CPA Lyapunov functions: Switched Systems vs. Differential Inclusions

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Abstract: We present an algorithm that uses linear programming to parameterize continuous and piecewise affine Lyapunov functions for switched systems. The novel feature of the algorithm is, that it can compute Lyapunov functions for switched system with a strongly asymptotically stable equilibrium, for which the equilibrium of the corresponding differential inclusion is merely weakly asymptotically stable. For the differential inclusion no such Lyapunov function exists. This is achieved by removing constraints from a linear programming problem of an earlier algorithm to compute Lyapunov functions, that are not necessary to assert strong stability for the switched system. We demonstrate the benefits of this new algorithm sing Artstein's circles as an example.

1 Introduction

We start with an informal introduction of the switched systems and differential inclusions we will be concerned with; the technical details follow later when we concretize our setting. Keep in mind that we are setting the stage to remove conditions on a Lyapunov function for a switched system with a strongly asymptotically stable equilibrium, which is merely weakly asymptotically stable for the corresponding differential inclusion.

We consider switched systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}_\alpha(\mathbf{x}), \ \ \alpha : [0,\infty) \to \mathcal{A}, \tag{1}$$

where for each $a \in \mathcal{A}$ the vector field $\mathbf{f}_a : \mathcal{D}(\mathbf{f}_a) \to \mathbb{R}^n$ is defined on $\mathcal{D}(\mathbf{f}_a) \subset \mathbb{R}^n$, $\mathcal{A}$ is a finite set equipped with the discrete topology, and $\alpha : [0,\infty) \to \mathcal{A}$ is a right-continuous switching signal. Solution trajectories of the system are continuous paths obtained by gluing together trajectory pieces of the individual systems $\dot{\mathbf{x}} = \mathbf{f}_a(\mathbf{x})$. Switched systems and their stability have been intensively studied, cf. the monograph (Liberzon, 2003; Sun and Ge, 2011). The main questions regarding the asymptotic stability of an equilibrium of a switched system, which we may assume is at the origin, is if a switching can be chosen such that solution trajectories are steered to the equilibrium (*weak asymptotic stability*) or if trajectories are attracted to it regardless of the switching (*strong asymptotic stability*). The latter case is referred to as *arbitrary switching*. Both types of stability are usually dealt with using Lyapunov functions, i.e. functions from the state space to the real numbers that are decreasing along solution trajectories. In (Hafstein, 2007) an algorithm to compute continuous and piecewise affine (CPA) Lyapunov functions for arbitrary switched systems was developed; we will discuss it below.

Closely related to the switched system (1) is the differential inclusion

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}) := \mathrm{co}\{\mathbf{f}_a(\mathbf{x}) : \mathbf{x} \in \mathcal{D}(\mathbf{f}_a)\}, \tag{2}$$

where $\mathrm{co}\,C$ denotes the convex hull of the set $C \subset \mathbb{R}^n$. Solution trajectories are absolutely continuous paths $t \mapsto \mathbf{x}(t)$ fulfilling $\dot{\mathbf{x}}(t) \in \mathbf{F}(\mathbf{x}(t))$ almost surely. One speaks of *weak asymptotic stability* of an equilibrium if there are solution trajectories that are asymptotically attracted towards the equilibrium and *strong asymptotic stability* if this is the case for all solution trajectories. In (Baier et al., 2010; Baier et al., 2012) an algorithm for the computation of CPA Lyapunov functions for strongly asymptotically stable inclusions was presented and in (Baier and Hafstein, 2014) a corresponding algorithm for the computation of control CPA Lyapunov functions for weakly asymptotically stable differential inclusions. See also (Baier et al., 2018) for a different approach including semiconcavity condition into the formulation of the optimization problem.

Let us give a short review of CPA Lyapunov functions in the context of switched systems and differential inclusions. To define a CPA Lyapunov function V, first a triangulation $\mathcal{T}$ of its domain, a subset of $\mathbb{R}^n$, must be fixed. The triangulation must have the property that any two different simplices intersect in a common face or not at all. The continuous and piecewise affine function V is then defined by assigning it values at the vertices of the simplices of $\mathcal{T}$ and linearly interpolating these values over the simplices. The resulting function is affine on each simplex $S_\nu \in \mathcal{T}$ and thus differentiable in its interior S_ν°. In particular, its gradient is a well defined constant vector ∇V_ν in the interior. At the boundaries, where two or more simplices intersect, the function V is not differentiable and its gradient is not defined. In (Baier et al., 2010; Baier et al., 2012; Baier and Hafstein, 2014) this was dealt with in the context of nonsmooth analysis using the Clarke subdifferential, which can be defined through

$$\partial_{\mathrm{Cl}} V(\mathbf{x}) := \mathrm{co}\{\lim_{\mathbf{x}_i \to \mathbf{x}} \nabla V(\mathbf{x}_i) : \exists \lim_{\mathbf{x}_i \to \mathbf{x}} \nabla V(\mathbf{x}_i)\} \tag{3}$$

for locally Lipschitz continuous V, cf. (Clarke, 1990). That is, Clarke's subdifferential $\partial_{\mathrm{Cl}} V(\mathbf{x}) \subset \mathbb{R}^n$ is the convex hull of all converging sequences $(\nabla V(\mathbf{x}_i))_{i\in\mathbb{N}}$, where $\mathbf{x}_i \to \mathbf{x}$ as $i \to \infty$. The establishing of the strong stability of an equilibrium of the differential inclusion (2) now essentially boils down to showing, where $\bullet$ denotes the dot product of vectors and sets of vectors $N \bullet M := \{\mathbf{x} \bullet \mathbf{y} : \mathbf{x} \in N, \mathbf{y} \in M\}$, that

$$\partial_{\mathrm{Cl}} V(\mathbf{x}) \bullet \mathbf{F}(\mathbf{x}) < 0 \tag{4}$$

in a punctuated neighbourhood of the equilibrium. In this context $<$ means that every element of the set on the left-hand side is less than zero. For a CPA function V this further simplifies to

$$\nabla V_\nu \bullet \mathbf{f}_a(\mathbf{x}) < 0 \tag{5}$$

for every ν such that $\mathbf{x} \in S_\nu$ and every $a \in \mathcal{A}$ such that $\mathbf{x} \in \mathcal{D}(\mathbf{f}_a)$. This comes as every element in $\partial_{\mathrm{Cl}} V(\mathbf{x})$ is the convex sum of such ∇V_ν and every element of $\mathbf{F}(\mathbf{x})$ is the convex sum of such $\mathbf{f}_a(\mathbf{x})$. Exactly the same condition (5) can be used to show the strong stability of the same equilibrium for the switched system (1).

Now, even for considerably more general differential inclusions than (2), i.e. compact, convex, upper semicontinuous $\mathbf{F}$, the strong asymptotic stability of an equilibrium was shown in (Clarke et al., 1998) to be equivalent to the existence of a Lyapunov function V fulfilling (4). However, there are arbitrary switched systems with a strongly asymptotically stable equilibrium, such that the equilibrium is only weakly stable for the corresponding differential inclusion. We will modify the CPA algorithm to compute Lyapunov functions for arbitrary switched system and differential inclusions to deal with this case. Our motivating example will be that of Artstein's circles.

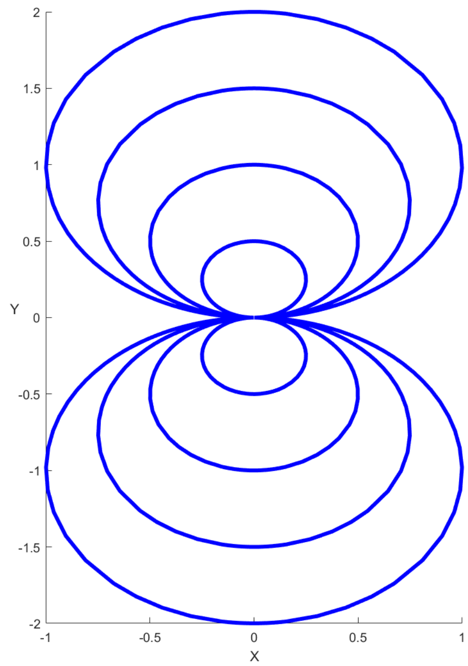

Figure 1: Trajectories of Artstein's circles (6). For an initial value (x_0, y_0) the trajectory is an arc of a circle with center on the y-axis and passing through $(0,0)$ and (x_0, y_0). Varying u changes the speed and the orientation of how the circles are traversed. For $u > 0$ the upper circles are traversed clockwise and the lower circles counter-clockwise until the equilibrium at zero is reached; for $u < 0$ vise versa. The speed of the traversing is proportional to $|u|$.

1.1 Artsteins's Circles

Artstein's circles (Artstein, 1983) are given by the differential inclusion

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} \in \left\{ u \begin{pmatrix} -x^2 + y^2 \\ -2xy \end{pmatrix} : u \in [-1, 1] \right\}. \tag{6}$$

See Fig. 1 for its solution trajectories. Clearly, the equilibrium at zero is not strongly asymptotically stable for the differential inclusion, fix e.g. $u = 0$. However, it is clearly weakly stable because, e.g. fixing $u = -1$ or $u = 1$, delivers an ODE with this property.

Let us define

$$\mathbf{f}_+(x, y) = \begin{pmatrix} -x^2 + y^2 \\ 2xy \end{pmatrix} \tag{7}$$

and $\mathbf{f}_-(x, y) = -\mathbf{f}_+(x, y)$. Then system (6) can be written $\dot{\mathbf{x}} \in \mathrm{co}\{\mathbf{f}_-(\mathbf{x}), \mathbf{f}_+(\mathbf{x})\}$ and the corresponding switched system is

$$\dot{\mathbf{x}} = \mathbf{f}_\alpha(\mathbf{x}), \quad \alpha : [0, \infty) \to \{-, +\}, \tag{8}$$

for which the origin is a weakly asymptotically stable equilibrium, and not strongly asymptotically stable!

From now on we limit the domains of $\mathbf{f}_-$ and $\mathbf{f}_+$ to $\mathcal{D}(\mathbf{f}_-) = (-\infty,0] \times \mathbb{R}$ and $\mathcal{D}(\mathbf{f}_+) = [0,\infty) \times \mathbb{R}$. Then the switched system (8) has a strongly asymptotically stable equilibrium at the origin. The reason for this is that the switching only has more than one option on $\mathcal{D}(\mathbf{f}_-) \cap \mathcal{D}(\mathbf{f}_+) = \{0\} \times \mathbb{R}$, and no matter which choice is made, the system state is asymptotically attracted to the origin. The decision only influences whether the left- or the right trajectory arc is traversed. Note that for the inclusion $\dot{\mathbf{x}} \in \mathrm{co}\{\mathbf{f}_-(\mathbf{x}), \mathbf{f}_+(\mathbf{x})\}$ with these domains for $\mathbf{f}_-(\mathbf{x})$ and $\mathbf{f}_+(\mathbf{x})$, the origin is weakly- but not strongly asymptotically stable, because $\mathbf{0} \in \mathrm{co}\{\mathbf{f}_-(\mathbf{x}), \mathbf{f}_+(\mathbf{x})\}$ for every $\mathbf{x} \in \{0\} \times \mathbb{R}$.

It follows from the discussion above, that there does not exists a CPA Lyapunov function V fulfilling (5) for the switched system (8), although the equilibrium is strongly asymptotically stable. The condition (5) is unnecessary strict for switched systems; it is used to assert

$$\limsup_{h \to 0+} \frac{V(\mathbf{x} + h\mathbf{f}_a(x)) - V(\mathbf{x})}{h} < 0 \tag{9}$$

for all $\mathbf{f}_a$ such that $\mathbf{x} \in \mathcal{D}(\mathbf{f}_a)$, but this may hold true although (5) fails for some of them. To see this consider two triangles, $S_\nu = \mathrm{co}\{(-c,c),(0,c),(0,2c)\}$ and $S_\mu = \mathrm{co}\{(c,c),(0,c),(0,2c)\}$ for $c > 0$, and $\mathbf{x} = (0,3c/2)$ for the switched system (8). Let V be a CPA function such that $\nabla V_\nu \bullet \mathbf{f}_-(\mathbf{x}) < 0$ and $\nabla V_\mu \bullet \mathbf{f}_+(\mathbf{x}) < 0$. Since $\mathbf{f}_-(\mathbf{x}) = -\mathbf{f}_+(\mathbf{x})$ this implies $\nabla V_\nu \bullet \mathbf{f}_+(\mathbf{x}) > 0$ and $\nabla V_\mu \bullet \mathbf{f}_-(\mathbf{x}) > 0$, but (9) still holds true. The reason is that $\mathbf{f}_-(\mathbf{x})$ points into S_ν and $\mathbf{f}_+(\mathbf{x})$ points into S_μ. In our modified CPA algorithm below we systematically remove such unnecessary constraints from the original linear programming problem.

2 The Setup

For formulating our modified CPA algorithm for switched system some preparation is needed. Let $\{S_\nu\}_{\nu \in T} = \mathcal{T}$, T an index set, be a set of simplices in $\mathbb{R}^n$, such that different simplices $S_\nu, S_\mu \in \mathcal{T}$ intersect in a common face or not at all and with $\mathcal{D}_\mathcal{T} = \bigcup_{\nu \in T} S_\nu$, $\mathcal{D}_\mathcal{T}^\circ$ is a simply connected neighborhood of the origin. The set $\mathcal{T}$ is the triangulation that we use to define a CPA function V by fixing its values at the vertices

$$\mathcal{V}_\mathcal{T} := \{\mathbf{x}_i : \mathbf{x}_i \text{ is a vertex of a simplex } S_\nu \in \mathcal{T}\}$$

of the triangulation.

For each $S_\nu \in \mathcal{T}$ denote by $C_\nu = \{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\}$ the set of its vertices. Thus $S_\nu = \mathrm{co}\, C_\nu$ and $S_\nu \cap S_\mu = \mathrm{co}(C_\nu \cap C_\mu)$. Define $I_\mathcal{T} : \mathbb{R}^n \rightrightarrows T$ by $I_\mathcal{T}(\mathbf{x}) := \{\nu \in T : \mathbf{x} \in S_\nu\}$. Thus $I_\mathcal{T}(\mathbf{x})$ is a set of the indices of the simplices in $\mathcal{T}$ containing $\mathbf{x}$. The notation $\rightrightarrows$ denotes a multivalued function, the values of $I_\mathcal{T}$ are subsets of T. We consider arbitrary switched systems as (1) and corresponding differential inclusions (2) that are adapted to the triangulation $\mathcal{T}$.

For defining what *adapted to the triangulation* $\mathcal{T}$ means some further definitions are useful: Let $\mathcal{A}$ be a finite set and $\mathbf{f}_a : \mathcal{D}(\mathbf{f}_a) \to \mathbb{R}^n$ be vector fields, such that for every $a \in \mathcal{A}$ the domain $\mathcal{D}(\mathbf{f}_a) \subset \mathbb{R}^n$ of $\mathbf{f}_a$ is the union of some of the simplices in $\mathcal{T}$. Thus, for every $a \in \mathcal{A}$ we have $\emptyset \neq \mathcal{D}(\mathbf{f}_a) = S_{\nu_1} \cup S_{\nu_2} \cup \cdots \cup S_{\nu_k}$, where $\nu_1, \nu_2, \ldots, \nu_k \in T$. Define for every $\nu \in T$ the set

$$\mathcal{A}_\nu := \{a \in \mathcal{A} : S_\nu \subset \mathcal{D}(\mathbf{f}_a)\}$$

and assume that $\mathcal{A}_\nu \neq \emptyset$ for all $\nu \in T$. Hence, on every simplex $S_\nu \in \mathcal{T}$ at least one of the vector fields $\mathbf{f}_a$ is defined.

For a simplex $S_\nu \in \mathcal{T}$ define the set

$$NS_\nu := \{S_\mu \in \mathcal{T} : S_\mu \neq S_\nu \text{ and } S_\mu \cap S_\nu \neq \emptyset\}$$

of its neighbouring simplices in $\mathcal{T}$.

Our modified CPA algorithm will eliminate unnecessary constraints from the original linear programming problem. For this we need to define for every simplex S_ν and every vector field $\mathbf{f}_a$ defined on the simplex S_ν, i.e. every $a \in \mathcal{A}_\nu$, the set of the *essential neighbouring simplices* ENS_ν^a with respect to the vector field $\mathbf{f}_a$. That is, ENS_ν^a contains the simplices $S_\mu \in NS_\nu$, such that solution trajectories of $\dot{\mathbf{x}} = \mathbf{f}_a(\mathbf{x})$ with an initial position in $\mathbf{x} \in S_\nu$ can move into the interior of S_μ in an infinitesimal time. In formula, for every $S_\nu \in \mathcal{T}$ and every $a \in \mathcal{A}_\nu$ define

$$ENS_\nu^a := \{S_\mu \in NS_\nu : \exists \mathbf{x} \in S_\nu, \exists h > 0 \\ \text{s.t. } \mathbf{x} + [0,h]\mathbf{f}_a(\mathbf{x}) \subset S_\mu\},$$

where

$$\mathbf{x} + [0,h]\mathbf{f}_a(\mathbf{x}) := \{\mathbf{x} + h'\mathbf{f}_a(\mathbf{x}) : 0 \leq h' \leq h\}.$$

Define

$$D^+V(\mathbf{x},\mathbf{y}) := \limsup_{h \to 0+} \frac{V(\mathbf{x} + h\mathbf{y}) - V(\mathbf{x})}{h}.$$

It is well known that $D^+V(\mathbf{x}, \mathbf{f}_a(\mathbf{x})$ is the orbital derivative of a locally Lipschitz V at point $\mathbf{x}$ along the solution trajectories of $\dot{\mathbf{x}} = \mathbf{f}_a(\mathbf{x})$, i.e. if $t \mapsto \phi(t)$ is the solution with $\phi(0) = \mathbf{x}$, then

$$D^+V(\mathbf{x}, \mathbf{f}_a(\mathbf{x})) = \limsup_{h \to 0+} \frac{V(\phi(h)) - V(\mathbf{x})}{h},$$

for a proof cf. e.g. (Marinósson, 2002, Th. 1.17). Further, CPA functions are obviously locally Lipschitz.

Lemma 2.1. *For every $S_\mu \in \mathcal{T}$ such that there exists an $h > 0$ with $\mathbf{x} + [0,h]\mathbf{f}_a(\mathbf{x}) \subset S_\mu$ we have $D^+V(\mathbf{x},\mathbf{f}_a(\mathbf{x})) = \nabla V_\mu \bullet \mathbf{f}_a(\mathbf{x})$.*

Proof. Assume $\mathbf{x} + [0,h]\mathbf{f}_a(\mathbf{x}) \subset S_\alpha \cap S_\beta$ for two simplices $S_\alpha, S_\beta \in \mathcal{T}$. Since V is affine on both simplices S_α and S_β we have for some constants $a_\alpha, a_\beta \in \mathbb{R}$ and ∇V_α and ∇V_β defined as above that $V(\mathbf{y}) = \nabla V_\alpha \bullet \mathbf{y} + a_\alpha$ for all $\mathbf{y} \in S_\alpha$ and $V(\mathbf{y}) = \nabla V_\beta \bullet \mathbf{y} + a_\beta$ for all $\mathbf{y} \in S_\beta$. In particular, we have for all $\mathbf{y} \in S_\alpha \cap S_\beta$ that

$$\nabla V_\alpha \bullet \mathbf{y} + a_\alpha = \nabla V_\beta \bullet \mathbf{y} + a_\beta.$$

Now substitute $\mathbf{x} + h'\mathbf{f}_a(\mathbf{x})$ for $\mathbf{y}$ and simplify to get

$$h'(\nabla V_\alpha - \nabla V_\beta) \bullet \mathbf{f}_a(\mathbf{x}) = a_\beta - a_\alpha - (\nabla V_\alpha - \nabla V_\beta) \bullet \mathbf{x}$$

and note that this equation must hold true for all $0 \leq h' \leq h$. Since the right-hand side is constant we must have $(\nabla V_\alpha - \nabla V_\beta) \bullet \mathbf{f}_a(\mathbf{x}) = 0$. The statement is now obvious. $\square$

The next lemma justifies the nomenclature for ENS^a_ν: *essential neighbouring simplices with respect to the vector field* $\mathbf{f}_a$

Lemma 2.2. *Let $\mathbf{x} \in \mathcal{D}^\circ_\mathcal{T}$. Then there is a $\nu \in T$ such that $\mathbf{x} \in S_\nu$ and for every $a \in \mathcal{A}_\nu$ we have*

$$D^+V(\mathbf{x},\mathbf{f}_a(\mathbf{x})) = \nabla V_\mu \bullet \mathbf{f}_a(\mathbf{x}) \tag{10}$$

where $\mu = \nu$ or $S_\mu \in ENS^a_\nu$.

Proof. That there is a $\nu \in T$ such that $\mathbf{x} \in S_\nu$ follows directly from our setup. Since simplices are convex and closed there surely exists an $h > 0$ and $\mu \in T$ such that $\mathbf{x} + [0,h]\mathbf{f}_a(\mathbf{x}) \subset S_\mu$ and by the definition of ENS^a_ν necessarily $S_\mu \in ENS^a_\nu$ if $\mu \neq \nu$. $\square$

3 The Modified Constraints

We now describe how we eliminate unnecessary constraints from the liner programming problem in (Baier and Hafstein, 2014). Note that this is the same linear programming problem as for the differential inclusion in (Baier et al., 2010; Baier et al., 2012), but when the differential inclusion is considered, then these constraints are necessary. Indeed, as discussed in Section 1.1 for Artstein's circles, the equilibrium might be merely weakly asymptotically stable for the differential inclusion, although it is strongly asymptotically stable for the corresponding switched system.

The constraints to enforce $\nabla V_\nu \bullet \mathbf{f}_a(\mathbf{x}) < 0$ for all $\mathbf{x} \in S_\nu$ are only verified at the vertices of the simplex S_ν. As a consequence one must verify $\nabla V_\nu \bullet \mathbf{f}_a(\mathbf{x}^\nu_i) < -a_i$ at the vertices $\mathbf{x}^\nu_i$ of the simplex S_ν for appropriate $a_i > 0$, to make sure that the inequality folds for all $\mathbf{x} \in S_\nu$. This motivates the following definitions.

For a set $C = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k\}$ of affinely independent vectors in $\mathbb{R}^n$ and a vector field $\mathbf{f}_a$ defined on $\operatorname{co} C$ with components $(f^a_1, f^a_2, \ldots, f^a_n) = \mathbf{f}_a$ define

$$B^a_{C,r,s} := \max_{\substack{\mathbf{x} \in \operatorname{co} C \\ m=1,2,\ldots,n}} \left| \frac{\partial^2 f^a_m}{\partial x_r \partial x_s}(\mathbf{x}) \right|. \tag{11}$$

Further, for each (vertex) $\mathbf{y} \in C$ define

$$C^{\max}_{\mathbf{y},s} := \max_{j=0,1,\ldots,k} |\mathbf{e}_s \bullet (\mathbf{x}_j - \mathbf{y})|$$

and let $E^{a,\mathbf{y}}_{C,\mathbf{x}_i}$, $i = 0,1,\ldots,k$, be constants such that

$$E^{a,\mathbf{y}}_{C,\mathbf{x}_i} \geq \frac{1}{2} \sum_{r,s=1}^{n} B^a_{C,r,s} |\mathbf{e}_r \bullet (\mathbf{x}_i - \mathbf{y})| (C^{\max}_{\mathbf{y},s} + |\mathbf{e}_s \bullet (\mathbf{x}_i - \mathbf{y})|). \tag{12}$$

A few comments are in order. As shown later, the constants $E^{a,\mathbf{y}}_{C,\mathbf{x}_i}$ are defined such that if for one fixed vertex $\mathbf{y}$ of the simplex $S_\nu = \operatorname{co} C_\nu \in \mathcal{T}$, $C_\nu = \{\mathbf{x}^\nu_0, \mathbf{x}^\nu_1, \ldots, \mathbf{x}^\nu_n\}$, we have

$$\nabla V_\nu \bullet \mathbf{f}_a(\mathbf{x}^\nu_i) < -E^{a,\mathbf{y}}_{C,\mathbf{x}^\nu_i} \|\nabla V_\nu\|_1, \quad i = 0,1,\ldots,k = n,$$

then $\nabla V_\nu \bullet \mathbf{f}_a(\mathbf{x}) < 0$ for all $\mathbf{x} \in S_\nu$.

For implementing the constraints it is of essential practical importance that $E^{a,\mathbf{y}}_{C,\mathbf{x}_i}$ is an *upper bound*, which in effect means that one must not compute the $B^a_{C,r,s}$ exactly, which can be difficult. Any upper bound on the second order derivatives of the components f^a_m can be used. However, less conservative bounds might mean that one needs smaller simplices to fulfill the constraints.

Finally, if

$$E^{a,\mathbf{y}}_{C,\mathbf{x}_i} \geq h^2_C \sum_{r,s=1}^{n} B^a_{C,r,s}$$

whith $h_C = \operatorname{diam}(C) := \max_{\mathbf{x},\mathbf{y} \in C} \|\mathbf{x} - \mathbf{y}\|_2$, then the estimate (12) follows. In particular one can for simplicity use a uniform bound

$$B^a \geq \max_{\substack{\mathbf{x} \in \mathcal{D}(\mathbf{f}_a) \\ m,r,s=1,2,\ldots,n}} \left| \frac{\partial^2 f^a_m}{\partial x_r \partial x_s}(\mathbf{x}) \right|$$

and set $E^{a,\mathbf{y}}_{C,\mathbf{x}^\nu_i} = n^2 B^a h^2_\nu$, where $h_\nu := \operatorname{diam}(S_\nu)$, for $\emptyset \neq C \subset C_\nu$. Indeed, a little more careful analysis using that $\|\mathbf{x}\|^2_1 \leq n\|\mathbf{x}\|^2_2$ shows that using

$$E^{a,\mathbf{y}}_{C,\mathbf{x}^\nu_i} = nB^a h^2_\nu$$

is enough to fulfill the estimate (12), cf. e.g. (Baier et al., 2012). However, although this more conservative formula is more pleasant to the eye, the implementation of (12) is not more involved in practice.

We therefore use the sharper estimate (12) in what follows. Note, however, that in order to understand the linear programming problems very little is lost if one just considers the $E^{a,\mathbf{y}}_{C,\mathbf{x}_i^\nu}$ as appropriate constants needed to interpolate inequality conditions from vertices over simplices and faces and of simplices, without following the details. Computing the $E^{a,\mathbf{y}}_{C,\mathbf{x}_i^\nu}$ algorithmically is simple given some upper bounds on the second order derivatives of the components of the vector fields $\mathbf{f}_a$.

We are now ready to state our linear programming problem to compute CPA Lyapunov functions. We first state it as in (Baier et al., 2010; Baier et al., 2012) and then outline a proof of why a feasible solution to it delivers a CPA Lyapunov function for the differential inclusion used in its construction. From the proof it becomes clear what conditions are unnecessary when we move from the differential inclusion (2) to the arbitrary switched system (1). In Section 3.2 we then discuss how the removing of constraints can be algorithmically implemented.

3.1 The Linear Programming Problem

Consider a triangulation $\mathcal{T}$ as in Section 2 and, adapted to the triangulation, the differential inclusion (2) and the corresponding arbitrary switched system (1). Assume that the equilibrium in question is at the origin. For every simplex $S_\nu \in \mathcal{T}$ let

$$C_\nu = \{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\}$$

denote its vertices, i.e. $S_\nu = \mathrm{co}\, C_\nu$. Assume that for every $S_\nu = \mathrm{co}\, C_\nu \in \mathcal{T}$ and every $\emptyset \neq C \subset C_\nu$ and every $a \in \mathcal{A}_\nu$ we have an upper bound $B^a_{C,r,s}$ as in (11) and that we have fixed a vertex $\mathbf{y}$ of C for the definition of $E^{a,\mathbf{y}}_{C,\mathbf{x}_i}$. If $\mathbf{0} \in C$ we must choose $\mathbf{y} = \mathbf{0}$ to avoid unsatisfiable constraints. Note that the sets $\mathrm{co}\, C$, where $\emptyset \neq C \subsetneq C_\nu$, are the faces of the simplex S_ν.

The variables of the linear programming problem are $V_\mathbf{x}$ for every $\mathbf{x}$ that is a vertex of a simplex in $\mathcal{T}$, i.e. $\mathbf{x} \in \mathcal{V}_\mathcal{T}$. From a feasible solution, where the variables $V_\mathbf{x}$ have been assigned values such that the linear constraints below are fulfilled, we then define a continuous function $V : \mathcal{D}_\mathcal{T} \to \mathbb{R}$ through parameterization using these values: for an $\mathbf{x} \in \mathcal{D}_\mathcal{T}$ we can find a simplex $S_\nu = \mathrm{co}\{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\}$ such that $\mathbf{x} \in S_\nu$ and $\mathbf{x}$ has a unique representation $\mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu$ as a convex sum of the vertices. For $\mathbf{x}$ we define

$$V(\mathbf{x}) := \sum_{i=0}^n \lambda_i V_{\mathbf{x}_i^\nu}.$$

If two different simplices in $\mathcal{T}$ intersect they do so in a common face, hence V is well-defined and continuous. By a slight abuse of notation we both write $V(\mathbf{x}_i^\nu)$ for the variable $V_{\mathbf{x}_i^\nu}$ of the linear programming problem and the value of the function V at $\mathbf{x}_i^\nu$, since after we have assigned a numerical value to the former it is the value of the function V at $\mathbf{x}_i^\nu$.

There are two groups of constrains in the linear programming problem. The first group is to assert that V has a minimum at the origin:

Linear constraints L1
If $\mathbf{0} \in \mathcal{V}_\mathcal{T}$ one sets $V(\mathbf{0}) = 0$. Then for all $\mathbf{x} \in \mathcal{V}_\mathcal{T}$:

$$V(\mathbf{x}) \geq \|\mathbf{x}\|_2.$$

Another possibility is to relax the condition of strong asymptotic stability of the origin to *practical strong asymptotic stability*. In this case one predefines an arbitrary small neighbourhood of the origin $\mathcal{N}$ and does not demand that V is decreasing along solution trajectories in this set. One must then make sure through constraints that

$$\max_{\mathbf{x} \in \partial \mathcal{N}} V(\mathbf{x}) < \min_{\mathbf{x} \in \partial \mathcal{D}_\mathcal{T}} V(\mathbf{x}),$$

because sublevel sets of V that are closed in $\mathcal{D}_\mathcal{T}^\circ$ are lower bounds on the basin of attraction. This is not difficult to implements and is discussed in detail in e.g. (Hafstein, 2004; Hafstein, 2007; Baier et al., 2012; Hafstein et al., 2015). In short, the implications of such a Lyapunov function are that solutions enter $\mathcal{N}$ in a finite time and either stay in $\mathcal{N}$ or stay close and enter it repeatedly.

The second group of linear constraints is to assert that V is decreasing along all solution trajectories. The simplest case is when $\mathcal{A}_\nu = \mathcal{A}$ for all $\nu \in T$ and then the appropriate constraints are:

Linear constraints L2 (simplest case)
For every $S_\nu \in \mathcal{T}$, we demand for every $a \in \mathcal{A}_\nu$ and $i = 0, 1, \ldots, n$ that:

$$\nabla V_\nu \bullet \mathbf{f}_a(\mathbf{x}_i^\nu) + \|\nabla V_\nu\|_1 E^{a,\mathbf{y}}_{C_\nu,\mathbf{x}_i^\nu} \leq -\|\mathbf{x}_i^\nu\|_2. \qquad (13)$$

In the case of practical strong asymptotic stability one disregards the constraints (13) for $S_\nu \subset \mathcal{N}$. Note that the constrains (13) are linear in the variables $V(\mathbf{x}_i^\nu)$, cf. e.g. (Giesl and Hafstein, 2014, Remarks 9 and 10), in particular $\|\nabla V_\nu\|_1$ can be modelled through linear constraint using auxiliary variables.

Now, let us consider how one uses the constraints (13) to show that $D^+V(\mathbf{x}, \mathbf{f}_a(\mathbf{x})) \leq -\|\mathbf{x}\|_2$. By (Marinósson, 2002, Lemma 4.16) we have for $a \in \mathcal{A}_\nu$ and $\mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu \in S_\nu$, $\sum_{i=0}^n \lambda_i = 1$, that

$$\left\| \mathbf{f}_a(\mathbf{x}) - \sum_{i=0}^n \lambda_i \mathbf{f}_a(\mathbf{x}_i^\nu) \right\|_\infty \leq \sum_{i=0}^n \lambda_i E^{a,\mathbf{y}}_{C_\nu,\mathbf{x}_i^\nu}. \qquad (14)$$

Hence, Hölder inequality, constraints (13), and the convexity of the norm imply that

$$\nabla V_\nu \bullet \mathbf{f}_a(\mathbf{x}) = \tag{15}$$
$$\sum_{i=0}^{n} \lambda_i \nabla V_\nu \bullet \mathbf{f}_a(\mathbf{x}_i^\nu) + \nabla V_\nu \bullet \left[\mathbf{f}_a(\mathbf{x}) - \sum_{i=0}^{n} \lambda_i \mathbf{f}_a(\mathbf{x}_i^\nu) \right]$$
$$\leq \sum_{i=0}^{n} \lambda_i \nabla V_\nu \bullet \mathbf{f}_a(\mathbf{x}_i^\nu) + \|\nabla V_\nu\|_1 \cdot \sum_{i=0}^{n} \lambda_i E^{a,\mathbf{y}}_{C_\nu,\mathbf{x}_i^\nu}$$
$$= \sum_{i=0}^{n} \lambda_i \left(\nabla V_\nu \bullet \mathbf{f}_a(\mathbf{x}_i^\nu) + \|\nabla V_\nu\|_1 E^{a,\mathbf{y}}_{C_\nu,\mathbf{x}_i^\nu} \right)$$
$$\leq \sum_{i=0}^{n} \lambda_i (-\|\mathbf{x}_i^\nu\|_2) \leq -\|\sum_{i=0}^{n} \lambda_i \mathbf{x}_i^\nu\|_2 = -\|\mathbf{x}\|_2.$$

For an $\mathbf{x} \in S_\nu^\circ$ we have the existence of an $h > 0$ such that $\mathbf{x} + h'\mathbf{f}_a(\mathbf{x}) \subset S_\nu$ for all $0 \leq h' \leq h$ and we get by Lemma 2.2 that

$$D^+V(\mathbf{f}_a(\mathbf{x}), \mathbf{x}) = \nabla V_\nu \bullet \mathbf{f}_a(\mathbf{x}) \leq -\|\mathbf{x}\|_2$$

for all $a \in \mathcal{A}_\nu$. For an $\mathbf{x} \in \partial S_\nu$ we cannot conclude this in general, because is is possible that $\mathbf{x} + [0,h]\mathbf{f}_a(\mathbf{x}) \not\subset S_\nu$ for any $h > 0$ and we need the constraints (13) to hold true with $\nu = \mu$, where S_μ is such that $\mathbf{x} + [0,h]\mathbf{f}_a(\mathbf{x}) \subset S_\mu$ for some $h > 0$. Note that if $a \in \mathcal{A}_\mu$ then this is assured, therefore nothing can go wrong if $\mathcal{A}_\mu = \mathcal{A}$ for all $\mu \in T$.

In (Baier et al., 2010; Baier et al., 2012) the general case, when $\mathcal{A}_\mu \neq \mathcal{A}$ for some $\mu \in T$, was dealt with by adding the constraints:

Linear constraints L2 (old)
For every $S_\nu = \mathrm{co}C_\nu \in \mathcal{T}$ we demand for every $a \in \mathcal{A}_\nu$, **in addition to the constraints** (13)**,** that for every $S_\mu \in NS_\nu^a$ such that $a \notin A_\mu$ we have for $C = C_\mu \cap C_\nu = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k\}$ that:

$$\nabla V_\mu \bullet \mathbf{f}_a(\mathbf{x}_i) + \|\nabla V_\mu\|_1 E^{a,\mathbf{y}}_{C,\mathbf{x}_i} \leq -\|\mathbf{x}_i\|_2 \tag{16}$$

for $i = 0, 1, \ldots, k$.

Note that the linear constraints (16) imply by computations analog to (15) that $\nabla V_\mu \bullet \mathbf{f}_a(\mathbf{x}) \leq -\|\mathbf{x}\|_2$ for every $\mathbf{x} \in \mathcal{D}(\mathbf{f}_a)$, even if $a \notin \mathcal{A}_\mu$, i.e. $S_\mu \not\subset \mathcal{D}(\mathbf{f}_a)$ and $\mathbf{f}_a$ is only defined on a face of S_μ. Just substitute n, ∇V_ν, $\mathbf{x}_i^\nu$, $E^{a,\mathbf{y}}_{C_\nu,\mathbf{x}_i^\nu}$ with k, ∇V_μ, $\mathbf{x}_i$, $E^{a,\mathbf{y}}_{C,\mathbf{x}_i}$, respectively, in the computations (15).

For the differential inclusion (2) these constraints is indeed necessary to show strong asymptotic stability of the equilibrium at the origin, or strong practical asymptotic stability of a set $\mathcal{N}$, because for a CPA function V we have

$$\partial_{\mathrm{Cl}} V(\mathbf{x}) := \mathrm{co}\{\nabla V_\mu : \mathbf{x} \in S_\mu\}$$

and

$$\mathbf{F}(\mathbf{x}) = \mathrm{co}\{\mathbf{f}_a(\mathbf{x}) : \mathbf{x} \in \mathcal{D}(\mathbf{f}_a(\mathbf{x}))\}.$$

Hence $\partial_{\mathrm{Cl}} V(\mathbf{x}) \bullet \mathbf{F}(\mathbf{x}) \subset \mathbb{R}$ consists of the elements

$$\left(\sum_{\mu : \mathbf{x} \in S_\mu} \alpha_\mu \nabla V_\mu \right) \bullet \left(\sum_{a : \mathbf{x} \in \mathcal{D}(\mathbf{f}_a)} \beta_a \mathbf{f}_a(\mathbf{x}) \right)$$
$$= \sum_{\substack{\mu : \mathbf{x} \in S_\mu \\ a : \mathbf{x} \in \mathcal{D}(\mathbf{f}_a)}} \alpha_\mu \beta_a \nabla V_\mu \bullet \mathbf{f}_a(\mathbf{x}),$$

for all $\alpha_\mu, \beta_a \geq 0$ fulfilling

$$\sum_{\mu : \mathbf{x} \in S_\mu} \alpha_\mu = \sum_{a : \mathbf{x} \in \mathcal{D}(\mathbf{f}_a(\mathbf{x}))} \beta_a = 1.$$

In the case $S_\nu \subset \mathcal{D}(\mathbf{f}_a)$, $S_\mu \not\subset \mathcal{D}(\mathbf{f}_a)$, and $\mathbf{x} \in S_\nu \cap S_\mu$, we have $S_\mu \in NS_\nu^a$ and the constraints (16) assure that $\nabla V_\mu \bullet \mathbf{f}_a(\mathbf{x}) < -\|\mathbf{x}\|_2$.

However, by Lemma 2.2, we can conclude $D^+V(\mathbf{f}_a(\mathbf{x}), \mathbf{x}) \leq -\|\mathbf{x}\|_2$ if the constraints (13) hold for all essential neighbours $S_\mu \in ENS_\nu^a$ of S_ν with respect to the vector field $\mathbf{f}_a$. We do not need to consider neighbouring simplices $S_\mu \in NS_\nu^a$ that are not essential! The modified constraints are :

New linear constraints L2
For every $S_\nu = \mathrm{co}C_\nu \in \mathcal{T}$ we demand for every $a \in \mathcal{A}_\nu$, **in addition to the constraints** (13)**,** that for every $S_\mu \in ENS_\nu^a$ such that $a \notin A_\mu$ we have for $C = C_\mu \cap C_\nu = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k\}$ that:

$$\nabla V_\mu \bullet \mathbf{f}_a(\mathbf{x}_i) + \|\nabla V_\mu\|_1 E^{a,\mathbf{y}}_{C,\mathbf{x}_i} \leq -\|\mathbf{x}_i\|_2 \tag{17}$$

for $i = 0, 1, \ldots, k$.

Note that the only difference between the old constraints and the new ones is that NS_ν^a in the old ones has been replaced by ENS_ν^a in the new ones.

For algorithmically implementing the new constraints we need to be able to generate the sets ENS_ν^a efficiently. In practice we can efficiently compute sets $(ENS_\nu^a)^*$,

$$NS_\nu^a \subset (ENS_\nu^a)^* \subset ENS_\nu^a,$$

using some of the same ideas as in the computations (15). The practical implementation of **New linear constraints L2** is then done by using these sets $(ENS_\nu^a)^*$ instead of the sets ENS_ν^a. We describe the details of how to compute the sets $(ENS_\nu^a)^*$ in the next section.

3.2 Computing Essential Neighbours

Consider a simplex $S_\nu = \mathrm{co}C_\nu \in \mathcal{T}$, where as before $C_\nu = \{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\}$. Another way to describe the simplex S_ν is to define it as the intersection of $n+1$ half-spaces. Recall that a half-space is defined as $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{n} \bullet (\mathbf{x} - \mathbf{y}) \geq 0\}$ for some vectors $\mathbf{n}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{n} \neq \mathbf{0}$.

These half-spaces for S_ν can be constructed using the set C_ν as follows, see also (Hafstein, 2017).

For $i = 0,1,\ldots,n$ we construct a half-space $H_{\nu,\mathbf{x}_i^\nu}$ such that $S_\nu \subset H_{\nu,\mathbf{x}_i^\nu}$ and $C_\nu \setminus \{\mathbf{x}_i^\nu\}$ is a subset of the boundary $\partial H_{\nu,\mathbf{x}_i^\nu}$ of $H_{\nu,\mathbf{x}_i^\nu}$ in the following way:

Set $\mathbf{y}_n := \mathbf{x}_i^\nu$ and pick an arbitrary, but fixed vector $\mathbf{y}_0 \in C_\nu \setminus \{\mathbf{x}_i^\nu\}$. Let $\{\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_{n-1}\} = C_\nu \setminus \{\mathbf{y}_0,\mathbf{y}_n\}$ and define the matrix $n \times n$ matrix

$$Y_{\nu,\mathbf{x}_i^\nu} := (\mathbf{y}_1 - \mathbf{y}_0, \mathbf{y}_2 - \mathbf{y}_0, \ldots, \mathbf{y}_n - \mathbf{y}_0)^T.$$

That is, the first row in the matrix $Y_{\nu,\mathbf{x}_i^\nu}$ is the vector $\mathbf{y}_1 - \mathbf{y}_0$, the second row is the vector $\mathbf{y}_2 - \mathbf{y}_0$, etc.

Because the vectors $\mathbf{y}_0,\mathbf{y}_1,\ldots,\mathbf{y}_n$ are affinely independent the matrix $Y_{\nu,\mathbf{x}_i^\nu}$ is non-singular and the equation

$$Y_{\nu,\mathbf{x}_i^\nu}\mathbf{y} = \mathbf{e}_n,$$

where $\mathbf{e}_n$ is the usual n-th unit basis vector, has a unique solution $\mathbf{n}_{\nu,\mathbf{x}_i^\nu}$ for $\mathbf{y}$. Define the half-space

$$H_{\nu,\mathbf{x}_i^\nu} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{n}_{\nu,\mathbf{x}_i^\nu} \bullet (\mathbf{x} - \mathbf{y}_0) \geq 0\}. \tag{18}$$

Note that $H_{\nu,\mathbf{x}_i^\nu}$ is a half-space such that $S_\nu \subset H_{\nu,\mathbf{x}_i^\nu}$ and the vertices $\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_{n-1}$ of the simplex S_ν, and hence the face $\mathrm{co}(C_\nu \setminus \{\mathbf{x}_i^\nu\})$, are in the hyper-plane $\partial H_{\nu,\mathbf{x}_i^\nu}$ dividing the space. This can be seen from $\mathbf{n}_{\nu,\mathbf{x}_i^\nu} = Y^{-1}_{\nu,\mathbf{x}_i^\nu}\mathbf{e}_n$ and then, since

$$\mathbf{n}_{\nu,\mathbf{x}_i^\nu} \bullet (\mathbf{x} - \mathbf{y}_0) = \mathbf{e}_n^T Y^{-T}_{\nu,\mathbf{x}_i^\nu}(\mathbf{x} - \mathbf{y}_0),$$

where $Y^{-T}_{\nu,\mathbf{x}_i^\nu} := \left(Y^{-1}_{\nu,\mathbf{x}_i^\nu}\right)^T$, and $Y^{-T}_{\nu,\mathbf{x}_i^\nu}(\mathbf{x} - \mathbf{y}_0) = \mathbf{e}_k$ for $\mathbf{x} = \mathbf{y}_k$, $k = 1,2,\ldots,n$, we have $\mathbf{n}_{\nu,\mathbf{x}_i^\nu} \bullet (\mathbf{y}_k - \mathbf{y}_0) = 0$ if $k = 1,2,\ldots,n-1$ and $\mathbf{n}_{\nu,\mathbf{x}_i^\nu} \bullet (\mathbf{y}_n - \mathbf{y}_0) = 1$.

Every point $\mathbf{x} \in S_\nu$ can be written uniquely as a convex combination $\mathbf{x} = \sum_{k=0}^n \lambda_k \mathbf{y}_k$ and thus

$$\mathbf{x} - \mathbf{y}_0 = \sum_{k=0}^n \lambda_k(\mathbf{y}_k - \mathbf{y}_0).$$

Hence,

$$\mathbf{n}_{\nu,\mathbf{x}_i^\nu} \bullet (\mathbf{x} - \mathbf{y}_0) = \lambda_n,$$

from which the propositions follow. Using these results it is easily verified that

$$S_\nu = \bigcap_{i=0}^n H_{\nu,\mathbf{x}_i^\nu}. \tag{19}$$

We now use this to compute a set $(ENS_\nu^a)^*$ such that $ENS_\nu^a \subset (ENS_\nu^a)^* \subset NS_\nu^a$. Fix an $S_\mu \in NS_\nu^a$ and let

$$\{\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_r\} = C_\nu \cap C_\mu$$

and

$$\{\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_s\} = C_\mu \setminus C_\nu.$$

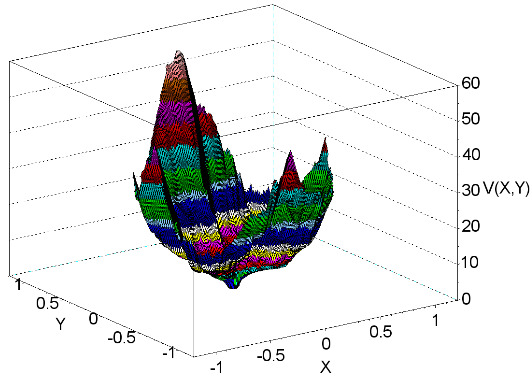

Figure 2: Continuous piecewise affine Lyapunov function computed for Artstein's circles with the new algorithm.

Corresponding to the half-spaces $H_{\mu,\mathbf{y}_j}$ are the vectors $\mathbf{n}_{\mu,\mathbf{y}_j}$ computed as above, but now for the simplex S_μ and its vertices $C_\mu \supset \{\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_s\}$. Assume that for any $j \in \{1,2,\ldots,s\}$ we have

$$0 > \mathbf{n}_{\mu,\mathbf{y}_j} \bullet \mathbf{f}_a(\mathbf{z}_i) + E^{a,\mathbf{y}}_{C_\nu \cap C_\mu,\mathbf{z}_i}\|\mathbf{n}_{\mu,\mathbf{y}_j}\|_1 \tag{20}$$

for all $i = 1,2,\ldots,r$. Then, for an arbitrary convex combination $\mathbf{x} = \sum_{i=1}^r \lambda_i \mathbf{z}_i$, we have

$$\begin{aligned}
\mathbf{n}_{\mu,\mathbf{y}_j} \bullet \mathbf{f}_a(\mathbf{x}) &= \sum_{i=1}^r \mathbf{n}_{\mu,\mathbf{y}_j} \bullet \lambda_i \mathbf{f}_a(\mathbf{z}_i) \\
&\quad + \mathbf{n}_{\mu,\mathbf{y}_j} \bullet \left(\mathbf{f}_a(\mathbf{x}) - \sum_{i=1}^r \lambda_i \mathbf{f}_a(\mathbf{z}_i)\right) \\
&\leq \sum_{i=1}^r \lambda_i \left(\mathbf{n}_{\mu,\mathbf{y}_j} \bullet \mathbf{f}_a(\mathbf{z}_i) + E^{a,\mathbf{y}}_{C_\nu \cap C_\mu,\mathbf{z}_i}\|\mathbf{n}_{\mu,\mathbf{y}_j}\|_1\right) \\
&< 0.
\end{aligned}$$

In particular, because $\mathbf{n}_{\mu,\mathbf{y}_j} \bullet (\mathbf{x} - \mathbf{z}_k) = 0$ as shown above, where $\mathbf{z}_k \in C_\nu \cap C_\mu$ corresponds to the vector $\mathbf{y}_0$ in (18), we have

$$\mathbf{n}_{\mu,\mathbf{y}_j} \bullet (h\mathbf{f}_a(\mathbf{x}) + \mathbf{x} - \mathbf{z}_k) = h\left[\mathbf{n}_{\mu,\mathbf{y}_j} \bullet \mathbf{f}_a(\mathbf{x})\right] < 0$$

for all $h > 0$. Hence, $h\mathbf{f}_a(\mathbf{x}) + \mathbf{x} \notin H_{\mu,\mathbf{y}_j}$ for any $h > 0$, which, by (19), implies $h\mathbf{f}_a(\mathbf{x}) + \mathbf{x} \notin S_\mu$ for any $h > 0$. Thus $S_\mu \notin ENS_\nu^a$.

For each $a \in \mathcal{A}_\nu$ we define $(ENS_\nu^a)^*$ as those $S_\mu \in NS_\nu$ that are not eliminated by this process. That is, $S_\mu \in (ENS_\nu^a)^*$ if for all $j = 1,2,\ldots,s$ the inequality (20) **fails to hold true** for at least one $i \in \{1,2,\ldots,r\}$. Note that this is easily checked algorithmically. For a visual illustration of the sets $(ENS_\nu^a)^*$ see Fig. 3.

Returning to our example of Artstein's circles, this new algorithm easily generates a CPA Lyapunov

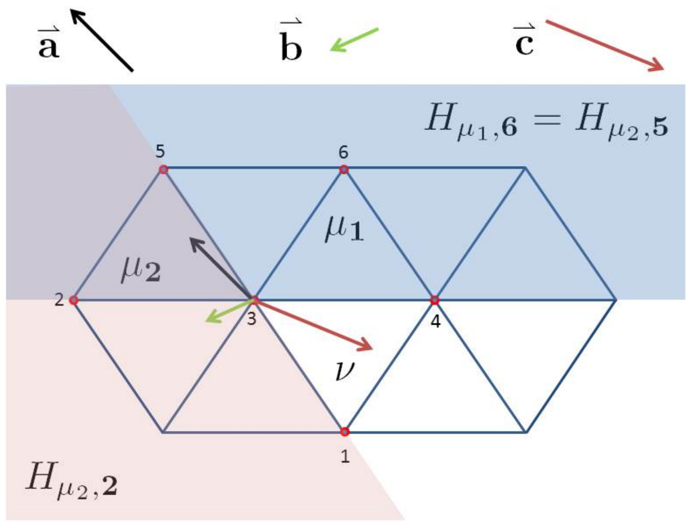

Figure 3: The simplex S_ν is the convex combination of the $3 = n+1$ vertices $\mathbf{1},\mathbf{3},\mathbf{4}$, i.e. $S_\nu = \mathrm{co}\{\mathbf{1},\mathbf{3},\mathbf{4}\}$, and $S_{\mu_1} = \mathrm{co}\{\mathbf{3},\mathbf{4},\mathbf{6}\}$, and $S_{\mu_2} = \mathrm{co}\{\mathbf{2},\mathbf{3},\mathbf{5}\}$. Clearly $\{S_{\mu_1}, S_{\mu_2}\} \subset NS_\nu$. We consider three constant $\mathbf{f}_a(\mathbf{x})$ on S_ν: $\mathbf{f}_1(\mathbf{x}) = \vec{\mathbf{a}}$, $\mathbf{f}_2(\mathbf{x}) = \vec{\mathbf{b}}$, and $\mathbf{f}_3(\mathbf{x}) = \vec{\mathbf{c}}$; depicted by arrows. Since the vector fields are constant the sets ENS_ν^a and $(ENS_\nu^a)^*$ coincide. Now $S_\nu \cap S_{\mu_1} = \mathrm{co}\{\mathbf{3},\mathbf{4}\}$ and the half-space $H_{\mu_1,\mathbf{6}}$, with $\mathrm{co}\{\mathbf{3},\mathbf{4}\}$ at its boundary and containing S_{μ_1}, is depicted in blue. We have $S_{\mu_1} \in (ENS_\nu^1)^*$ because $\mathbf{f}_1(\mathbf{x}) = \vec{\mathbf{a}}$ points into $H_{\mu_1,\mathbf{6}}$ for (some) $\mathbf{x} \in \mathrm{co}\{\mathbf{3},\mathbf{4}\}$ but $S_{\mu_1} \notin (ENS_\nu^a)^*$ for $a = 2,3$ because $\mathbf{f}_2(\mathbf{x}) = \vec{\mathbf{b}}$ and $\mathbf{f}_3(\mathbf{x}) = \vec{\mathbf{c}}$ do not point into $H_{\mu_1,\mathbf{6}}$ for any $\mathbf{x} \in \mathrm{co}\{\mathbf{3},\mathbf{4}\}$. Similarly, $S_\nu \cap S_{\mu_2} = \{\mathbf{3}\}$ and the $(n-1)$-faces $\mathrm{co}\{\mathbf{2},\mathbf{3}\}$ and $\mathrm{co}\{\mathbf{3},\mathbf{5}\}$ of S_{μ_2} contain $S_\nu \cap S_{\mu_2}$. The half-spaces $H_{\mu_2,\mathbf{5}}$ (blue) and $H_{\mu_2,\mathbf{2}}$ (red) are supersets of S_{μ_2} and with $\mathrm{co}\{\mathbf{2},\mathbf{3}\}$ and $\mathrm{co}\{\mathbf{3},\mathbf{5}\}$ at their boundaries, respectively, are depicted. Now $S_{\mu_2} \in (ENS_\nu^1)^*$ because $\mathbf{f}_1(\mathbf{x}) = \mathbf{a}$ points into both $H_{\mu_2,\mathbf{5}}$ (blue) and $H_{\mu_2,\mathbf{2}}$ for $\mathbf{x}$ at the vertex $\mathbf{3}$, but $S_{\mu_2} \notin (ENS_\nu^2)^*$ because $\mathbf{f}_2(\mathbf{x}) = \vec{\mathbf{b}}$ does not point into $H_{\mu_2,\mathbf{5}}$ and $S_{\mu_2} \notin (ENS_\nu^3)^*$ because $\mathbf{f}_2(\mathbf{x}) = \vec{\mathbf{c}}$ does neither point into $H_{\mu_2,\mathbf{5}}$ nor $H_{\mu_2,\mathbf{2}}$.

function for the system (8) in a few seconds after the domains of $\mathbf{f}_+$ and $\mathbf{f}_-$ have been fixed as $\mathcal{D}(\mathbf{f}_-) = (-\infty,0] \times \mathbb{R}$ and $\mathcal{D}(\mathbf{f}_+) = [0,\infty) \times \mathbb{R}$, see Fig. 2. With the old algorithm or when $\mathcal{D}(\mathbf{f}_+) = \mathcal{D}(\mathbf{f}_-) = \mathbb{R}^2$ no such CPA Lyapunov function exists and the algorithm reports that the corresponding linear programming problem is infeasible.

4 Conclusions

We presented a novel algorithm that uses linear programming to parameterize continuous and piecewise affine Lyapunov functions asserting strong asymptotic stability of equilibria for arbitrary switched systems, for which the same equilibria of the corresponding differential inclusion is merely weakly asymptotically stable. For the differential inclusion no such Lyapunov function can exists. This algorithm is an adaptation of an earlier algorithm for differential inclusions presented in (Baier et al., 2010; Baier et al., 2012; Baier and Hafstein, 2014). Artstein's circles were studied and used as motivation for the new approach in the paper.

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