SMOOTH COMPLETE LYAPUNOV FUNCTIONS FOR ODEs

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Abstract. We establish a link between complete Lyapunov functions in dynamical systems and time functions in general relativity. This result is the first converse theorem for smooth complete Lyapunov functions for general autonomous ODEs and a novel characterization of the chain recurrent set using cone fields.

1. Introduction

Lyapunov functions play an essential role in the stability theory of dynamical systems. However, a classical Lyapunov function is only defined for one attractor. Complete Lyapunov functions, introduced by Auslander [2] and Conley [4] for ODEs on compact state spaces, generalize the concept to characterize the qualitative behaviour of solutions on the entire state space; see also Franks [8] and Akin [1] for comparable results for mappings and relations, respectively. A classical Lyapunov function for a dynamical system, whose dynamics is given by an ODE $\dot{\gamma}(t) = X_{\gamma(t)}$, where $X_{\gamma(t)}$ denotes the value of the vector field $X\colon \mathbb{R}^m \supset U \to \mathbb{R}^m$ at the point $\gamma(t) \in U$, is a continuous function defined on the neighbourhood N_A of one attractor A, decreasing along all solution trajectories $t \mapsto \gamma(t)$ in $N_A \setminus A$. A complete Lyapunov function (see [4] and Definition 4.5) is defined on the whole state space U and is decreasing along solution trajectories whenever possible. The state space is divided into two disjoint areas, where the flow is fundamentally different: on the chain recurrent set the motion is (almost) recurrent, e.g. stationary, periodic or almost periodic, and on this set no continuous function can be decreasing along solution trajectories. On the complement of the chain recurrent set the flow is gradient-like, i.e. resembles $\dot{\gamma}(t) = -\nabla S_{\gamma(t)}$ for a scalar field $S\colon \mathbb{R}^m \supset U \to \mathbb{R}$, and solutions flow through. Whereas the chain recurrent set is sensitive to infinitesimal perturbations, the gradient-like flow is not. Hurley later proved the existence of complete Lyapunov functions for dynamical systems on noncompact spaces [11, 12, 13], see also Patrão [15], and recently the existence of smooth complete Lyapunov functions on compact state spaces was proved by Fathi and Pageault [5]. Here and elsewhere in the paper smooth means C^∞.

Time functions in general relativity were introduced by Hawking [10], where he showed that stable causality is the necessary and sufficient condition for the existence of a cosmic time function which increases along every future directed timelike or null curve. Given stable causality there are no closed timelike curves in

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any Lorentz metric that is sufficiently close to the spacetime metric. Time functions are closely related to Lyapunov functions as first noted by Fathi and Siconolfi in [7]. One significant difference, however, is that by definition time functions are concerned with causal evolutions in the given spacetime and therefore are always gradient-like, i.e. only exist in the absence of (almost) recurrent events. A notational difference is that time functions are increasing along solution trajectories, rather than decreasing like Lyapunov functions in dynamical systems.

In [3] Lyapunov functions for ODEs and time functions for general relativity were joined into a single theory of Lyapunov functions for cone fields, see Definition 3.7. Roughly speaking a cone field is the pointwise convex hull of the values of a family of vector fields and a Lyapunov function for a cone field is the negative of a simultaneous Lyapunov function for all vector fields defining the given cone field, see Definition 4.3. Note that Lyapunov functions for cone fields have the same sign convention as time functions, i.e. opposite to the sign convention of Lyapunov functions for ODEs. The main existence result for Lyapunov functions for cone fields in [3] states that for a given cone field there exists a smooth Lyapunov function which is regular whenever possible.

In this paper we establish a link between complete Lyapunov functions for dynamical systems with dynamics defined by an ODE and Lyapunov functions for cone fields. This link allows us to use results from Bernard and Suhr [3] to deliver a smooth converse theorem for complete Lyapunov functions for general autonomous ODEs. Apart from its theoretical importance, the existence of a smooth complete Lyapunov function is advantageous for applications and essential for proving that numerical methods for the computation of complete Lyapunov functions work as intended, cf. Giesl et al. [9].

In more detail: Just as in the case of classical Lyapunov functions, the differentiability of the Lyapunov function $V\colon U \to \mathbb{R}$ allows for the characterization of the decrease condition of solution trajectories to the ODE $\dot\gamma(t) = X_{\gamma(t)}$ through the formula $V'(x) := \langle \nabla V(x), X_x\rangle$ for the orbital derivative. This formula does not contain an explicit reference to the solution of the system and this is one of the key benefits of the Lyapunov stability theory: one can study the qualitative behaviour of a dynamical system through a Lyapunov function without knowing the solution to the associated ODE. Further, for a smooth Lyapunov function the orbital derivative inherits the smoothness properties from the vector field X, which is of great advantage when studying numerical methods and robustness.

The paper is organized as follows: In Section 2 we set the stage and show that it is sufficient to consider ODEs, of which all solution trajectories are defined on the entire real line. Note that for our application it is more natural to refer to solutions trajectories of the ODE $\dot\gamma(t) = X_{\gamma(t)}$ as X-orbits because we study their properties in relation to the curves tangent to the cone field C_X defined in (1). Further, by the results in §2 it seems more natural to define complete Lyapunov functions for vector fields X, cf. Definition 4.5, rather than for the related ODEs $\dot\gamma(t) = X_{\gamma(t)}$.

In Section 3 we define the relevant concepts of chain recurrence for X-orbits in §3.1 and in §3.2 we recall definitions of cone fields and stable recurrence from [3].

Section 4 states the main results. In Theorem 4.1 we establish that the notions of chain recurrence for the X-orbits and stable recurrence of the associated cone field $\mathcal{C}_X$, cf. (1), are equivalent. We then show that a smooth Lyapunov function

τ for the cone field $\mathcal{C}_X$ delivers a function $V = -\tau$ that is almost a complete Lyapunov function for the vector field X and we close the gap; recall that a Lyapunov function for $\mathcal{C}_X$ is increasing, not decreasing, along X-orbits. In more detail, [3, Theorem 2] establishes the existence of a particular Lyapunov function τ for the cone field $\mathcal{C}_X$ in the sense of Definition 4.3, cf. Theorem 4.4 (a),(b),(c). The only property $V = -\tau$ is missing to be a complete Lyapunov function for X in the sense of Definition 4.5 is that $V(\mathcal{R}_X^{\mathrm{chain}})$ should additionally be nowhere dense in $\mathbb{R}$. This is established by Theorem 4.4 (d) for τ and in Corollaries 4.7 and 4.8 we explicitly state analogous useful propositions about the complete Lyapunov function $V = -\tau$ for the vector field X. The difference between the different kinds of (complete) Lyapunov functions in the literature is somewhat involved and we explain the differences with a simple example in Remark 4.6. To express it absolutely clear: V is a C^∞ complete Lyapunov function for the ODE $\dot\gamma(t) = X_{\gamma(t)}$ that fulfills all the criteria stated by Conley [4].

In Section 5 we then prove the theorems. In §5.1 and §5.2 we prove the equivalence of the notions of recurrence for the vector field X and the cone field $\mathcal{C}_X$, i.e. Theorem 4.1, and in §5.3 we prove Theorem 4.4 (d).

2. The Setting

Recurrence for dynamical systems is a ubiquitous phenomenon. We will therefore be as general as reasonably possible in our exposition while keeping an applicator-friendly point-of-view. We will always consider a connected and open set $U \subset \mathbb{R}^m$ and continuous $X\colon U \to \mathbb{R}^m$. Note that we explicitly do not assume X to be uniquely integrable, i.e. we do not exclude ODEs $\dot\gamma(t) = X_{\gamma(t)}$ where different solutions trajectories can emerge from the same initial value.

Definition 2.1. *A curve $\gamma\colon I \to U$ is an* **integral curve of X** *or an* **X-orbit***, if γ is continuously differentiable with $\dot\gamma(t) = X_{\gamma(t)}$ for all $t \in I$.*

Note that it suffices to assume that $\dot\gamma(t)$ exists for almost all $t \in I$ and satisfies $\dot\gamma(t) = X_{\gamma(t)}$ whenever it exists. Since X is continuous a solution in the weaker sense is continuously differentiable and $\dot\gamma(t) = X_{\gamma(t)}$ holds everywhere.

An X-orbit $\gamma\colon (a,b) \to U$ is **inextensible** if $\gamma(t)$ for neither $t \downarrow a$ nor $t \uparrow b$ accumulates in U, cf. e.g. [18, §6,§7]. Then the vector field $X\colon U \to \mathbb{R}^m$ is **complete** if all inextensible X-orbits are defined on the entire real line, i.e. $\gamma\colon \mathbb{R} \to U$ for all inextensible X-orbits.

Remark 2.2. *We will see below that assuming that $X\colon U \to \mathbb{R}^m$ is complete poses no restriction on the scope of our results.*

For a more detailed treatment of the following we refer to [17] or e.g. [14, Chapter 1] or [16, §3.1] for a more accessible discussion. Given a continuous $X\colon U \to \mathbb{R}^m$ there exists a smooth function $r\colon U \to (0,1)$ such that $Y := rX$ is complete. Note that Y has the same orbits as X. The essential idea is to choose r fulfilling, cf. [3, Lemma 5.4],

$$r_x \le \frac{1}{1+\|X_x\|}\frac{\operatorname{dist}(x,\partial U)}{1+\operatorname{dist}(x,\partial U)},$$

where $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$ is the Euclidian norm and $\operatorname{dist}(x,M) := \inf_{y\in M}\|x-y\|$ for $M \ne \emptyset$ and $:= 1$ if $M = \emptyset$. We claim that every inextensible Y-orbit is complete.

Let $\gamma\colon (a,b) \to U$ be a Y-orbit. It suffices to consider the case $b < \infty$. Fix $t_0 \in (a,b)$. If $b < \infty$ we have for the length of the trajectory

$$L(\gamma|_{[t_0,b)}) \le b - t_0 < \infty$$

since $\|\dot\gamma(t)\| = \|Y_{\gamma(t)}\| \le 1$. This implies that $z := \lim_{t\uparrow b}\gamma(t)$ exists; either in U or in ∂U. The former case is a contradiction to γ being inextensible in U. In the latter case we have by the Cauchy-Schwarz inequality

$$\begin{aligned}-\frac{d}{dt}\|z-\gamma(t)\| = -\frac{\langle z-\gamma(t), \dot\gamma(t)\rangle}{\|z-\gamma(t)\|} &\le \|\dot\gamma(t)\| \\ &\le \operatorname{dist}(\gamma(t), \partial U) \le -(-\|z-\gamma(t)\|)\end{aligned}$$

since $z \in \partial U$. Now Gronwall's Lemma implies for all $t \in [t_0, b)$ that

$$-\|z-\gamma(t)\| \le e^{-(t-t_0)}(-\|z-\gamma(t_0)\|),$$

i.e.

$$\|z-\gamma(t)\| \ge e^{t_0-t}\|z-\gamma(t_0)\| \ge e^{t_0-b}\|z-\gamma(t_0)\| > 0,$$

which clearly contradicts the definition of z.

Now, for a smooth function $V\colon U \to \mathbb{R}$ we have for $Y = rX$ and all $p \in U$ that

$$\langle \nabla V(p), Y_p\rangle = r_p\langle \nabla V(p), X_p\rangle.$$

In particular, with $r\colon U \to (0,\infty)$ the orbital derivatives $\langle \nabla V(p), Y_p\rangle$ and $\langle \nabla V(p), X_p\rangle$ have the same sign and vanish on the same set.

3. Two notions of recurrence

We first put down the necessary definitions of chain recurrence in dynamical systems and stable recurrence for time functions, before we state our results in the next section.

3.1. **Chain recurrence and chain equivalence.** The following definitions are inspired by [4, 12]. We assume that $X\colon U \to \mathbb{R}^m$ is continuous and complete.

Definition 3.1. *Let $T > 0$ and $\varepsilon\colon U \to (0,\infty)$ be continuous. A finite collection of points $p_0, \dots, p_n \in U$ $(n \ge 1)$ is an* (ε, T)**-chain** *if there exist $t_i \ge T$ and X-orbits $\gamma_i\colon [0, t_i] \to U$ with*

$$\gamma_i(0) = p_i \quad \textit{and} \quad \|\gamma_i(t_i) - p_{i+1}\| \le \varepsilon(\gamma_i(t_i))$$

for all $0 \le i \le n-1$.

Definition 3.2. *A point $p \in U$ is* **chain recurrent for X** *if for all $T > 0$ and $\varepsilon\colon M \to (0,\infty)$ there exists an (ε, T)-chain $p_0 = p, p_1, \dots, p_n = p$.*

Denote with

$$\mathcal{R}_X^{\text{chain}}$$

the set of chain recurrent points for the complete vector field X. If X is not complete, choose $r\colon U \to (0,\infty)$ such that rX is complete (see Section 2) and set

$$\mathcal{R}_X^{\text{chain}} := \mathcal{R}_{rX}^{\text{chain}}.$$

We will see a posteriori (Remark 4.2) that the definition of chain recurrence does not depend on the chosen function r.

Definition 3.3. *The point* **q is chain reachable from p**, *denoted* $\mathbf{p} \to_{\mathbf{X}} \mathbf{q}$, *if there exists* $T > 0$ *such that for all continuous* $\varepsilon\colon U \to (0,\infty)$ *there exists an* (ε, T)*-chain* $p = p_0, p_1, \dots, p_n = q$.

Note that for every $p \in \mathcal{R}_X^{\text{chain}}$ we have $p \to_X p$, i.e.

$$\mathcal{R}_X^{\text{chain}} \subset \{p \in U \mid p \to_X p\}.$$

The equality of these sets will be a consequence of the results below. The equality is analogue to the equivalence of the chain recurrent set of a semiflow and its time one map in [12, Theorem 5].

The set of chain reachable points from p together with the point p itself is denoted with

$$\mathcal{F}_X^{\text{chain}}(p) := \{q \in U \mid q = p \text{ or } p \to_X q\}.$$

If X is not complete choose $r\colon U \to (0,\infty)$ such that rX is complete. Set

$$\mathcal{F}_X^{\text{chain}}(p) := \mathcal{F}_{rX}^{\text{chain}}(p)$$

and "$\to_X := \to_{rX}$". We will see a posteriori (Remark 4.2) that the definition of chain reachability does not depend on the chosen function r.

Note that if $p \to_X q$ and $T > 0$ is chosen according to Definition 3.3 then for all $0 < T' \le T$ there exists a (ε, T')-chain from p to q for all continuous $\varepsilon\colon U \to (0,\infty)$.

Example 3.4. *Let* $U = \mathbb{R}^m$ *and* $X \equiv e_1$. *Then* X *is uniquely integrable (i.e. exactly one solution trajectory through each point) and complete. Its flow is given for* $(x,y) \in \mathbb{R} \times \mathbb{R}^{m-1}$ *by* $\varphi_t(x,y) = (x+t, y)$, *a horizontal translation. We claim that* (z,w) *is chain reachable from* (x,y), *iff* $w = y$ *and* $z \ge x$. *It is obvious that under these conditions* (z,w) *is chain reachable from* (x,y), *since it lies on the forward orbit through* (x,y). *Conversely assume that* (z,w) *is chain reachable from* (x,y). *Then for some* $T \in (0,1/2)$ *we have for every* $\varepsilon \equiv \varepsilon_0 \le T^2$ *that there exists an* (ε,T)*-chain* $(x_0,y_0) = (x,y), (x_1,y_1), \dots, (x_n,y_n) = (z,w)$. *Since* $\varepsilon_0 \le T^2 \le T/2$ *and* $x_i - x_{i-1} \ge T - \varepsilon_0$ *we have* $x_i \ge x_{i-1} + T/2$, *i.e.* $z \ge x$. *Further we have* $\|y_i - y_{i-1}\| \le \varepsilon_0$. *Since in the interval* $[x,z]$ *there can occur no more than* $2\frac{z-x}{T}$*-many jumps we have*

$$\|w - y\| \le \sum_{i=1}^{n} \|y_i - y_{i-1}\| \le 2\frac{z-x}{T}\varepsilon_0 = 2(z-x)\sqrt{\varepsilon_0}.$$

For $\varepsilon_0 \downarrow 0$ *the claim follows.*

Definition 3.5. *Two points* $p, q \in R_X^{chain}$ *are* **chain transitive** *if* $p \to_X q$ *and* $q \to_X p$, *i.e. there exists* $T > 0$ *such that for all* $\varepsilon\colon U \to (0,\infty)$ *there exists an* (ε,T)*-chain* $p = p_0, p_1, \dots, p_n = p$ *containing* q.

It is easy to see that the property to be chain transitive is an equivalence relation on the chain recurrent set. For $p \in \mathcal{R}_X^{\text{chain}}$ denote with $\mathcal{R}_X^{\text{chain}}(p)$ the **chain transitive components of** p, cf. e.g. [13, §2.3].

3.2. **Stable recurrence.** The definitions in this section are taken from [3]. A *convex cone* in the vector space E is a convex subset $C \subset E$ such that $tx \in C$ for each $t > 0$ and $x \in C$. The convex cone C is called *regular* if it is not empty and it is contained in an open half-space, or equivalently, if there exists a linear form τ on E such that $\tau \cdot v > 0$ for each $v \in C$. The full cone $C = E$ will be called the *singular cone.*

Definition 3.6. *We say that $\Omega \subset E$ is an* **open cone** *if it is a convex cone which is open as a subset of E.*

We say that $C \subset E$ is a **closed cone** *if it is a convex cone which is regular and $C \cup \{0\}$ is a closed subset of E, or if it is singular.*

A *cone field* $\mathcal{C}$ on the open set U is a subset of $U \times \mathbb{R}^m$ such that $\mathcal{C}(p) := (\{p\} \times \mathbb{R}^m) \cap \mathcal{C}$ is a convex cone for each p.

Definition 3.7. *We say that $\mathcal{E} \subset U \times \mathbb{R}^m$ is an* **open cone field** *if it is a cone field which is open as a subset of $U \times \mathbb{R}^m$.*

We say the $\mathcal{C} \subset U \times \mathbb{R}^m$ is a **closed cone field** *if it is a cone field such that $\mathcal{C} \cup (U \times \{0\})$ is a closed subset of $U \times \mathbb{R}^m$ and such that $\mathcal{C}(p)$ is a closed cone for each p.*

For an open cone field $\mathcal{E}$ the cones $\mathcal{E}(p)$ are open cones in $\{p\} \times \mathbb{R}^m$.

Given a closed cone field $\mathcal{C}$, each point $p \in U$ is of one and only one of the following types:

- Regular, which means that $\mathcal{C}(p)$ is a regular cone, or
- singular, which means that $\mathcal{C}(p) = \{p\} \times \mathbb{R}^m$, or
- degenerate, which means that $\mathcal{C}(p)$ is empty.

For a subset A of a vector space define $\mathrm{pos}(A) := \{\lambda v |\ v \in A, \lambda > 0\}$. Then we associate with every continuous vector field X the cone field $\mathcal{C}_X \subset U \times \mathbb{R}^m$ by

$$\mathcal{C}_X(p) := \begin{cases} \{p\} \times \mathrm{pos}(X_p), & \text{if } X_p \neq 0 \\ \{p\} \times \mathbb{R}^m, & \text{if } X_p = 0. \end{cases} \tag{1}$$

Following [3] $\mathcal{C}_X$ is an example of a closed cone field.

We say that the cone field $\mathcal{C}'$ is **wider** than the cone field $\mathcal{C}$ if $\mathcal{C} \subset \mathcal{C}'$. We say that $\mathcal{C}'$ is an **enlargement** of $\mathcal{C}$, written $\mathcal{C} \prec \mathcal{C}'$, if there exists an open cone field $\mathcal{E}$ and a closed cone field $\mathcal{D}$ such that $\mathcal{C} \subset \mathcal{D} \subset \mathcal{E} \subset \mathcal{C}'$. An **open enlargement** of a closed cone field $\mathcal{C}$ is just an open cone field wider than $\mathcal{C}$. For a vector field $X : U \to \mathbb{R}^m$ and continuous $\varepsilon : U \to (0, \infty)$ the cone field $\mathcal{E}_{X,\varepsilon} \subset U \times \mathbb{R}^m$ defined by

$$\mathcal{E}_{X,\varepsilon}(p) := \{p\} \times \mathrm{pos}(B_{\varepsilon(p)}(X_p))$$

is an open enlargement of $\mathcal{C}_X$, where $B_\delta(x)$ denotes an open ball centered at x and with radius $\delta > 0$.

Given an open cone field $\mathcal{E}$, we say that the curve $\gamma : I \to U$ is **$\mathcal{E}$-timelike** (or just **timelike**) if it is piecewise smooth and if $\dot\gamma(t) \in \mathcal{E}(\gamma(t))$ for all t in I where γ is smooth. At non smooth points, the inclusion is required to hold for left and right differentials. The last condition insures that tangents to timelike curves do not accumulate on the boundary of $\mathcal{E}$. This intuition is inspired by Lorentzian geometry. The **chronological future** $\mathcal{I}^+_{\mathcal{E}}(p)$ of $p \in U$ relative to $\mathcal{E}$ is the set of points q such that there exists an $\mathcal{E}$-timelike curve from p to q.

Definition 3.8. *The* **stable future of p (with respect to** $\mathcal{C}$*) is the set*

$$\mathcal{F}^+_{\mathcal{C}}(p) := \{p\} \cup \bigcap_{\mathcal{C} \prec \mathcal{E}} \mathcal{I}^+_{\mathcal{E}}(p),$$

where the intersection is taken over all open enlargements $\mathcal{E}$ of $\mathcal{C}$.

Definition 3.9. *A point $p \in U$ is said to be* **stably recurrent (for $\mathcal{C}$)** *if $p \in \bigcap_{\mathcal{C} \prec \mathcal{E}} \mathcal{I}^+_{\mathcal{E}}(p)$, i.e. for each open enlargement $\mathcal{E}$ of $\mathcal{C}$, there exists an $\mathcal{E}$-timelike curve $\gamma\colon I \to U$ and $s < t \in I$ with $p = \gamma(s) = \gamma(t)$.*

We denote with

$$\mathcal{R}_{\mathcal{C}}$$

the set of stably recurrent points of the cone field $\mathcal{C}$. Note that if $\mathcal{C} = \mathcal{C}_X$ then $\mathcal{R}_{\mathcal{C}_X} = \mathcal{R}_{\mathcal{C}_Y}$ for $Y := rX$, $r\colon U \to (0, \infty)$ continuous.

Definition 3.10. *Two points $p, q \in \mathcal{R}_{\mathcal{C}}$ are* **stably equivalent** *if for every open enlargement $\mathcal{E}$ of $\mathcal{C}$ both points lie on a common $\mathcal{E}$-timelike loop.*

The sets of stably equivalent points are called **stable classes**. The **stable class of $\mathbf{p}$** $\in \mathcal{R}_{\mathcal{C}}$ is denoted with $\mathcal{R}_{\mathcal{C}}(p)$.

4. Main Results

With all definitions in place we can state our main results. Theorem 4.1 establishes the connection between the notions of recurrence and chain transitivity for vector fields and cone fields. Together with [3, Theorem 2] (here Theorem 4.4(a)-(c)) and the new Theorem 4.4(d), we obtain useful consequences on the existence of smooth complete Lyapunov functions for vector fields; stated in Corollary 4.7 and 4.8.

Theorem 4.1. *Let $U \subset \mathbb{R}^m$ be open and $X\colon U \to \mathbb{R}^m$ be a continuous complete vector field. Then both notions of recurrence coincide, i.e.*

$$\mathcal{R}_X^{chain} = \mathcal{R}_{\mathcal{C}_X},$$

and the chain transitive components and the stable classes coincide, i.e.

$$\mathcal{R}_X^{chain}(p) = \mathcal{R}_{\mathcal{C}_X}(p)$$

for every $p \in \mathcal{R}_X^{chain}$.

Further,

$$\mathcal{F}_X^{chain}(p) = \mathcal{F}^+_{\mathcal{C}_X}(p)$$

for all $p \in U$.

Remark 4.2. *An immediate corollary of the theorem is the equivalence:*

$$\mathcal{R}_X^{chain} = \{p \in U |\, p \to_X p\}$$

The theorem further shows that the definition of the chain recurrent set and of chain reachability for an incomplete vector field X is independent of the "rescaling" function $r\colon U \to (0, \infty)$ for which rX is complete.

For a function $\tau : U \to \mathbb{R}$ we call a point $p \in U$ critical if the differential of τ vanishes at p, i.e. $d\tau_p = 0$ and refer to $y = \tau(p)$ as a critical value of τ. A point $p \in U$ that is not critical is called regular and a value y of τ that is not critical, i.e. there exists no critical $p \in U$ such that $y = \tau(p)$, is said to be a regular value.

The next definition unifies time functions and Lyapunov functions for cone fields. Note however, that this definition of a Lyapunov function τ for a cone field only forces τ to be increasing along curves tangent to $\mathcal{C}_X$ and critical at (almost) recurring points, e.g. equilibria or the points on a periodic orbits or almost periodic orbits, etc. As explained in Remark 4.6 it corresponds to the Lyapunov functions studied by Auslander for ODEs [2] and not the more sophisticated complete Lyapunov functions introduced later by Conley in [4].

Definition 4.3. [3, Definition 1.4] *Let $\mathcal{C}$ be a cone field on U. The function $\tau: U \to \mathbb{R}$ is a* **Lyapunov function for the cone field** *$\mathcal{C}$ if it is smooth, $d\tau_p(v) \geq 0$ for each $p \in U$ and $v \in \mathcal{C}(p)$, and if, at each regular point p of τ (i.e. $d\tau_p \neq 0$), we have $d\tau_p(v) > 0$ for each $v \in \mathcal{C}(p)$.*

We have the following existence result for Lyapunov functions.

Theorem 4.4. *Let $X: U \to \mathbb{R}^m$ be a continuous vector field. Then there exists a Lyapunov function $\tau: U \to \mathbb{R}$ for the cone field $\mathcal{C}_X$ with the following properties:*

(a) *The function τ is regular at each point of $U \setminus \mathcal{R}_{\mathcal{C}_X}$ and critical at each point of $\mathcal{R}_{\mathcal{C}_X}$.*
(b) *Two points p and p' of $\mathcal{R}_{\mathcal{C}_X}$ belong to the same stable class iff $\tau(p') = \tau(p)$.*
(c) *If p and p' are two points of U such that $p' \in \mathcal{F}^+_{\mathcal{C}_X}(p)$ and $p \notin \mathcal{F}^+_{\mathcal{C}_X}(p')$, then $\tau(p') > \tau(p)$.*
(d) *The set $\tau(\mathcal{R}_{\mathcal{C}_X})$ of critical values of τ is nowhere dense in $\mathbb{R}$.*

The assertion is [3, Theorem 2] for the present special case $\mathcal{C} = \mathcal{C}_X$, except for point (d) which is proved in Section 5.3. Properties (a) and (b) imply that $\mathcal{R}_{\mathcal{C}_X}$ is a closed set, as well as the stable classes. Together with Theorem 4.1 we obtain that $\mathcal{R}_X^{\text{chain}}$ as well as the chain transitive components are closed.

The following definition is adapted from [4, §6.4].

Definition 4.5. *A* **complete Lyapunov function for the continuous vector field X** *is a continuous function $V: U \to \mathbb{R}$, which is strictly decreasing along orbits outside of the chain recurrent set and such that (1) $V(\mathcal{R}_X^{chain})$ is nowhere dense and (2) for $t \in V(\mathcal{R}_X^{chain})$ the set $V^{-1}(t) \cap \mathcal{R}_X^{chain}$ is a chain transitive component.*

Remark 4.6. *Consider the vector field $X_{(x,y)} = (-y, x)$ in the plane $U = \mathbb{R}^2$. The solution trajectories of $\dot{\gamma}(t) = X_{\gamma(t)}$ are circles centered at the origin. It is not difficult to see that $\mathcal{R}_X^{chain}(p) = \mathcal{R}_X^{chain} = U$ for all $p \in U$. Note that the function $V(p) = \|p\|^2$, although nonincreasing along all solution trajectories and strictly decreasing on $U \setminus \mathcal{R}_X^{chain} = \emptyset$, is not a complete Lyapunov function for X in the sense of Definition 4.5 (nor Conley [4, §6.4]) because it neither fulfills (1) nor (2). Additionally, it is not difficult to see that $\tau = -V$ does not fulfill the properties (a), (b), (d) of the Lyapunov function from Theorem 4.4. However, $\tau = -V$ is a Lyapunov function for the cone field $\mathcal{C}_X$ according to Definition 4.3* and *V fulfills the conditions of Theorem 2 in [2] by Auslander, where prototypes of a complete Lyapunov functions were first introduced.*

A more sophisticated complete Lyapunov function as introduced by Conley in [4], cf. Definition 4.5, also has to separate the different chain transitive components. The only functions fulfilling all properties of a complete Lyapunov functions in Definition 4.5 are $V = -\tau \equiv const$.

Combining the previous theorems and setting $V = -\tau$ we obtain the following corollary.

Corollary 4.7. *Let $X: U \to \mathbb{R}^m$ be a continuous vector field. Then there exists a smooth complete Lyapunov function $V: U \to \mathbb{R}$ such that*

(a) *two points $p, q \in \mathcal{R}_X^{chain}$ are chain transitive iff $V(p) = V(q)$ and*
(b) *if $p' \in \mathcal{F}_X^{chain}(p)$ and $p \notin \mathcal{F}_X^{chain}(p')$, then $V(p') < V(p)$.*

The novel results in the last corollary is the smoothness of V, from which we obtain an obvious corollary useful for applications.

Corollary 4.8. *Let $X\colon U \to \mathbb{R}^m$ be a C^k vector field, $k \in \mathbb{N}_0 \cup \{\infty\}$. For $V\colon U \to \mathbb{R}$ from Corollary 4.7 the orbital derivative $V'(x) := \langle \nabla V(x), X_x\rangle$ is C^k and nonpositive on U and $\{x \in U \mid V'(x) = 0\} = \mathcal{R}_X^{chain}$.*

5. Proof of the Main Results

In the first two sections we prove Theorem 4.1. For readability we split the proof into several assertions. Then we prove part (d) of Theorem 4.4 in Section 5.3.

5.1. **Proof of Theorem 4.1**: $\subset$. We show that $\mathcal{R}_X^{\text{chain}} \subset \mathcal{R}_{\mathcal{C}_X}$, $\mathcal{R}_X^{\text{chain}}(p) \subset \mathcal{R}_{\mathcal{C}_X}(p)$ for all $p \in \mathcal{R}_X^{\text{chain}}$, and that for every $p \in U$ we have $\mathcal{F}_X^{\text{chain}}(p) \subset \mathcal{F}_{\mathcal{C}_X}^{+}(p)$.

Let $\mathcal{E}$ be an arbitrary, but fixed, open enlargement of $\mathcal{C}_X$ till the end of this section.

Lemma 5.1. *For all $T > 0$ there exists a locally finite cover $\{V_k\}_{k\in\mathbb{N}}$ of U and a sequence $\varepsilon_k > 0$ such that $B_{\varepsilon_k}(z) \subset \mathcal{I}_{\mathcal{E}}^{+}(\gamma(0))$ for all $z \in V_k$, where $\gamma\colon [0,t] \to U$ is an X-orbit ending at z and $t \geq T$.*

Proof: Since $\mathcal{E}$ is an open enlargement of $\mathcal{C}_X$ the set

$$S_{\mathcal{E}} := \{z \in U \mid \mathcal{E}(z) = \mathbb{R}^m\}$$

is an open neighborhood of $S_X := \{z \in U \mid X_z = 0\}$. Therefore we can choose for every $y \in S_X$ an $0 < \varepsilon_y \leq 1$ such that $B_{2\varepsilon_y}(y) \subset S_{\mathcal{E}}$. Since S_X is paracompact, cf. [?, Thm. 1.15], there exists a locally finite refinement of nonempty sets $\{V_l^* \mid V_l^* \subset B_{\varepsilon_{y_l}}(y_l)\}_{l\in\mathbb{N}}$ of the cover $\{B_{\varepsilon_y}(y)\}_{y\in S_X}$. Since $\varepsilon_y \leq 1$ it follows that $\{V_l^S := B_{\varepsilon_{y_l}}(y_l)\}_{l\in\mathbb{N}}$, where $V_l^* \subset V_l^S$, is a locally finite subcover of S_X. Set $\varepsilon_l^S := \varepsilon_{y_l}$. For $z \in V_l^S$ we have

$$B_{\varepsilon_l^S}(z) \subset \mathcal{I}_{\mathcal{E}}^{+}(z) \subset \mathcal{I}_{\mathcal{E}}^{+}(\eta(0))$$

for every X-orbit $\gamma\colon [0,t] \to U$ ending at z.

Set $U_R := U \setminus \bigcup_{l\in\mathbb{N}} V_l^S$. For $y \in U_R$ choose $r_y > 0$ such that $B_{3r_y}(y) \subset U \setminus S_X$. For

$$0 < \lambda_y \leq \frac{r_y}{2\|X|_{B_{2r_y}(y)}\|_\infty}$$

every X-orbit $\eta\colon [0,\lambda_y] \to U$ ending at $z \in B_{r_y}(y)$ is contained in $B_{2r_y}(y)$. Indeed let $\eta(\varepsilon)$ be the last entry point of η into $B_{2r_y}(y)$. Then we have

$$\begin{aligned}
\|\eta(\varepsilon) - y\| &\leq \|\eta(\varepsilon) - \eta(\lambda_y)\| + \|z - y\| \\
&\leq \|\eta(\varepsilon) - \eta(\lambda_y)\| + r_y \\
&\leq \|X|_{B_{2r_y}(y)}\|_\infty \frac{r_y}{2\|X|_{B_{2r_y}(y)}\|_\infty} + r_y \\
&= \frac{3r_y}{2},
\end{aligned}$$

which implies $\eta \subset B_{2r_y}(y)$.

Note that X is bounded away from 0 on $B_{2r_y}(y)$. After diminishing r_y we can choose, by the continuity of X, a $\delta_y > 0$ such that $B_{\delta_y}(X_z) \subset \mathcal{E}(z')$ for all $z, z' \in B_{2r_y}(y)$. After diminishing λ_y, if necessary, we can assume that $\lambda_y \leq T$.

We claim that

$$B_{\lambda_y\delta_y}(z) \subset \mathcal{I}_{\mathcal{E}}^{+}(\gamma(0))$$

for all X-orbits $\gamma\colon [0,t] \to U$ with endpoint $z \in B_{r_y}(y)$ and $t \geq T$.

Indeed let $w \in B_{\lambda_y \delta_y}(0)$ and $\eta\colon [0,t] \to U$ be an X-orbit with endpoint $z \in B_{r_y}(y)$ and $t \geq T$. Consider the curve $\eta_w\colon [0,t] \to U$ with $\eta_w(s) = \eta(s)$ for $s \leq t - \lambda_y$ and

$$\eta_w(s) = \eta(s) + \frac{s + \lambda_y - t}{\lambda_y} w$$

for $s \geq t - \lambda_y$. Since $\eta|_{[t-\lambda_y, t]} \subset B_{2r_y}(y)$ and

$$\|\dot\eta_w(s) - X_{\eta(s)}\| = \|\dot\eta_w(s) - \dot\eta(s)\| \leq \frac{\|w\|}{\lambda_y} \leq \delta_y$$

we conclude that $\dot\eta_w \in \mathcal{E}$ everywhere. Thus the curve is $\mathcal{E}$-timelike, i.e. $z + w \in \mathcal{I}^+_{\mathcal{E}}(\eta(0))$.

Next choose a locally finite subcover $\{V_l^R\} := \{B_{r_{y_l}}(y_l)\}_{l \in \mathbb{N}}$ of U_R and set $\varepsilon_l := \lambda_{y_l}\delta_{y_l}$. We have that

$$B_{\varepsilon_l}(z) \subset \mathcal{I}^+_{\mathcal{E}}(\gamma(0))$$

for all X-orbits $\gamma\colon [0,t] \to U$ with endpoint $z \in V_l^R$ and $t \geq T$.

Define the family $\{V_k\}_{k\in\mathbb{N}}$ by $V_k := V^S_{k/2}$ for k even and $V_k := V^R_{(k+1)/2}$ for k odd. It forms a locally finite cover of U. Further define the sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ by $\varepsilon_k := \varepsilon^S_{k/2}$ for k even and $\varepsilon_k := \varepsilon^S_{(k+1)/2}$ for k odd. The sequence has the property that $B_{\varepsilon_k}(z) \subset \mathcal{I}^+_{\mathcal{E}}(\gamma(0))$ where $\gamma\colon [0,t] \to U$ is an X-orbit ending at $z \in V_k$ and $t \geq T$. □

Lemma 5.2. *If q is chain reachable from p, then q is in chronological future of p relative to all open enlargements $\mathcal{E}'$ of $\mathcal{C}_X$. That is*

$$p \to_X q \Rightarrow q \in \bigcap_{\mathcal{C}_X \prec \mathcal{E}'} \mathcal{I}^+_{\mathcal{E}'}(p),$$

in particular $\mathcal{F}^{chain}_X(p) \subset \mathcal{F}^+_{\mathcal{C}_X}(p)$ for every $p \in U$.

Proof. Let $p, q \in U$ and $T > 0$ such that for all continuous $\varepsilon\colon U \to (0,\infty)$ there exits an (ε, T)-chain $p_0 = p, p_1, \ldots, p_n = q$. Choose $\varepsilon\colon U \to (0,\infty)$ continuous with $\varepsilon|_{V_k} \leq \varepsilon_k$ for all $k \in \mathbb{N}$. By construction there exist $t_i \geq T$ and an $\mathcal{E}$-timelike path $\eta_i\colon [0,t_i] \to M$ from p_i to p_{i+1} for all $0 \leq i \leq n-1$. A concatenation of these paths gives an $\mathcal{E}$-timelike path from p to q. Since $\mathcal{E}$ is an arbitrary open enlargement of $\mathcal{C}$ the lemma follows. □

Lemma 5.3. *$\mathcal{R}^{chain}_X \subset \mathcal{R}_{\mathcal{C}_X}$ and $\mathcal{R}^{chain}_X(p) \subset \mathcal{R}_{\mathcal{C}_X}(p)$ for all $p \in \mathcal{R}^{chain}_X$.*

Proof. First note that every chain recurrent point is chain reachable from itself, i.e. $p \to_X p$. This implies by Lemma 5.2 that

$$p \in \bigcap_{\mathcal{C}_X \prec \mathcal{E}'} \mathcal{I}^+_{\mathcal{E}'}(p),$$

i.e. $p \in \mathcal{R}_{\mathcal{C}_X}$.

Now let $p, q \in \mathcal{R}^{\mathrm{chain}}_X$ be chain transitive, i.e. $q \in \mathcal{R}^{\mathrm{chain}}_X(p)$. Since $q \in \bigcap_{\mathcal{C} \prec \mathcal{E}'} \mathcal{I}^+_{\mathcal{E}'}(p)$ and $p \in \bigcap_{\mathcal{C} \prec \mathcal{E}'} \mathcal{I}^+_{\mathcal{E}'}(q)$ by Lemma 5.2, we can for an arbitrary open enlargement $\mathcal{E}'$ of $\mathcal{C}$ concatenate $\mathcal{E}'$-timelike paths from p to q and from q to p to obtain an $\mathcal{E}'$-timelike loop around p containing q. Hence, p and q are stably equivalent, i.e. $q \in \mathcal{R}_{\mathcal{C}_X}(p)$. □

5.2. **Proof of Theorem 4.1 : $\supset$.** We complete the proof of Theorem 4.1 by showing that $\mathcal{R}_X^{\text{chain}} \supset \mathcal{R}_{\mathcal{C}_X}$, $\mathcal{R}_X^{\text{chain}}(p) \supset \mathcal{R}_{\mathcal{C}_X}(p)$ for all $p \in \mathcal{R}_X^{\text{chain}}$, and that for every $p \in U$ we have $\mathcal{F}_X^{\text{chain}}(p) \supset \mathcal{F}_{\mathcal{C}_X}^+(p)$.

Fix a compact exhaustion $\{K_l\}_{l\in\mathbb{N}}$ of U with $K_{l-1} \subset K_l^\circ$, i.e. every K_l is compact, K_{l-1} is contained in the interior K_l° of K_l, and $\bigcup_{l\in\mathbb{N}} K_l = U$.

Proposition 5.4. *For every $S > 0$ and every continuous function $\varepsilon\colon U \to (0,\infty)$ there exists a continuous function $\delta\colon U \to (0,\infty)$, such that for all $\eta\colon [0,s] \to U$, $s \le S$, with*

$$\|\dot\eta(\sigma) - X_{\eta(\sigma)}\| < \delta(\eta(\sigma))$$

for all $\sigma \in [0,s]$, there exists an X-orbit $\gamma\colon [0,s] \to U$ with $\gamma(0) = \eta(0)$ and

$$\|\gamma(s) - \eta(s)\| < \varepsilon(\gamma(s)).$$

Proof: Fix $l, \lambda \in \mathbb{N}$, $l < \lambda$. Consider a sequence $\{s_n^\lambda\}_n$, $s_n^\lambda \in [0,S]$, and a sequence of curves

$$\{\eta_n^\lambda\colon [0,s_n^\lambda] \to K_\lambda\}_{n\in\mathbb{N}}$$

with $\eta_n(0) \in K_l$ and $\|\dot\eta_n(\sigma) - X_{\eta_n(\sigma)}\| < 1/n$ for all $\sigma \in [0,s_n^\lambda]$.

By the Theorem of Arzela-Ascoli the sequence $\{\eta_n^\lambda\}_n$ includes a subsequence converging uniformly to a Lipschitz curve $\eta^\lambda\colon [0,s^\lambda] \to K_\lambda$, where s^λ is the limes inferior over the s_n^λ's in the converging subsequence. If $s^\lambda = 0$ there is nothing to prove. Therefore we can assume $s^\lambda > 0$. By Rademacher's Theorem η^λ is differentiable almost everywhere. For all $t \in [0,s^\lambda]$ such that $\dot\eta^\lambda(t)$ exists we have $\dot\eta^\lambda(t) = X_{\eta^\lambda(t)}$. Since X is continuous we conclude that $\dot\eta^\lambda(t) = X_{\eta^\lambda(t)}$ for all $t \in [0,s^\lambda]$, i.e. η^λ is an X-orbit.

The argument shows as a special case: For every $l \in \mathbb{N}$ there exist $n(l), \lambda(l) \in \mathbb{N}$ such that for all $s \le S$ and all curves $\eta\colon [0,s] \to U$ with $\eta(0) \in K_l$ and $\|\dot\eta(\sigma) - X_{\eta(\sigma)}\| < 1/n(l)$ for all $\sigma \in [0,s]$ we have $\eta([0,s]) \subset K_{\lambda(l)}$.

Hence, we conclude by the same argument that for a given $\varepsilon_l > 0$ there exists an $N(l) \in \mathbb{N}$, i.g. larger than $n(l)$, such that for all curves $\eta\colon [0,s] \to U$, $s \le S$, with $\eta(0) \in K_l$ and $\|\dot\eta(\sigma) - X_{\eta(\sigma)}\| < 1/N(l)$ for all $\sigma \in [0,s]$ there exits an X-orbit $\gamma\colon [0,s] \to U$ with

$$\|\gamma(\sigma) - \eta(\sigma)\| < \varepsilon_l$$

for all $\sigma \in [0,s]$.

By passing to a sub-exhaustion we can assume that $\lambda(l) = l+1$. Then choose $\varepsilon_l < \min \varepsilon|_{K_{l+1}}$. For a continuous function $\delta\colon U \to (0,\infty)$ with $\delta|_{K_{l+1}\setminus K_{l-1}} < 1/N(l)$ we have: If $s \le S$ and $\eta\colon [0,s] \to U$ is a curve with $\|\dot\eta(\sigma) - X_{\eta(\sigma)}\| < \delta(\eta(\sigma))$ for all $\sigma \in [0,s]$ then there exists an X-orbit $\gamma\colon [0,s] \to U$ with $\gamma(0) = \eta(0)$ and $\|\gamma(s) - \eta(s)\| < \varepsilon(\gamma(s))$. □

Lemma 5.5. *If q is in chronological future of p relative to all open enlargements $\mathcal{E}'$ of $\mathcal{C}_X$,. then q is chain reachable from p. That is*

$$q \in \bigcap_{\mathcal{C}_X \prec \mathcal{E}'} \mathcal{I}_{\mathcal{E}'}^+(p) \Rightarrow p \to_X q,$$

in particular $\mathcal{F}_X^{chain}(p) \supset \mathcal{F}_{\mathcal{C}_X}^+(p)$ for every $p \in U$.

Proof: Let a continuous function $\varepsilon\colon U \to (0,\infty)$ be given. Following Proposition 5.4 we can choose for ε and $S = 2$ a continuous function $\delta\colon U \to (0,\infty)$.

Set

$$\mathcal{E} := \bigcup_{z\in U} \{z\} \times \mathrm{pos}(B_{\delta(z)}(X_z)).$$

Note that the cone field $\mathcal{E}$ is an open enlargement of $\mathcal{C}_X$. By assumption there exists an $\mathcal{E}$-timelike curve $\eta\colon I \to U$ from p to q. We can assume without loss of generality that η is smooth. By the construction of $\mathcal{E}$ we have $\|\dot\eta(\sigma) - X_{\eta(\sigma)}\| < \delta(\eta(\sigma))$ for all $\sigma \in I$. Let $T_0 \in (0,1)$ be a lower bound on the length $|I|$ of the interval I.

Now construct an (ε, T_0)-chain from p to q as follows: By construction there exists for all $r \in I$ and $s \le \min\{r+1, \max I\}$ an X-orbit $\gamma\colon [r,s] \to U$ with $\gamma(r) = \eta(r)$ and $\|\eta(s) - \gamma(s)\| < \varepsilon(\gamma(s))$. Divide I into subintervals $[a_0,a_1],[a_1,a_2],\dots,[a_{k-1},a_k]$ with $T_0 \le a_{i+1} - a_i \le 2$. Then the points $\eta(a_0) = p, \eta(a_1), \dots, \eta(a_k) = q$ form an (ε, T_0)-chain from p to q.

□

Lemma 5.6.

$$\mathcal{R}_{\mathcal{C}_X} \subset \mathcal{R}_X^{chain}$$

Proof: Let $T > 0$ and a continuous function $\varepsilon\colon U \to (0,\infty)$ be given. Choose a continuous function $\delta\colon U \to (0,\infty)$ according to Proposition 5.4 for ε and $S = 2T$. Set

$$\mathcal{E} := \bigcup_{z\in U} \{z\} \times \mathrm{pos}(B_{\delta(z)}(X_z)).$$

As before the cone $\mathcal{E}$ is an open enlargement of $\mathcal{C}_X$.

Assume $p \in \mathcal{R}_{\mathcal{C}_X}$ and choose an $\mathcal{E}$-timelike loop $\eta\colon I \to U$ around p with

$$\|\dot\eta(s) - X_{\eta(s)}\| < \delta(\eta(s))$$

for all $s \in I$. By iterating the loop η we can assume that $T \le |I|$. Divide I into subintervals $[a_0,a_1],\dots,[a_{k-1},a_k]$ with $T \le a_{i+1} - a_i \le 2T$. Then the points $p = \eta(a_0), \eta(a_1), \dots, \eta(a_k) = p$ form an (ε, T)-chain.

Since $T > 0$ and $\varepsilon\colon U \to (0,\infty)$ were arbitrary it follows that $p \in \mathcal{R}_X^{\mathrm{chain}}$. □

Remark 5.7. *The inclusion $\mathcal{R}_{\mathcal{C}_X}(p) \subset \mathcal{R}_X^{chain}(p)$ for all $p \in \mathcal{R}_{\mathcal{C}_X}$ is now obvious from the two preceding lemmas.*

5.3. **Proof of Theorem 4.4.** As mentioned above the following theorem is [3, Theorem 2] for the special case $\mathcal{C} = \mathcal{C}_X$.

Theorem 5.8. *Let $X: U \to \mathbb{R}^m$ be a continuous vector field. Then there exists a Lyapunov function $\rho\colon U \to \mathbb{R}$ for the cone field $\mathcal{C}_X$ with the following properties:*

- (a) *The function ρ is regular at each point of $U \setminus \mathcal{R}_{\mathcal{C}_X}$ and critical at each point of $\mathcal{R}_{\mathcal{C}_X}$.*
- (b) *Two points p and p' of $\mathcal{R}_{\mathcal{C}_X}$ belong to the same stable class iff $\rho(p') = \rho(p)$.*
- (c) *If p and p' are two points of U such that $p' \in \mathcal{F}^+_{\mathcal{C}_X}(p)$ and $p \notin \mathcal{F}^+_{\mathcal{C}_X}(p')$, then $\rho(p') > \rho(p)$.*

Lemma 5.9. *Let $X: U \to \mathbb{R}^m$ be a continuous vector field. Then there exists a Lyapunov function $\tau\colon U \to \mathbb{R}$ for the cone field $\mathcal{C}_X$ satisfying the conclusion of Theorem 5.8 as well as*

- (d') *For all $r_1, r_2 \in \tau(\mathcal{R}_{\mathcal{C}_X})$, $r_1 < r_2$, there exist $r_1 \le s_1 < s_2 \le r_2$ such that every value $t \in (s_1, s_2)$ is regular for τ.*

Proof of Theorem 4.4. It only remains to prove (d) of Theorem 4.4. Assume the contrary, i.e.

$$\left(\overline{\tau(\mathcal{R}_{\mathcal{C}_X})}\right)^{\circ} \neq \emptyset.$$

Choose a nonempty open interval $I \subset \overline{\tau(\mathcal{R}_{\mathcal{C}_X})}$. Then there exist $r_1, r_2 \in I \cap \tau(\mathcal{R}_{\mathcal{C}_X})$, $r_1 < r_2$. Choose $r_1 \leq s_1 < s_2 \leq r_2$ according to Lemma 5.9(d'). Then we have $(s_1, s_2) \cap \tau(\mathcal{R}_{\mathcal{C}_X}) = \emptyset$, which clearly represents a contradiction to Lemma 5.9. □

Proof of Lemma 5.9. By Theorem 5.8 we can fix a Lyapunov function $\rho\colon U \to \mathbb{R}$ satisfying properties (a)-(c) in Theorem 4.4. Keep in mind throughout the construction that $\mathcal{R}_{\mathcal{C}_X}$ is the set of critical points of ρ. If the set $\rho(\mathcal{R}_{\mathcal{C}_X})$ is finite there is nothing to prove.

If $\sigma\colon U \to \mathbb{R}$ is a second Lyapunov function such that

$$\operatorname{sgn}[\sigma(p') - \sigma(p)] = \operatorname{sgn}[\rho(p') - \rho(p)] \in \{0, \pm 1\} \tag{2}$$

for all $p, p' \in \mathcal{R}_{\mathcal{C}_X}$, then the Lyapunov function $\tau := \rho + \sigma$ satisfies (a)-(c) as well. This can be seen as follows: First note that τ is regular at a point p iff ρ or σ are regular at p. Second, if $p, p' \in \mathcal{R}_{\mathcal{C}_X}$ belong to the same stable component then $\tau(p) = \tau(p')$. Conversely if $\tau(p) = \tau(p')$ then

$$\rho(p) - \rho(p') = \sigma(p') - \sigma(p), \tag{3}$$

which by our assumption implies $\rho(p) = \rho(p')$, i.e. p and p' belong to the same stable component. Third $p' \in \mathcal{F}^+_{\mathcal{C}_X}(p)$ implies $\sigma(p') \geq \sigma(p)$. If $p \notin \mathcal{F}^+_{\mathcal{C}_X}(p')$ we have

$$\tau(p') = \rho(p') + \sigma(p') > \rho(p) + \sigma(p) = \tau(p).$$

The remainder of the proof is the construction of a Lyapunov function $\sigma\colon U \to \mathbb{R}$ satisfying (2) and such that $\rho + \sigma$ has property (d').

We first establish the existence of a countable set $\mathcal{Z} \subset \mathbb{R}$ of regular values of ρ such that for all $r_1, r_2 \in \rho(\mathcal{R}_{\mathcal{C}_X})$, $r_1 < r_2$, there exists $z \in \mathcal{Z}$ with $r_1 < z < r_2$. Note that $\{\rho = z\} \neq \emptyset$ for such z by the Intermediate Value Theorem because U is connected.

We construct $\mathcal{Z}$ as follows: Define

$$\rho(\mathcal{R}_{\mathcal{C}_X})^{\triangle} := \{(r_1, r_2) |\, r_i \in \rho(\mathcal{R}_{\mathcal{C}_X}),\, r_1 < r_2\}$$

and

$$\mathbb{Q}^{\triangle} := \{(s_1, s_2) |\, s_i \in \mathbb{Q},\ s_1 < s_2\}$$

and choose a map

$$\varphi\colon \rho(\mathcal{R}_{\mathcal{C}_X})^{\triangle} \to \mathbb{Q}^{\triangle}$$

such that if $\varphi(r_1, r_2) = (s_1, s_2)$ we have $r_1 \leq s_1 < s_2 \leq r_2$. Further choose a map

$$\psi\colon \mathbb{Q}^{\triangle} \to \mathbb{R} \setminus \rho(\mathcal{R}_{\mathcal{C}_X})$$

such that $s_1 < \psi(s_1, s_2) < s_2$. Now define

$$\mathcal{Z} := \psi(\varphi(\rho(\mathcal{R}_{\mathcal{C}_X})^{\triangle}))$$

and note that since $\varphi(\rho(\mathcal{R}_{\mathcal{C}_X})^{\triangle}) \subset \mathbb{Q}^{\triangle}$ is countable so is $\mathcal{Z}$. Choose an injective enumeration $i \mapsto z_i$ of $\mathcal{Z}$, i.e. $\mathcal{Z} = \{z_i\}_{i \in \mathbb{N}}$.

Now we construct a Lyapunov function $\sigma\colon U \to \mathbb{R}$ as described. Since every $z_i \in \mathcal{Z}$ is a regular value of ρ we can choose $\varepsilon\colon \{\rho = z_i\} \to (0, \infty)$ continuous such that the gradient flow $\Phi(p, t)$ of ρ, i.e. $\dot{\Phi}(t, p) = \nabla\rho(\Phi(t, p))$ and $\Phi(0, p) = p$, is well defined for $|t| \leq \varepsilon_i(p)$. Since ρ is smooth it is locally Lipschitz and it follows by the Theorem of Picard-Lindelöff that solution trajectories of the gradient flow do not intersect and thus every Φ-orbit starting at a regular point stays in the regular set, i.e. cannot enter the stably recurrent set where solution trajectories are stationary. It follows that all points $\Phi(p,t)$, $|t| \leq \varepsilon_i(p)$, are regular points of ρ. Set $V_i := \{\Phi(p,t)|\ \rho(p) = z_i,\ |t| < \varepsilon_i(p)\}$. Since ρ is a Lyapunov function the set $\{\rho > z_i\}$ is a *trapping domain* in the sense of [3, Definition 3.1]. Set $I_i := (U \setminus V_i) \cap \{\rho > z_i\}$ and $O_i := (U \setminus V_i) \cap \{\rho < z_i\}$. Then according to [3, Corollary 5.2] there exists a smooth Lyapunov function $\sigma_i \colon U \to [0,2]$ such that $\{\sigma_i = 1\} = \{\rho = z_i\}$, σ_i is regular on $\{0 < \sigma_i < 2\}$, and $\sigma_i|_{I_i} \equiv 2$ as well as $\sigma_i|_{O_i} \equiv 0$. Next we choose a positive sequence $\{a_i\}$ (see [6]) such that $\sigma := \sum_i a_i \sigma_i$ is smooth.

First, we verify that σ satisfies (2). Let $p, p' \in \mathcal{R}_{\mathcal{C}_X}$. If p and p' are in the same stable component we have $\sigma'(p) = \sigma'(p')$ for every Lyapunov function $\sigma' \colon U \to \mathbb{R}$. If p and p' are not in the same stable component we can assume $\rho(p') > \rho(p)$. By construction we have $\sigma_i(p') \geq \sigma_i(p)$ for all $i \in \mathbb{N}$ and there exists $j \in \mathbb{N}$ with $\sigma_j(p') > \sigma_j(p)$. It thus follows that $\sigma(p') > \sigma(p)$. We conclude that (2) holds for all $p, p' \in \mathcal{R}_{\mathcal{C}_X}$.

Second, we verify that $\tau = \rho + \sigma$ satisfies (d'). Recall that $\mathcal{R}_{\mathcal{C}_X}$ is the set of critical points of τ and $\tau(\mathcal{R}_{\mathcal{C}_X})$ is the set of its critical values. Let $r_1, r_2 \in \tau(\mathcal{R}_{\mathcal{C}_X})$, $r_1 < r_2$, be given. Choose $p_1 \in \{\tau = r_1\} \cap \mathcal{R}_{\mathcal{C}_X}$ and $p_2 \in \{\tau = r_2\} \cap \mathcal{R}_{\mathcal{C}_X}$. By (2) we conclude that $\rho(p_1) < \rho(p_2)$. Choose $z_i \in \mathcal{Z}$ with $\rho(p_1) < z_i < \rho(p_2)$ and set $A_i := \{j \in \mathbb{N}|\ z_j < z_i\}$. Let $q, q' \in \mathcal{R}_{\mathcal{C}_X}$ be given. We can assume that $\rho(q) < z_i < \rho(q')$. Then we have

$$\tau(q) \leq \rho(q) + \sum_{j \in A_i} 2a_j < z_i + \sum_{j \in A_i} 2a_j$$

and

$$\tau(q') \geq \rho(q') + 2a_i + \sum_{j \in A_i} 2a_j > z_i + 2a_i + \sum_{j \in A_i} 2a_j.$$

Therefore the interval $[z_i + \sum_{j \in A_i} 2a_j, z_i + 2a_i + \sum_{j \in A_i} 2a_j]$ contains only regular values of τ. By setting $s_1 := z_i + \sum_{j \in A_i} 2a_j$ and $s_2 := z_i + 2a_i + \sum_{j \in A_i} 2a_j$ the claim follows. $\square$

6. Conclusions

By establishing a link between complete Lyapunov functions in dynamical systems and time functions in general relativity we are able to apply results from Bernard and Suhr [3] to dynamical systems. This delivers a novel characterization of the chain recurrent set using cone fields and a first smooth converse theorem on complete Lyapunov functions for general ODEs on noncompact state spaces. In addition to the theoretical significance, these results have direct applications in computational methods for complete Lyapunov functions as shown in Giesl et al. [9].

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