Common Lyapunov function computation for discrete-time systems

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***Abstract*— We describe an algorithm to compute a common Lyapunov function for a finite set of nonlinear discrete-time systems. In this algorithm a compact neighbourhood of a common equilibrium of the systems is subdivided into simplices and a linear programming problem is constructed. We prove that any feasible solution to this linear programming problem can be used to parameterize a common Lyapunov function for the systems that is continuous and affine on each of the simplices of the triangulation. We conclude the paper by applying our algorithm to two planar examples.**

I. INTRODUCTION

We consider a finite set of discrete-time systems

$$\mathbf{x}_{k+1} = \mathbf{g}_i(\mathbf{x}_k), \quad \mathbf{g}_i(\mathbf{0}) = \mathbf{0}, \quad i = 1, 2, \ldots, N. \tag{I.1}$$

The components of the vector fields $\mathbf{g}_i$ are assumed to be C^1, but are otherwise arbitrary. A common Lyapunov function for this set of systems is a continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ that is decreasing along solution trajectories of each of the systems on a neighbourhood of the origin. From the existence of a common Lyapunov function it follows that the origin is an asymptotically stable equilibrium for all the systems $\mathbf{x}_{k+1} = \mathbf{g}_i(\mathbf{x}_k)$. Further, one gets lower bounds on their common basin of attraction $\mathcal{N}$, i.e. the set of initial points $\mathbf{x}$ such that $\lim_{k\to\infty} \mathbf{g}_i^{\circ k}(\mathbf{x}) = \mathbf{0}$ for $i = 1, 2, \ldots, N$, where $\mathbf{g}^{\circ k}$ stands for the k-th iterate of the vector field $\mathbf{g}$. Hence, V is also a Lyapunov function and $\mathcal{N}$ is a lower bound for the basin of attraction for the arbitrary switched system

$$\mathbf{x}_{k+1} \in \bigcup_{i=1}^{n} \mathbf{g}_i(\mathbf{x}_k),$$

which has been intensively studied, cf. e.g. [2], [12], [16], [17], [1], [13], [19]. Analytical methods to generate Lyapunov functions for nonlinear systems are few and far between and only work for systems of some special algebraic structure. Hence, numerical methods are called for to generate Lyapunov functions, cf. e.g. the recent review article [5].

In [10] an algorithm was proposed to compute continuous and piecewise affine (CPA) Lyapunov functions for nonlinear continuous-time systems $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. In this method a subset of the state-space was subdivided into simplices, i.e. triangulated, and then a linear programming (LP) problem was constructed for the system. A feasible solution to this LP problem was used to parameterize a function that, if some a posteriori analysis delivered desired results, could be shown to be a Lyapunov function for the system. In [14], [15] a similar algorithm was proposed with a different LP problem and it was proved that any feasible solution to the LP problem parameterized a CPA Lyapunov function for the system (no a posteriori analysis needed). This algorithm has since been improved and adapted to different kinds of systems by various authors, including discrete-time systems [4], [11], [8]. In this paper we construct an LP problem for the systems (I.1), of which every feasible solution parameterizes a common Lyapunov functions for the systems.

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II. NOTATION AND PRELIMINARIES

We denote by $\mathbb{Z}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{R}_+$ the sets of the integers, the nonnegative integers, the real numbers, and the nonnegative real numbers respectively. For integers $r, s \in \mathbb{Z}$, $r < s$, we write $r : s$ for $r, r+1, \ldots, s$ and $\{r : s\}$ for the set $\{r, r+1, \ldots, s\}$. We write vectors in boldface, e.g. $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{Z}^n$, and their components as $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. All vectors are assumed to be column vectors unless specified otherwise. The null vector in $\mathbb{R}^n$ is written as $\mathbf{0}$ and the standard orthonormal basis as $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, i.e. the i-th component of $\mathbf{e}_j$ is equal to $\delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta, equal to 1 if $i = j$ and 0 otherwise. The scalar product of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is denoted by $\mathbf{x} \cdot \mathbf{y}$ and $\|\mathbf{x}\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ denotes the p-norm of $\mathbf{x}$ with the usual convention that $\|\mathbf{x}\|_\infty := \max_{i=1:n} |x_i|$. For a matrix $A \in \mathbb{R}^{n\times n}$ the induced matrix norm $\|A\|_p$ is the smallest number such that $\|A\mathbf{x}\|_p \leq \|A\|_p \|\mathbf{x}\|_p$ for all $\mathbf{x} \in \mathbb{R}^n$. The transpose of a vector $\mathbf{x}$ is denoted by $\mathbf{x}^T$.

We write sets $\mathcal{K} \subset \mathbb{R}^n$ in calligraphic and we denote the closure, interior, and the boundary of $\mathcal{K}$ by $\overline{\mathcal{K}}$, $\mathcal{K}^\circ$, and $\partial\mathcal{K}$ respectively. A function $\alpha : \mathbb{R}^n \to \mathbb{R}_+$ is said to be *positive definite* if $\alpha(\mathbf{0}) = 0$ and $\alpha(\mathbf{x}) > 0$ if $\mathbf{x} \neq \mathbf{0}$. A function $\alpha : \mathbb{R}^n \to \mathbb{R}$ is said to be *convex* if $\alpha(\sum_{i=1}^m \lambda_i \mathbf{x}_i) \leq \sum_{i=1}^m \lambda_i \alpha(\mathbf{x}_i)$ for all convex combinations $\sum_{i=1}^m \lambda_i \mathbf{x}_i$, i.e. $\lambda_i \geq 0$ for $i = 1 : m$ and $\sum_{i=1}^m \lambda_i = 1$.

The *convex hull* of an $(m+1)$-tuple $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m)$ of vectors $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ is defined by

$$\mathrm{co}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m) := \left\{ \sum_{i=0}^m \lambda_i \mathbf{v}_i : 0 \leq \lambda_i, \sum_{i=0}^m \lambda_i = 1 \right\}.$$

If $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ are affinely independent, i.e. the vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_m - \mathbf{v}_0$ are linearly independent, the set $\mathrm{co}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m)$ is called an m-*simplex*. For a subset $\{\mathbf{v}_{i_0}, \mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}\}$, $0 \leq k < m$, of affinely independent vectors $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$, the k-simplex $\mathrm{co}(\mathbf{v}_{i_0}, \mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k})$ is called a k-*face* of the simplex $\mathrm{co}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m)$.

Definition 1: A set $\mathcal{T}$ of n-simplices in $\mathbb{R}^n$ is called a *suitable triangulation* if the interior $\mathcal{D}_{\mathcal{T}}^\circ$ of $\mathcal{D}_{\mathcal{T}} :=$

$\bigcup_{\mathfrak{S}_\nu \in \mathcal{T}} \mathfrak{S}_\nu$ is a simply connected open neighbourhood of the origin, the origin is a vertex of a simplex in $\mathcal{T}$, and two different simplices $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$ intersect in a common face or not at all. We call the set $\mathcal{D}_\mathcal{T}$ the domain of the triangulation $\mathcal{T}$ and we denote the set of all vertices of the simplices in $\mathcal{T}$ by $\mathcal{V}_\mathcal{T}$. A *sub-triangulation* $\mathcal{T}^*$ of $\mathcal{T}$ is a collection of some of its simplices, that is $\mathcal{T}^* \subset \mathcal{T}$.

Given a suitable triangulation $\mathcal{T}$ we can parameterize a continuous function $f : \mathcal{D}_\mathcal{T} \to \mathbb{R}$ that is affine on each of its simplices by specifying its values on $\mathcal{V}_\mathcal{T}$. For such a continuous and piecewise affine function we write $f \in \text{CPA}[\mathcal{T}]$.

The regular triangulation $\mathcal{T}^{\text{std}}$ of $\mathbb{R}^n$, defined in Definition 2, and whose vertices are the set $\mathbb{Z}^n$, is particularly important. For some illustrations of the simplices in $\mathcal{T}^{\text{std}}$ together with a discussion on how they can be efficiently generated, see e.g. [6], [7].

Remark 1: For the construction of our triangulations we use the set S_n of all permutations of the set $\{1 : n\}$, the characteristic function $\chi_\mathcal{J}(i)$ equal to one if $i \in \mathcal{J}$ and equal to zero if $i \notin \mathcal{J}$, and the standard orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathbb{R}^n$. Further, we use the functions $\mathbf{R}^\mathcal{J} : \mathbb{R}^n \to \mathbb{R}^n$, defined for every $\mathcal{J} \subset \{1 : n\}$ by

$$\mathbf{R}^\mathcal{J}(\mathbf{x}) := \sum_{i=1}^{n} (-1)^{\chi_\mathcal{J}(i)} x_i \mathbf{e}_i.$$

$\mathbf{R}^\mathcal{J}(\mathbf{x})$ puts a minus in front of the coordinate x_i of $\mathbf{x}$ whenever $i \in \mathcal{J}$.

Definition 2 (Basic triangulation $\mathcal{T}^{\text{std}}$): The triangulation $\mathcal{T}^{\text{std}}$ consists of the n-dimensional simplices

$$\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \text{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$$

for all $\mathbf{z} \in \mathbb{N}_0^n$, all $\mathcal{J} \subset \{1 : n\}$, and all $\sigma \in S_n$ (cf. Remark 1 for notations), where

$$\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} := \mathbf{R}^\mathcal{J}\left(\mathbf{z} + \sum_{j=1}^{i} \mathbf{e}_{\sigma(j)}\right) \text{ for } i \in \{1 : n\}. \quad \text{(II.1)}$$

Finally, we give two definitions of common Lyapunov functions used in this paper. The first one (II.2) is the usual one and the second one (II.3) is a variant more suited for numerical construction.

Definition 3 (Common Lyapunov function): Let $\mathcal{D}, \mathcal{O}^*, \mathcal{E}^* \subset \mathbb{R}^n$ be simply connected neighbourhoods of the origin, $\mathcal{E}^* \subset \mathcal{O}^* \subset \mathcal{D}$, and $\alpha_1, \alpha_2 : \mathbb{R}^n \to \mathbb{R}_+$ be positive definite convex functions. A continuous function $V : \mathcal{D} \to \mathbb{R}_+$ such that

$$V(\mathbf{0}) = 0 \text{ and } V(\mathbf{x}) \geq \alpha_1(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{D}$$

is said to be a common Lyapunov function for the systems (I.1) if for $i = 1 : N$ it fulfills

$$V(\mathbf{g}_i(\mathbf{x})) - V(\mathbf{x}) \leq -\alpha_2(\mathbf{x}) \text{ for all all } \mathbf{x} \in \mathcal{O}^*. \quad \text{(II.2)}$$

It is said to be a common Lyapunov function with target $\mathcal{E}^*$ if for $i = 1 : N$ it fulfills

$$V(\mathbf{g}_i(\mathbf{x})) - V(\mathbf{x}) < 0 \text{ for all all } \mathbf{x} \in \mathcal{O}^* \setminus (\mathcal{E}^*)^\circ. \quad \text{(II.3)}$$

The conditions (II.2) and (II.3) are referred to as the decrease condition and the sets $\mathcal{O}^*$ and $\mathcal{O}^* \setminus (\mathcal{E}^*)^\circ$ are referred to as the decrease regions respectively.

The implications of the existence of such Lyapunov functions are proved in Thm. 2.2 and Prop. 2.4 in [3]. In short, for a Lyapunov function as in (II.3), every sub-level set $\mathcal{L}_R := \{\mathbf{x} \in \mathcal{D} : V(\mathbf{x}) \leq R\}$ such that $\mathcal{E}^* \subset \mathcal{L}_R$ and $\partial\mathcal{L}_R \subset \mathcal{O}^* \setminus (\mathcal{E}^*)^\circ$ is forward invariant and solution trajectories of every system move from such level-sets to smaller such level-sets.

III. Linear Programming Problem

After this preparation we can immediately state the algorithm that constructs our LP problem for the systems (I.1).

Algorithm 1 Constructs an LP problem for the switched systems (I.1), of which every feasible solution parameterizes a CPA Lyapunov function for the system.

Input: Systems $\mathbf{x}_{k+1} = \mathbf{g}_i(\mathbf{x}_k)$ as in (I.1). Suitable triangulation $\mathcal{T}$ as in Definition 1 with a convex domain $\mathcal{D}_\mathcal{T}$ and and two sub-triangulations $\mathcal{E}, \mathcal{O}$ of $\mathcal{T}$ that are also suitable triangulations, $\mathbf{g}_i(\mathcal{D}_\mathcal{O}) \subset \mathcal{D}_\mathcal{T}$ for $i = 1 : N$, and $\mathcal{D}_\mathcal{O}^\circ \supset \mathcal{D}_\mathcal{E}$
A positive definite convex function $\alpha_1 : \mathbb{R}^n \to \mathbb{R}_+$.

Constants:
For all $i = 1 : N$ and all $\mathfrak{S}_\nu \in \mathcal{T}$:

$$G_{\nu,i} \geq \max_{\substack{j=1:n \\ \mathbf{x}\in\mathfrak{S}_\nu}} \|\nabla[\mathbf{g}_i]_j(\mathbf{x})\|_2$$

$[\mathbf{g}_i]_j$ denotes the j-th component of the vector field $\mathbf{g}_i$.
For all $\mathfrak{S}_\nu \in \mathcal{T}$:

$$h_\nu := \text{diam}(\mathfrak{S}_\nu) = \max_{\mathbf{x},\mathbf{y}\in\mathfrak{S}_\nu} \|\mathbf{x} - \mathbf{y}\|_2$$

A small number $\varepsilon > 0$.

Variables: $V_\mathbf{x} \in \mathbb{R}$ For each $\mathbf{x} \in \mathcal{V}_\mathcal{T}$ and $L \in \mathbb{R}$.

Constraints:
$V_\mathbf{0} = 0$ and for all $\mathbf{x} \in \mathcal{V}_\mathcal{T}$:

$$V_\mathbf{x} \geq \alpha_1(\mathbf{x}) \quad \text{(III.1)}$$

For all $\mathfrak{S}_\nu \in \mathcal{T}$:

$$\|\nabla V_\nu\|_1 \leq L \quad \text{(III.2)}$$

For all $i = 1 : N$ and all $\mathbf{x} \in \mathcal{V}_\mathcal{O}$ such that $\mathbf{x} \notin \mathcal{D}_\mathcal{E}^\circ$:

$$\sum_{j=0}^{n} \mu_j^i V_{\mathbf{y}_j} - V_\mathbf{x} + G_{\nu,i} h_\nu L \leq -\varepsilon, \quad \text{(III.3)}$$

where (recall $\mathbf{g}_i(\mathcal{D}_\mathcal{O}) \subset \mathcal{D}_\mathcal{T}$)

$$\mathbf{g}_i(\mathbf{x}) = \sum_{j=0}^{n} \mu_j^i \mathbf{y}_j \text{ for a simplex } \text{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n) \in \mathcal{T}.$$

Remark 2: The constraints (III.2) are implemented using the auxiliary variables C_j^ν and enforcing the linear constraints

$$-C_j^\nu \le \mathbf{e}_j^T X_\nu^{-1}\mathbf{v}_\nu \le C_j^\nu \tag{III.4}$$

for all $\mathfrak{S}_\nu \in \mathcal{T}$ and $j = 1:n$ and

$$\sum_{i=j}^{n} C_j^\nu \le L$$

for all $\mathfrak{S}_\nu \in \mathcal{T}$. In (III.4) the matrix $X_\nu \in \mathbb{R}^{n\times n}$ is the so-called shape matrix of the simplex $\mathrm{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n) \in \mathcal{T}$, obtained by writing the vectors

$$(\mathbf{y}_1 - \mathbf{y}_0)^T, (\mathbf{y}_2 - \mathbf{y}_0)^T, \ldots, (\mathbf{y}_n - \mathbf{y}_0)^T$$

in its rows subsequently, and

$$\mathbf{v}_\nu = \begin{pmatrix} V_{\mathbf{y}_1} - V_{\mathbf{y}_0} \\ V_{\mathbf{y}_2} - V_{\mathbf{y}_0} \\ \vdots \\ V_{\mathbf{y}_n} - V_{\mathbf{y}_0} \end{pmatrix}. \tag{III.5}$$

It is easily shown that

$$\nabla V_\nu = X_\nu^{-1}\mathbf{v}_\nu$$

and thus $\mathbf{e}_i^T X_\nu^{-1}\mathbf{v}_\nu$ is the i-th component $[\nabla V_\nu]_i$ of ∇V_ν, $|[\nabla V_\nu]_i| \le C_i^\nu$, and $\|\nabla V_\nu\|_1 \le L$.

We now show that a feasible solution to the LP problem from Algorithm 1 delivers a common Lyapunov function for the systems (I.1).

Theorem 1: Assume the LP problem constructed by Algorithm 1 has a feasible solution $(V_\mathbf{x})$, $\mathbf{x} \in \mathcal{V}_\mathcal{T}$. Then the function $V \in \mathrm{CPA}[\mathcal{T}]$, defined by fixing $V(\mathbf{x}) = V_\mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}_\mathcal{T}$, is a common Lyapunov function for the system (I.1) as in Definition 3 fulfilling the decrease condition (II.3) on the set $\mathcal{D}_\mathcal{O} \setminus \mathcal{D}_\mathcal{E}^\circ$ and with $\mathcal{D}_\mathcal{E}$ as target.

Proof: Clearly $V(\mathbf{0}) = 0$ and for an $\mathbf{x} \in \mathcal{D}_\mathcal{T}$ there is a $\mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{T}$ such that $\mathbf{x} = \sum_{k=0}^n \lambda_k \mathbf{x}_k$ is the convex combination of its vertices. Hence, by (III.1) and the convexity of α_1 we get

$$V(\mathbf{x}) = \sum_{k=0}^{n} \lambda_k V_{\mathbf{x}_k} \ge \sum_{k=0}^{n} \lambda_k \alpha_1(\mathbf{x}_k) \ge \alpha_1\left(\sum_{k=0}^{n} \lambda_k \mathbf{x}_k\right) = \alpha_1(\mathbf{x}).$$

Fix an $i \in \{1:N\}$ and an $\mathbf{x} \in \mathcal{D}_\mathcal{O} \setminus \mathcal{D}_\mathcal{E}^\circ$. Now there are simplices $\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{O} \setminus \mathcal{E}$ and $\mathrm{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n) \in \mathcal{T}$ such that $\mathbf{x} = \sum_{k=0}^n \lambda_k \mathbf{x}_k$ and $\mathbf{g}_i(\mathbf{x}) = \sum_{j=0}^n \mu_j \mathbf{y}_j$ are convex combination of their vertices. Note that (III.3) implies that for $\mathbf{x}_k$, $k = 0:n$, we have

$$V(\mathbf{g}_i(\mathbf{x}_k)) - V(\mathbf{x}_k) + G_{\nu,i}h_\nu L < 0. \tag{III.6}$$

Hence

$$V(\mathbf{g}_i(\mathbf{x})) - V(\mathbf{x}) = V(\mathbf{g}_i(\mathbf{x})) - \sum_{k=0}^{n} \lambda_k V(\mathbf{x}_k) \tag{III.7}$$
$$= \sum_{k=0}^{n} \lambda_k \left[V(\mathbf{g}_i(\mathbf{x})) - V(\mathbf{g}_i(\mathbf{x}_k)) + V(\mathbf{g}_i(\mathbf{x}_k)) - V(\mathbf{x}_k)\right].$$

In the proof of [4, Thm. 2.10] the estimate

$$|V(\mathbf{g}_i(\mathbf{x})) - V(\mathbf{g}_i(\mathbf{x}_k))| \le L\|\mathbf{g}_i(\mathbf{x}) - \mathbf{g}_i(\mathbf{x}_k)\|_\infty$$

is established. Let $j \in \{1:n\}$ be such that

$$\|\mathbf{g}_i(\mathbf{x}) - \mathbf{g}_i(\mathbf{x}_k)\|_\infty = |[\mathbf{g}_i]_j(\mathbf{x}) - [\mathbf{g}_i]_j(\mathbf{x}_k)|.$$

By the Mean-Value-Theorem there is a $\mathbf{z}$ on the line-segment between $\mathbf{x}$ and $\mathbf{x}_k$ such that

$$[\mathbf{g}_i]_j(\mathbf{x}) - [\mathbf{g}_i]_j(\mathbf{x}_k) = \nabla[\mathbf{g}_i]_j(\mathbf{z}) \cdot (\mathbf{x} - \mathbf{x}_k).$$

Hence, by the Cauchy-Schwartz inequality,

$$\begin{aligned}\|\mathbf{g}_i(\mathbf{x}) - \mathbf{g}_i(\mathbf{x}_k)\|_\infty &= |\nabla[\mathbf{g}_i]_j(\mathbf{z}) \cdot (\mathbf{x} - \mathbf{x}_k)| \\ &\le \|\nabla[\mathbf{g}_i]_j(\mathbf{z})\|_2 \|\mathbf{x} - \mathbf{x}_k\|_2.\end{aligned}$$

Thus, since $\mathbf{x}, \mathbf{x}_k \in \mathfrak{S}_\nu$ we have

$$\|\mathbf{g}_i(\mathbf{x}) - \mathbf{g}_i(\mathbf{x}_k)\|_\infty \le G_{\nu,i}h_\nu$$

for $k = 0:n$. It now follows from (III.7), (III.6), and the constraints (III.3) that

$$\begin{aligned}&V(\mathbf{g}_i(\mathbf{x})) - V(\mathbf{x}) \\ &\quad = \sum_{k=0}^{n} \lambda_k \left[LG_{\nu,i}h_\nu + V(\mathbf{g}_i(\mathbf{x}_k)) - V(\mathbf{x}_k)\right] \le -\varepsilon < 0.\end{aligned}$$

■

We now show that if there exists a common Lyapunov function for the systems (I.1), then we can compute one with our algorithm.

Theorem 2: Assume there exists a common Lyapunov function $W : \mathbb{R}^n \to \mathbb{R}_+$ as in Definition 3 for the system (I.1) fulfilling the decrease condition (II.2) on an open neighbourhood $\mathcal{O}_W$ of the origin. Then, for every simply connected and compact neighbourhoods $\mathcal{C}_\mathcal{O}, \mathcal{C}_\mathcal{E} \subset \mathcal{O}_W$ of the origin, $\mathcal{C}_\mathcal{O}^\circ \supset \mathcal{C}_\mathcal{E}$, we can construct a triangulation $\mathcal{T}$ and sub-triangulations $\mathcal{E}$ and $\mathcal{O}$ that fulfill the conditions of Algorithm 1 in addition to $\mathcal{D}_\mathcal{O} \supset \mathcal{C}_\mathcal{O}$ and $\mathcal{C}_\mathcal{E} \supset \mathcal{D}_\mathcal{E}$.
Note: This theorem is of course especially interesting for large $\mathcal{C}_\mathcal{O}$ and small $\mathcal{C}_\mathcal{E}$.

Proof: Let $R > 0$ be so large that $\|\mathbf{g}_i(\mathcal{C}_\mathcal{O})\|_\infty \le R$ for $i = 1:N$ and set

$$L_W := \max\left\{\left|\frac{\partial W}{\partial x_k}(\mathbf{x})\right| \;:\; \|\mathbf{x}\|_\infty \le R+1 \text{ and } k = 1:n\right\}.$$

Fix $r > 0$ be so small that $\{\mathbf{x} : \|\mathbf{x}\|_\infty \le r\} \subset \mathcal{C}_\mathcal{E}$. Set

$$\gamma := \inf_{\mathbf{x} \in \mathcal{O}_W \setminus [-r/2, r/2]^n} \alpha_2(\mathbf{x})$$

and note that $\gamma > 0$. Further, fix the constant $G > 0$ so large that that

$$G \ge \|\nabla[\mathbf{g}_i]_j(\mathbf{x})\|_2 \tag{III.8}$$

for all $i = 1 : N$, $j = 1 : n$, and $\|\mathbf{x}\|_\infty \le R+1$.

By the Mean-Value-Theorem there exists a $\mathbf{z}$ on the line-segment between $\mathbf{x}, \mathbf{y} \in [-(R+1), R+1]^n$ such that

$$W(\mathbf{x}) - W(\mathbf{y}) = \nabla W(\mathbf{z}) \cdot (\mathbf{x} - \mathbf{y})$$

and by the Cauchy-Schwartz inequality it follows that

$$|W(\mathbf{x}) - W(\mathbf{y})| \le \sqrt{n} L_W \|\mathbf{x} - \mathbf{y}\|_2. \tag{III.9}$$

Define

$$\beta \mathcal{T}^{\mathrm{std}} := \left\{ \beta \mathfrak{S}_\nu \,:\, \mathfrak{S}_\nu \in \mathcal{T}^{\mathrm{std}} \right\},$$

where $\mathcal{T}^{\mathrm{std}}$ is the triangulation from Definition 2 and $\beta > 0$ is so small that $\beta < 1$, $\beta < r/2$, and

$$\beta < \frac{\gamma}{2} \left[2n^2 L_W \sqrt{n}\, G + n L_W (2n+1) \right]^{-1} \tag{III.10}$$

and fix the constant $\varepsilon = \gamma/2$.

Define the triangulation

$$\mathcal{T} := \{ \mathfrak{S}_\nu \in \beta \mathcal{T}^{\mathrm{std}} \,:\, \mathfrak{S}_\nu \cap [-R, R]^n \neq \emptyset \}$$

and the sub-triangulations $\mathcal{O}$ and $\mathcal{E}$ through

$$\mathcal{O} := \{ \mathfrak{S}_\nu \in \mathcal{T} \,:\, \mathfrak{S}_\nu \cap \mathcal{O}_W \neq \emptyset \}$$

and

$$\mathcal{E} := \{ \mathfrak{S}_\nu \in \mathcal{T} \,:\, \mathfrak{S}_\nu \cap [-r/2, r/2]^n \neq \emptyset \}.$$

It is not difficult to see that these triangulations fulfill the conditions of Algorithm 1 and that $\mathcal{D}_\mathcal{E} \subset \mathcal{C}_\mathcal{E}$ and $\mathcal{D}_\mathcal{O} \supset \mathcal{C}_\mathcal{O}$. Fix the constants $G_{\nu,i} := G$ for all $i = 1 : N$ and all $\mathfrak{S}_\nu \in \mathcal{T}$.

Note that for $\mathbf{x}, \mathbf{y} \in \mathfrak{S}_\nu \in \mathcal{T}$ we have from (III.9) that

$$|W(\mathbf{x}) - W(\mathbf{y})| \le \sqrt{n} L_W h_\nu = n L_W \beta \tag{III.11}$$

Now assign values to the variables of the LP problem through $L = 2n^2 L_W$ and $V_\mathbf{x} = W(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{V}_\mathcal{T}$. Clearly $V_\mathbf{x} = W(\mathbf{x}) \ge \alpha_1(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{V}_\mathcal{T}$, i.e. the constraints (III.1) are fulfilled. Now

$$\|\nabla V_\nu\|_1 = \|X_\nu^{-1} \mathbf{v}_\nu\|_1 \le \|X_\nu^{-1}\|_1 \|\mathbf{v}_\nu\|_1 \le 2n^2 L_W = L,$$

where we used $\|X_\nu^{-1}\|_1 \le 2/\beta$ shown in Remark 2 in [18] and $\|\mathbf{v}_\nu\|_1 \le n^2 L_W \beta$ which follows immediately from the definition of $\mathbf{v}_\nu$ in (III.5) and (III.11). Hence, the constraints (III.2) are fulfilled. Additionally, we get the estimate

$$|V(\mathbf{x}) - W(\mathbf{x})| \le n L_W (2n+1)\beta$$

for any $\mathbf{x} \in \mathcal{D}_\mathcal{T}$, because with $\mathbf{y}$ as vertex of $\mathfrak{S}_\nu$, where $\mathbf{x} \in \mathfrak{S}_\nu$, we get from the Hölder inequality

$$\begin{aligned}
|V(\mathbf{x}) - W(\mathbf{x})| &\le |V(\mathbf{x}) - V(\mathbf{y})| + |W(\mathbf{y}) - W(\mathbf{x})| \\
&\le |\nabla V_\nu \cdot (\mathbf{x} - \mathbf{y})| + n L_W \beta \\
&\le \|\nabla V_\nu\|_1 \|\mathbf{x} - \mathbf{y}\|_\infty + n L_W \beta \\
&= 2n^2 L_W \beta + n L_W \beta \\
&= n L_W (2n+1) \beta
\end{aligned}$$

because $V(\mathbf{y}) = W(\mathbf{y})$.

Now, for any vertex $\mathbf{x} \in \mathcal{D}_\mathcal{O}$, $\mathbf{x} \notin \mathcal{D}_\mathcal{E}^\circ$, we have for $i = 1 : N$ that $W(\mathbf{g}_i(\mathbf{x})) - W(\mathbf{x}) \le -\gamma$. Hence,

$$\begin{aligned}
&V(\mathbf{g}_i(\mathbf{x})) - V(\mathbf{x}) \\
&\quad = V(\mathbf{g}_i(\mathbf{x})) - W(\mathbf{g}_i(\mathbf{x})) + W(\mathbf{g}_i(\mathbf{x})) - W(\mathbf{x}) \\
&\quad \le n L_W (2n+1)\beta - \gamma
\end{aligned}$$

and the constants $\beta, \varepsilon, G > 0$ were precisely fixed such that this implies

$$\begin{aligned}
&V(\mathbf{g}_i(\mathbf{x})) - V(\mathbf{x}) + G_{\nu,i} h_\nu L \\
&\quad \le 2n^2 L_W \sqrt{n}\, G\beta + n L_W (2n+1)\beta - \gamma \quad \le -\varepsilon,
\end{aligned}$$

i.e. the constraints (III.3) are fulfilled, which concludes our proof. ■

IV. Examples

We consider two planar examples to show how our method works. The LP problems were generated with a program in C++ and then solved using the barrier method in Gurobi, a state of the art solver free for academic use. Programs for generating simplicial complexes and similar LP problems have been discussed by the author in [6], [7]. The computer we used has a i9-7900X processor (3.3 GHz, 10 cores) and 128 GB RAM. Both examples are taken from [13] and both are also considered in [19]. In these papers methods to compute Lyapunov functions for polynomial systems, i.e. the components of the vector fields $\mathbf{g}_i$ in (I.1) are polynomials in the variables, are considered. Our method is not limited to polynomials, the only assumption on the components of the vector fields $\mathbf{g}_i$ is that they are C^1, but it is interesting to compare the results from the methods.

For constructing the triangulation $\mathcal{T}$ and its sub-triangulations $\mathcal{O}$ and $\mathcal{E}$ in the examples below we start with a grid of the form $\{-T : T\}^2 \subset \mathbb{Z}^2$ for a given integer $T > 0$. This grid is then mapped to $\mathbb{R}^2$ by using a mapping $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, dependent on two parameters $c > 0$ and $p > 0$. The formula for $\mathbf{F}$ is

$$\mathbf{F}(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad \mathbf{F}(\mathbf{x}) = \frac{c\|\mathbf{x}\|_\infty^p}{\|\mathbf{x}\|_2} \mathbf{x} \quad \text{if } \mathbf{x} \neq \mathbf{0}.$$

The triangulation $\mathcal{T}$ using the grid $\{-T : T\}^2$ and the mapping $\mathbf{F}$ is now constructed in the following way:

For every simplex $\mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{T}^{\mathrm{std}}$ such that $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\} \subset \{-T : T\}^2$ we add the simplex $\mathrm{co}\,(\mathbf{F}(\mathbf{x}_0), \mathbf{F}(\mathbf{x}_1), \ldots, \mathbf{F}(\mathbf{x}_n))$ to the triangulation $\mathcal{T}$.

By using this procedure we create triangulations or areas that are approximately circular discs with radius cT^p, compare Figure 1 taken from [9]. The sub-triangulations $\mathcal{O}$ and $\mathcal{E}$ are constructed identically using the grid $\{-O : O\}^2 \subset \mathbb{Z}^2$ for $\mathcal{O}$ and $\{-E : E\}^2 \subset \mathbb{Z}^2$ for $\mathcal{E}$, $0 < E < O < T$.

Example 1

The first system is $\mathbf{x}_{k+1} = \mathbf{g}_i(\mathbf{x}_k)$ with

$$\mathbf{g}_1(x, y) = \begin{pmatrix} 0.5x \\ -0.8y - x^2 \end{pmatrix} \tag{IV.1}$$

$$\mathbf{g}_2(x, y) = \begin{pmatrix} 0.5x + xy \\ -0.8y \end{pmatrix}$$

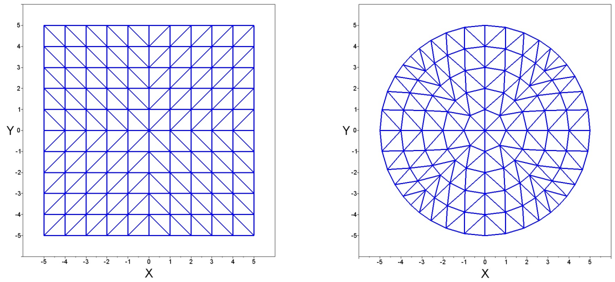

Fig. 1. On the left are the simplices $\operatorname{co}(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) \in \mathcal{T}^{\text{std}}$ such that $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\} \subset [-5, 5]^2$. On the right we depict the simplices $\operatorname{co}(\mathbf{F}(\mathbf{x}_0), \mathbf{F}(\mathbf{x}_1), \mathbf{F}(\mathbf{x}_2))$ with $c = p = 1$. Note that the square $[-5, 5]^2$ becomes approximately a circular disc with radius 5.

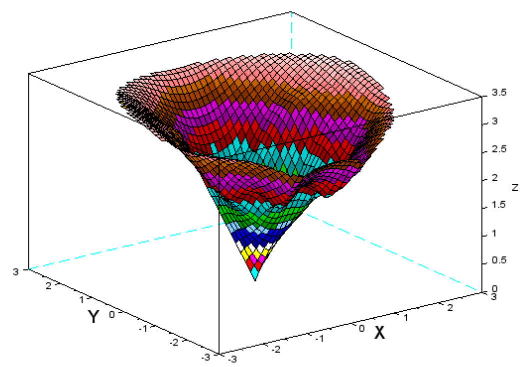

Fig. 2. The Lyapunov function computed for system (IV.1).

The triangulations $\mathcal{T}$, $\mathcal{O}$, and $\mathcal{E}$ were constructed using the procedure described above and with $T = 90$, $O = 58$, and $E = 15$. For the mapping $\mathbf{F}$ we used the parameters $c = 0.0075$ and $p = 1.3$. The sets $\mathcal{D}_\mathcal{T}$, $\mathcal{D}_\mathcal{O}$, and $\mathcal{D}_\mathcal{E}$ were thus approximately circular discs with radii 2.60, 1.47, and 0.25 respectively. The LP problem was constructed and solved in 1413 seconds. In Figure 2 the Lyapunov function computed is depicted and in Figure 3 some level-sets of it are plotted together with the set $\mathcal{D}_\mathcal{E}$.

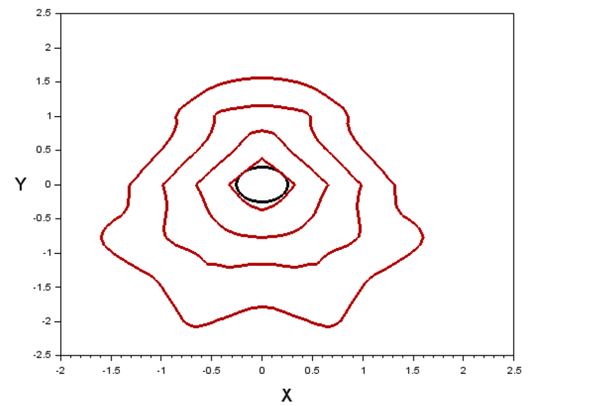

Fig. 3. Selected level-sets of the Lyapunov function computed for system (IV.1) (red) and the target $\mathcal{D}_\mathcal{E}$ (black). All these level-sets have boundaries in $\mathcal{D}_\mathcal{O} \setminus \mathcal{D}_\mathcal{E}^\circ$ and therefore forward invariant. A solution starting in one of them will go to $\mathcal{D}_\mathcal{E}$ as time evolves and stay within the smallest level-set for all times.

Example 2

The second system we considered is $\mathbf{x}_{k+1} = \mathbf{g}_i(\mathbf{x}_k)$ with

$$\mathbf{g}_1(x, y) = \begin{pmatrix} y \\ 0.6x - xy \end{pmatrix} \tag{IV.2}$$
$$\mathbf{g}_2(x, y) = \begin{pmatrix} y \\ 0.2x - 0.2y - y^2 \end{pmatrix}$$

The triangulations $\mathcal{T}$, $\mathcal{O}$, and $\mathcal{E}$ were constructed using the procedure described above and with $T = 80$, $O = 60$, and $E = 14$. For the mapping $\mathbf{F}$ we used the parameters $c = 0.003$ and $p = 1.1$. The sets $\mathcal{D}_\mathcal{T}$, $\mathcal{D}_\mathcal{O}$, and $\mathcal{D}_\mathcal{E}$ were thus approximately circular discs with radii 0.372, 0.271, and 0.0547 respectively. The LP problem was constructed and solved in 540 seconds. In Figure 4 the Lyapunov function computed is depicted and in Figure 5 some level-sets of it are plotted together with the set $\mathcal{D}_\mathcal{E}$.

V. CONCLUSIONS AND FUTURE WORKS

We presented an algorithm to compute a common Lyapunov function for nonlinear discrete-time systems. In Example 1 the Lyapunov function computed delivered a considerably larger lower bounds on the basins of attraction than the bound in [13] on a given Lyapunov function and seems to be similar to [19], although a detailed analysis is difficult because many details are missing from the second reference. In Example 2 our method does not deliver better estimates. The proposed approach can certainly be improved in many ways. In particular, one might want to include

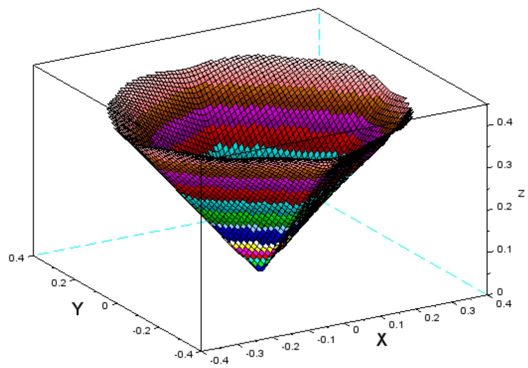

Fig. 4. The Lyapunov function computed for system (IV.2).

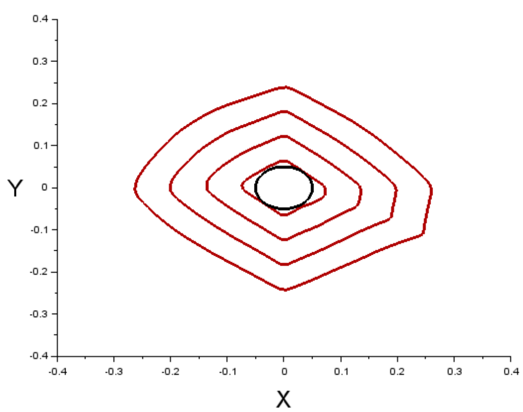

Fig. 5. Selected level-sets of the Lyapunov function computed for system (IV.2) (red) and the target $\mathcal{D}_\mathcal{E}$ (black). All these level-sets have boundaries in $\mathcal{D}_\mathcal{O} \setminus \mathcal{D}_\mathcal{E}^\circ$ and therefore forward invariant. A solution starting in one of them will go to $\mathcal{D}_\mathcal{E}$ as time evolves and stay within the smallest level-set for all times.

some mechanism to maximize the lower bounds on basins of attraction. This will be studied in forthcoming work.

VI. ACKNOWLEDGMENTS

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