where $D := 0.0174653$ is the value of the corresponding variable D in the LP problem. These results are comparable to the ones in [7], but computed using a slightly longer time-interval.

4.4. Example 4

The fourth example is the Duffing equation modeling a mass-spring system with a hardening spring, linear viscous damping, and a periodic external force

$$m\ddot{x} + c\dot{x} + kx + ka^2x^3 = A\cos(\omega t),$$

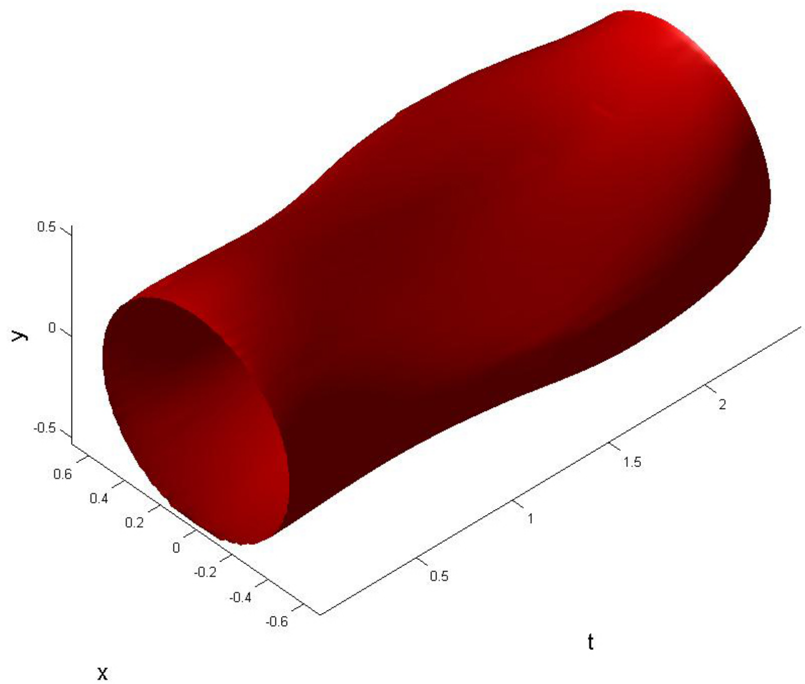


Figure 9: The M-forward invariant set $\mathcal{L}_{V,D}$ for the FT Lyapunov function computed for system (12).

see, e.g. Section 1.2.3 and Examples 4.5, 4.6, and 4.24 in [17]. With $y = \dot{x}$ and $c = k = m = 1$ we can write it as the system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x(1 + a^2x^2) - y + A\cos(\omega t) \end{pmatrix}. \tag{13}$$

For $A = 0$ the system is autonomous and the zero solution is asymptotically stable. For $A \neq 0$ the system does not posses a stationary solution, however, one can show that solutions are globally uniformly bounded. This means that there exists a constant $C > 0$ such that $\limsup_{t\to\infty} \|\boldsymbol{\phi}(t, s, \boldsymbol{\xi})\| < C$ for all $s \in \mathbb{R}$ and $\boldsymbol{\xi} \in \mathbb{R}^n$.

We fixed the parameters as $a = 0.1$, $A = 0.15$, and $\omega = 1$ and computed an FT Lyapunov function for the system using the mapping

$$\mathbf{F}(i_t, \mathbf{i_x}) = \left(\frac{6\pi}{80} i_t, \frac{0.6}{7}\mathbf{F_x}(\mathbf{i_x})\right)$$

and the grid $[0, 80]_{\mathbb{Z}} \times [-7, 7]^2_{\mathbb{Z}}$ for $\mathcal{T}_M$, $[0, 80]_{\mathbb{Z}} \times [-2, 2]^2_{\mathbb{Z}}$ for $\mathcal{T}_I$, and $[-4, 4]^2_{\mathbb{Z}}$ for $\mathcal{T}_S$ and $\mathcal{T}_E$. Thus M is approximately the cylinder $[0, 6\pi] \times \mathcal{B}^2_{0.6}$, I approximately the cylinder $[0, 6\pi] \times \mathcal{B}^2_{0.1714286}$, S approximately the circular disc $\{0\} \times \mathcal{B}^2_{0.3428571}$, and E approximately the circular disc $\{6\pi\} \times \mathcal{B}^2_{0.3428571}$. We set $B_\nu := \max_{(t,x,y)\in\mathfrak{S}_\nu} \max(6a^2|x|, |A|\omega^2)$. Setting up the LP problem with 300,486 variables and 1,198,601 constraints took 8.99 sec. Solving the LP problem took 51.97 sec. using the Gurobi barrier method. In Figure 10 the M-forward set $\mathcal{L}_{V,D}$ delivered by the computed FT Lyapunov function is depicted, where $D := 0.00816909$ is the value of the corresponding variable D in the LP problem. Note, that since the system is 2π-periodic and $\mathcal{L}_{V,D}(6\pi) \subset \mathcal{L}_{V,D}(0)$ we have by Corollary 4 that $\mathcal{L}_{V,D}^{\equiv_{6\pi}}$ is forward invariant.

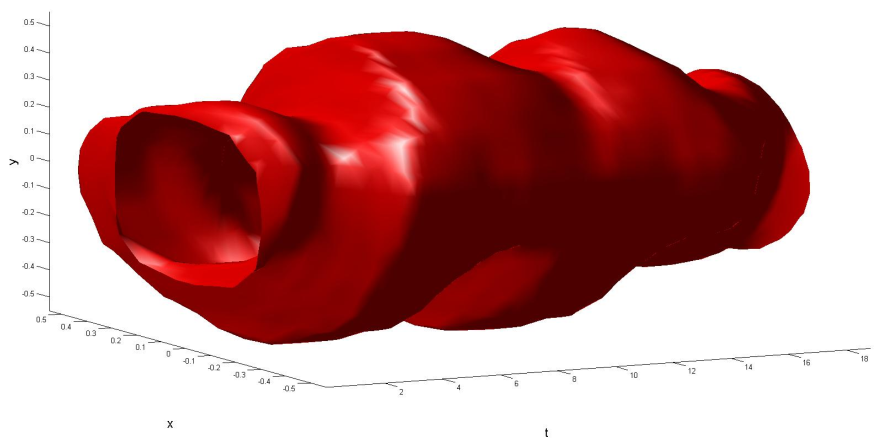


Figure 10: The M-forward invariant set $\mathcal{L}_{V,D}$ for the FT Lyapunov function computed for system (13).

5. Conclusions

We proposed a linear programming based algorithm for the computation of Lyapunov functions for nonlinear, nonautonomous systems on finite time intervals. Such a finite-time Lyapunov function as defined in Definition 1 delivers through its sub-level sets invariant sets of the dynamics of the system on the time-interval and thus delivers valuable information about the behaviour of the system's solution trajectories on the interval, cf. Theorem 3. For time-periodic systems one can prove the forward invariance of a neighbourhood of the line $\mathbb{R} \times \{\mathbf{0}\} \subset \mathbb{R} \times \mathbb{R}^n$, cf. Corollary 4, and thus get global information on the solution trajectories.

We state a system dependent linear programming (LP) problem in LP Problem 5 and we demonstrated in Theorem 7 that a CPA function parameterized from any feasible solution to the LP problem is a finite-time Lyapunov function for system (1). Finally, we gave four examples where we used the proposed method to compute finite-time Lyapunov functions for nonlinear systems.

References

[1] J. Anderson and A. Papachristodoulou. Advances in computational Lyapunov analysis using sum-of-squares programming. *Discrete Contin. Dyn. Syst. Ser. B*, 20(8):2361–2381, 2015.

[2] R. Baier, L. Grüne, and S. Hafstein. Linear programming based Lyapunov function computation for differential inclusions. *Discrete Contin. Dyn. Syst. Ser. B*, 17(1):33–56, 2012.

[3] F. Camilli, L. Grüne, and F. Wirth. A regularization of Zubov's equation for robust domains of attraction. In *Nonlinear control in the year 2000, Vol. 1 (Paris)*, volume 258 of *Lecture Notes in Control and Inform. Sci.*, pages 277–289. Springer, London, 2001.

[4] G. Chesi. LMI techniques for optimization over polynomials in control: a survey. *IEEE Trans. Automat. Control*, 55(11):2500–2510, 2010.

[5] A. Doban. *Stability domains computation and stabilization of nonlinear systems: implications for biological systems.* PhD thesis: Eindhoven University of Technology, 2016.

[6] P. Giesl. *Construction of Global Lyapunov Functions Using Radial Basis Functions*, volume 1904 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2007.

[7] P. Giesl. Construction of a finite-time Lyapunov function by meshless collocation. *Discrete Contin. Dyn. Syst. Ser. B*, 17(7):2387–2412, 2012.

[8] P. Giesl and S. Hafstein. Computation of Lyapunov functions for nonlinear discrete systems by linear programming. *J. Difference Equ. Appl.*, 20:610–640, 2014.

[9] P. Giesl and S. Hafstein. Implementation of a fan-like triangulation for the CPA method to compute Lyapunov functions. In *Proceedings of the 2014 American Control Conference*, pages 2989–2994 (no. 0202), Portland (OR), USA, 2014.

[10] P. Giesl and S. Hafstein. Revised CPA method to compute Lyapunov functions for nonlinear systems. *J. Math. Anal. Appl.*, 410:292–306, 2014.

[11] P. Giesl and S. Hafstein. Review of computational methods for Lyapunov functions. *Discrete Contin. Dyn. Syst.-Series B*, 20(8):2291–2331, 2015.

[12] P. Giesl and M. Rasmussen. Areas of attraction for nonautonomous differential equations on finite time intervals. *J. Math. Anal. Appl.*, 390:27–46, 2012.

[13] S. Hafstein. *An algorithm for constructing Lyapunov functions.* Monograph. Electron. J. Diff. Eqns., 2007.

[14] S. Hafstein. Implementation of simplicial complexes for CPA functions in C++11 using the Armadillo linear algebra library. In *Proceedings of SIMULTECH*, pages 49–57, Reykjavik, Iceland, 2013.

[15] W. Hahn. *Stability of Motion.* Springer, Berlin, 1967.

[16] P. Julian, J. Guivant, and A. Desages. A parametrization of piecewise linear Lyapunov functions via linear programming. *Int. J. Control*, 72(7-8):702–715, 1999.

[17] H. K. Khalil. *Nonlinear systems.* Prentice Hall, 3 edition, 2002.

[18] S. Margolis and W. Vogt. Control engineering applications of V. I. Zubov's construction procedure for Lyapunov functions. *IEEE Trans. Automat. Control*, 8(2):104–113, Apr 1963.

[19] S. Marinósson. Lyapunov function construction for ordinary differential equations with linear programming. *Dynamical Systems: An International Journal*, 17:137–150, 2002.

[20] A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Pranja, P. Seiler, and P. Parrilo. *SOSTOOLS: Sum of Squares Optimization Toolbox for MATLAB*. User's guide. Version 3.00 edition, 2013.

[21] P. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimiziation.* PhD thesis: California Institute of Technology Pasadena, California, 2000.

[22] C. Sanderson. Armadillo: An open source C++ linear algebra library for fast prototyping and computationally intensive experiments. Technical report, NICTA, 2010.

[23] S. Sastry. *Nonlinear Systems: Analysis, Stability and Control.* Springer, 1999.

[24] M. Vidyasagar. *Nonlinear Systems Analysis.* Number 42 in Classics in Applied Mathematics. SIAM, 2 edition, 2002.

[25] W. Walter. *Ordinary Differential Equation.* Springer, 1998.

[26] W. Walter. *Analysis 1.* Springer, 7. edition, 2004.