Study of dynamical systems by fast numerical computation of Lyapunov functions

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Abstract: In this paper we discuss a computational method of numerically searching for Lyapunov functions for nonlinear systems and demonstrate its efficacy. The method is built upon applying various theoretical Lyapunov functions, given by integrating some specific positive functions along solution trajectories in the state space, to the vertices of a simplical complex. Then we assign the remaining values by convex interpolation over the simplices. The benefits of explicitly constructing the candidate functions in this manner are twofold. Firstly it is computationally inexpensive, growing linearly with the number of vertices we calculate a candidate function on, and secondly the freedom in choosing a positive function allows us flexibility to not be overly constrained by the shape of the attractor. Finally we will demonstrate the method on two planar examples. Most notably we will see that the constructed Lyapunov functions give us lower bounds on basins of attraction that are significantly larger than those found by other methods in the literature.

1. Introduction

Consider the dynamical system, whose dynamics are given by the ODE

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{f} : \mathcal{D} \to \mathbb{R}^n$, $\mathcal{D} \subset \mathbb{R}^n$, is locally Lipschitz. We denote the (unique) solution to (1) with initial value $\boldsymbol{\xi} \in \mathcal{D}$ at $t = 0$ with $t \mapsto \boldsymbol{\phi}(t, \boldsymbol{\xi})$. If $\boldsymbol{\eta} \in \mathcal{D}$ is an equilibrium point for (1), i.e. $\mathbf{f}(\boldsymbol{\eta}) = \mathbf{0}$ and consequently $\boldsymbol{\phi}(t, \boldsymbol{\eta}) = \boldsymbol{\eta}$ for all $t \in \mathbb{R}$ a constant solution, its stability properties are of much practical interest. The equilibrium point $\boldsymbol{\eta}$ is said to be asymptotically stable if it is stable (in the sense of Lyapunov) and attractive. The former means that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\boldsymbol{\xi} - \boldsymbol{\eta}\| < \delta$ implies $\|\boldsymbol{\phi}(t, \boldsymbol{\xi}) - \boldsymbol{\eta}\| < \varepsilon$ for all $t \geq 0$ and the latter denotes that there exists a neighbourhood $\mathcal{N}_{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}$ such that $\boldsymbol{\xi} \in \mathcal{N}_{\boldsymbol{\eta}}$ implies $\lim_{t\to\infty} \|\boldsymbol{\phi}(t, \boldsymbol{\xi}) - \boldsymbol{\eta}\| = 0$. The set of all points that are attracted to the asymptotically stable equilibrium $\boldsymbol{\eta}$ as $t \to \infty$, i.e. the largest possible $\mathcal{N}_{\boldsymbol{\eta}}$, is called its *basin of attraction* and its spatial extension is a measure of the robustness of the equilibrium's stability.

Stability of equilibrium points and basins of attraction are concepts of fundamental relevance in applications of dynamical systems. They are usually dealt with using the Lyapunov stability theory. Some good references are [12, 16, 18]. The centerpiece of the Lyapunov stability theory is the so-called Lyapunov function, a scalar-valued function from the state-space of the dynamical system that is decreasing along all solutions of the system in a neighbourhood of the equilibrium in question. Lyapunov functions deliver lower bounds on basins of attraction through their sublevel sets and for linear systems $\mathbf{x}' = A\mathbf{x}$ they can be constructed explicitly using algebraic methods. For nonlinear systems there is no general method, but one can resort to linearization around the equilibrium in question and construct a Lyapunov function for the linearization. This Lyapunov function is also a Lyapunov function for the nonlinear system in a neighbourhood of the equilibrium, but it is not a good Lyapunov function in the sense that it does in general deliver very conservative lower bounds on the equilibrium's basin of attraction. For exact formulas see, e.g. [9].

2. Method to Compute Lyapunov Functions

For the reasons discussed in the last section there have been numerous methods proposed in the literature to generate Lyapunov functions for nonlinear systems [8]. One approach is to approximate numerically formulas for Lyapunov functions [1, 4, 5, 10] from classical converse theorems [11, 14, 19] in the Lyapunov stability theory. These converse theorems assert the existence of Lyapunov functions for systems with asymptotically stable equilibria and give formulas, in terms of the systems's solution, for these Lyapunov functions. Because these formulas include solutions to the systems, that are in general not obtainable for nonlinear systems, one resorts to approximate their values at a finite number of points. The Lyapunov function must be decreasing along solution trajectories in a whole neighbourhood of the equilibrium in question. If this cannot be asserted the constructed (Lyapunov) function is of little use, i.e. an approximation to a Lyapunov function is of little value. Therefore the computed values must be interpolated such that the resulting function is a Lyapunov function in a whole area. This can be achieved by using the linear programming (LP) problem from [7], but instead of using LP to compute the values of the Lyapunov function at the vertices of a simplicial complex, one uses a formula from a converse theorem to assign values at the vertices and then verifies if the linear constraints of the LP problem are fulfilled using these values. If the linear constraints are fulfilled for all vertices of a certain simplex, then the affine interpolation of these values over the simplex defines a function, whose orbital derivative is negative along all solution trajectories passing through this simplex. This was already shown in [1].

We improve this method in two ways. First, we incorporate sharper error estimates in the next section for the LP problem from [7], which leads to less conservative conditions in its linear constraints. Second, we tune the positive definite function in an integral formula from [14] to enlarge the lower bound on the basin of attraction, i.e. we approximate the Lyapunov function

$$V(\mathbf{x}) = \int_0^T \frac{\|\boldsymbol{\phi}(\tau,\mathbf{x})\|^2}{\delta + \|\boldsymbol{\phi}(\tau,\mathbf{x})\|^p} d\tau \tag{2}$$

for some appropriately chosen $T, \delta, p > 0$ at the vertices, instead of using

$$V(\mathbf{x}) = \int_0^T \|\boldsymbol{\phi}(\tau,\mathbf{x})\|^2 d\tau. \tag{3}$$

3. Sharper Error Bounds

The error bounds in the LP problem form [7, Def. 6] that served as basis for the constructions in [1, 10] can be sharpened using more regular triangulations and results from [13]. Further, the statement of the essential part of the LP problem can be considerably simplified.

To achieve this the linear constraints LC4 from [13] must first be rewritten in the notation of [7]. Denote by Sym_n the set of the permutations of $\{1:n\} := \{1,2,\ldots,n\}$, by $\mathfrak{P}(\{1:n\})$ the powerset of $\{1:n\}$, and set $\mathcal{Z} := \mathbb{N}_0^n \times \mathfrak{P}(\{1:n\})$. Let Γ and $\mathrm{PS_i}$, $i = 1:n$, be strictly increasing functions $\mathbb{R} \to \mathbb{R}$ that vanish at zero and define $\mathbf{PS}: \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{PS} = (\mathrm{PS_1}, \mathrm{PS_1}, \ldots, \mathrm{PS_n})^\top$. Define $\mathbf{R}^{\mathcal{J}}(\mathbf{x}) = \sum_{i=1}^n (-1)^{\chi(i)} x_i \mathbf{e}_i$ for every $\mathcal{J} \in \mathfrak{P}(\{1:n\})$,

$$\mathbf{x}_i^\sigma := \sum_{j=i}^n \mathbf{e}_{\sigma(j)} \quad \text{for every } \sigma \in \mathrm{Sym}_n \text{ and every } i = 1:n+1, \text{ and} \tag{4}$$

$$\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} := \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + \mathbf{x}_i^\sigma)) \quad \text{for every } (\mathbf{z},\mathcal{J}) \in \mathcal{Z}, \text{ every } \sigma \in \mathrm{Sym}_n \text{ and every } i = 1:n+1.$$

Assume that the components of $\mathbf{f}$ in the system (1) are C^2 and let $B_{rs}^{(\mathbf{z},\mathcal{J})}$ for every $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$ and $r, s = 1:n$ be a constant fulfilling

$$B_{rs}^{(\mathbf{z},\mathcal{J})} \geq \max_{\substack{\mathbf{x} \in \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}+[0,1]^n)) \\ k=1:n}} \left| \frac{\partial^2 f_k(\mathbf{x})}{\partial x_r \partial x_s} \right| \tag{5}$$

For every $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$, every $k, i = 1:n$, and every $\sigma \in \mathrm{Sym}_n$, define

$$A_{\sigma,k,i}^{(\mathbf{z},\mathcal{J})} := |\mathbf{e}_k \cdot (\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})|. \tag{6}$$

The constraints LC4 from [13] can now be written as: For every $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$, every $\sigma \in \mathrm{Sym}_n$, and every $i = 1:n+1$:

$$\begin{aligned} -\Gamma[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|] \geq & \sum_{j=1}^n \frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\ & + \frac{1}{2} \sum_{r,s=1}^n B_{rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})} (A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})}) \sum_{j=1}^n \left| \frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} \right| \end{aligned} \tag{7}$$

In [13] the vectors

$$\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma} := \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{j=1}^{i} \mathbf{e}_{\sigma(j)}\right) \quad \text{for } i = 0 : n \text{ are used.} \tag{8}$$

The relationship between the (4) and (8) is with $\mathbf{1} := \mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_n$ that

$$\mathbf{x}_{i-1}^{\mathbf{z}\mathcal{J}\sigma} + \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} = \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z})) + \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}+\mathbf{1})) \tag{9}$$

for every $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$, every $\sigma \in \mathrm{Sym}_n$, and every $i = 1 : n+1$. Thus with $\alpha \in \mathrm{Sym}_n$ defined through $\alpha(i) = \sigma(n+1-i)$ for $i = 1 : n$, we have $\sigma(i) = \alpha(n+1-i)$ and $\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} = \mathbf{x}_{n+1-i}^{\mathbf{z}\mathcal{J}\alpha}$. Hence, from (6)

$$A_{\sigma,k,i}^{(\mathbf{z},\mathcal{J})} = |\mathbf{e}_k \cdot (\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})| = |\mathbf{e}_k \cdot (\mathbf{x}_{n+1-i}^{\mathbf{z}\mathcal{J}\alpha} - \mathbf{x}_0^{\mathbf{z}\mathcal{J}\alpha})| =: A_{k,n+1-i}^{\mathbf{z}\mathcal{J}\alpha}$$

and (7) can be rewritten as

$$\begin{aligned}
-\Gamma[\|\mathbf{x}_{n+1-i}^{\mathbf{z}\mathcal{J}\alpha}\|] \geq{}& \sum_{j=1}^{n} \frac{V[\mathbf{x}_{n+1-j}^{\mathbf{z}\mathcal{J}\alpha}] - V[\mathbf{x}_{n-j}^{\mathbf{z}\mathcal{J}\alpha}]}{\mathbf{e}_{\alpha(n+1-j)} \cdot (\mathbf{x}_{n+1-j}^{\mathbf{z}\mathcal{J}\alpha} - \mathbf{x}_{n-j}^{\mathbf{z}\mathcal{J}\alpha})} f_{\alpha(n+1-j)}(\mathbf{x}_{n+1-i}^{\mathbf{z}\mathcal{J}\alpha}) \\
&+ \frac{1}{2} \sum_{r,s=1}^{n} B_{rs}^{(\mathbf{z},\mathcal{J})} A_{r,n+1-i}^{\mathbf{z}\mathcal{J}\alpha}(A_{s,n+1-i}^{\mathbf{z}\mathcal{J}\alpha} + A_{s,n}^{\mathbf{z}\mathcal{J}\alpha}) \sum_{j=1}^{n} \left| \frac{V[\mathbf{x}_{n+1-j}^{\mathbf{z}\mathcal{J}\alpha}] - V[\mathbf{x}_{n-j}^{\mathbf{z}\mathcal{J}\alpha}]}{\mathbf{e}_{\alpha(n+1-j)} \cdot (\mathbf{x}_{n+1-j}^{\mathbf{z}\mathcal{J}\alpha} - \mathbf{x}_{n-j}^{\mathbf{z}\mathcal{J}\alpha})} \right|.
\end{aligned}$$

Thus by renaming $i \leftarrow n+1-i$ and $\sigma \leftarrow \alpha$, the linear constraints LC4 from [13] in (7) are fulfilled, if and only if for every $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$, every $\sigma \in \mathrm{Sym}_n$, and every $i = 0 : n$, we have

$$\begin{aligned}
-\Gamma[\|\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}\|] \geq{}& \sum_{j=1}^{n} \frac{V[\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma}] - V[\mathbf{x}_{j-1}^{\mathbf{z}\mathcal{J}\sigma}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma} - \mathbf{x}_{j-1}^{\mathbf{z}\mathcal{J}\sigma})} f_{\sigma(j)}(\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}) \\
&+ \frac{1}{2} \sum_{r,s=1}^{n} B_{rs}^{(\mathbf{z},\mathcal{J})} A_{r,i}^{\mathbf{z}\mathcal{J}\sigma}(A_{s,i}^{\mathbf{z}\mathcal{J}\sigma} + A_{s,n}^{\mathbf{z}\mathcal{J}\sigma}) \sum_{j=1}^{n} \left| \frac{V[\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma}] - V[\mathbf{x}_{j-1}^{\mathbf{z}\mathcal{J}\sigma}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma} - \mathbf{x}_{j-1}^{\mathbf{z}\mathcal{J}\sigma})} \right|.
\end{aligned} \tag{10}$$

We now show the connection between (10) and the statement of the constraints using the gradient of the Lyapunov function ∇V as in [7, Def. 6]. The so-called shape-matrix $X_{\mathbf{z}\mathcal{J}\sigma}$ of the simplex $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \mathrm{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right)$ is defined by writing the vectors $\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma} - \mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}$, $\mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma} - \mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma} - \mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}$ consecutively in its rows. For the affine function $V_{\mathbf{z}\mathcal{J}\sigma} : \mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} \to \mathbb{R}$ defined through

$$V_{\mathbf{z}\mathcal{J}\sigma}\left(\sum_{j=0}^{n} \lambda_j \mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma}\right) = \sum_{j=0}^{n} \lambda_j V[\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma}] \tag{11}$$

for all convex combinations of the vertices of $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$, it is not difficult to see that with

$$\mathbf{v}_{\mathbf{z}\mathcal{J}\sigma} := \left(V[\mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}] - V[\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}],\ V[\mathbf{x}_2^{\mathbf{z}\mathcal{J}\sigma}] - V[\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}], \ldots,\ V[\mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}] - V[\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}]\right)^{\top}$$

we have

$$V_{\mathbf{z}\mathcal{J}\sigma}(\mathbf{x}) = (X_{\mathbf{z}\mathcal{J}\sigma}^{-1}\mathbf{v}_{\mathbf{z}\mathcal{J}\sigma})\bullet(\mathbf{x}-\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}) + V[\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}] = \mathbf{v}_{\mathbf{z}\mathcal{J}\sigma}^\top X_{\mathbf{z}\mathcal{J}\sigma}^{-T}(\mathbf{x}-\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}) + V[\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}] \tag{12}$$

for all $\mathbf{x} \in \mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$. This is a simple consequence of the fact that (11) and (12) are affine functions with identical values at the vertices of $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$. Thus the gradient of $V_{\mathbf{z}\mathcal{J}\sigma}$ is given by (the column vector) $\nabla V_{\mathbf{z}\mathcal{J}\sigma} := X_{\mathbf{z}\mathcal{J}\sigma}^{-1}\mathbf{v}_{\mathbf{z}\mathcal{J}\sigma}$. The linear constraints in [7, Def. 6] corresponding to (10), but for more general triangulations than discussed here, can be formulated as

$$-\Gamma[\|\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}\|] \geq \nabla V_{\mathbf{z}\mathcal{J}\sigma}\bullet\mathbf{f}(\mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}) + E^{\mathbf{z}\mathcal{J}\sigma}\|\nabla V_{\mathbf{z}\mathcal{J}\sigma}\|_1, \tag{13}$$

where $E^{\mathbf{z}\mathcal{J}\sigma}$ is a simplex-dependent error bound.

To shorten formulas in the following computations we fix the simplex $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$ and thus $\mathbf{z}$, $\mathcal{J}$, and σ and set $X := X_{\mathbf{z}\mathcal{J}\sigma}$. It is not difficult to see that $X = LSP$, where $S := \mathrm{diag}(s_1, s_2, \ldots, s_n)$ is a diagonal matrix with $s_i = \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}+\mathbf{e}_i)) - \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}))$,

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{with} \quad L^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}, \tag{14}$$

is a lower-triangular matrix $L_{ij} = 1$ if $i \geq j$, and P is a permutation matrix, $\mathbf{e}_i^\top P = \mathbf{e}_{\sigma(i)}^\top$ for $i = 1:n$. Especially $P^{-1} = P^\top$. Now set $\mathbf{x}_i := \mathbf{x}_i^{\mathbf{z}\mathcal{J}\sigma}$, $V_i := V[\mathbf{x}_i]$, $\nabla V := \nabla V_{\mathbf{z}\mathcal{J}\sigma}$, and $\mathbf{v} := (V_1 - V_0, V_2 - V_0, \ldots, V_n - V_0)^\top = \mathbf{v}_{\mathbf{z}\mathcal{J}\sigma}$ and note that

$$\begin{aligned}
\nabla V \bullet \mathbf{f}(\mathbf{x}_i) &= \mathbf{v}^\top X^{-T}\mathbf{f}(\mathbf{x}_i) = \left(\mathbf{v}^\top X^{-\top}\mathbf{f}(\mathbf{x}_i)\right)^\top = \mathbf{f}(\mathbf{x}_i)^\top X^{-1}\mathbf{v} = \mathbf{f}(\mathbf{x}_i)^\top P^\top S^{-1}L^{-1}\mathbf{v} \\
&= \mathbf{f}(\mathbf{x}_i)^\top P^\top S^{-1}L^{-1}\begin{pmatrix} V_1 - V_0 \\ V_2 - V_0 \\ \vdots \\ V_n - V_0 \end{pmatrix} = \mathbf{f}(\mathbf{x}_i)^\top P^\top S^{-1}\begin{pmatrix} V_1 - V_0 \\ V_2 - V_1 \\ \vdots \\ V_n - V_{n-1} \end{pmatrix} \\
&= \sum_{j=1}^n \frac{V_j - V_{j-1}}{s_j}\mathbf{f}(\mathbf{x}_i)^\top P^\top \mathbf{e}_j = \sum_{j=1}^n \frac{V_j - V_{j-1}}{s_j}\left(\mathbf{e}_j^\top P\mathbf{f}(\mathbf{x}_i)\right) \\
&= \sum_{j=1}^n \frac{V_j - V_{j-1}}{s_j}\mathbf{e}_{\sigma(j)}^\top\mathbf{f}(\mathbf{x}_i) = \sum_{j=1}^n \frac{V_j - V_{j-1}}{s_j} f_{\sigma(j)}(\mathbf{x}_i).
\end{aligned}$$

This implies that in our setting (10) is equivalent to (13) and we can replace the error bound $E^{\mathbf{z}\mathcal{J}\sigma}$ in [7, Def. 6] with the sharper estimate from (10):

$$\frac{1}{2}\sum_{r,s=1}^n B_{rs}^{(\mathbf{z},\mathcal{J})} A_{r,i}^{\mathbf{z}\mathcal{J}\sigma}(A_{s,i}^{\mathbf{z}\mathcal{J}\sigma} + A_{s,n}^{\mathbf{z}\mathcal{J}\sigma}), \quad \text{which is always} \leq E^{\mathbf{z}\mathcal{J}\sigma}. \tag{15}$$

Remark 1: Notionally it is often more convenient to suppress the dependance on $\mathbf{z}\mathcal{J}\sigma$ and just refer to a simplex $\mathfrak{S}_\nu$ rather than $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$. When using this simplified notation one then refers to B^ν_{rs} and not $B^{(\mathbf{z},\mathcal{J})}_{rs}$ for all simplices $\mathfrak{S}_\nu$ such that $\mathfrak{S}_\nu \subset \mathbf{PS}(\mathbf{R}^\mathcal{J}(\mathbf{z}+[0,1]^n))$, and it is not difficult to see that one can use different estimates B^ν_{rs} for the different $\mathfrak{S}_\nu \subset \mathbf{PS}(\mathbf{R}^\mathcal{J}(\mathbf{z}+[0,1]^n))$, although this hardly justifies the effort.

Remark 2: From the decomposition $X = LSP$ one can easily derive concrete upper bounds on some matrix norms of $X^{-1} = X^{-1} = P^T S^{-1} L^{-1}$. For any matrix norm induced by a vector norm we have $\|X^{-1}\| \leq \|P^T\|\,\|S^{-1}\|\,\|\mathrm{L}^{-1}\|$. For $\|\cdot\| = \|\cdot\|_1$ and $\|\cdot\| = \|\cdot\|_\infty$ one can easily see from (14) that for $n \geq 2$ we have

$$\|L^{-1}\|_1 = \|L^{-1}\|_\infty = 2,\ \|S^{-1}\|_1 = \|S^{-1}\|_\infty = \max_{i=1,2,\dots,n} |s_i|^{-1},\ \text{and } \|P^T\|_1 = \|P^T\|_\infty = 1.$$

It follows that $\|X^{-1}\|_1 \leq 2s^*$ and $\|X^{-1}\|_\infty \leq 2s^*$ with $s^* := \max_{i=1:n} |s_i|^{-1}$ and from the well known $\|X^{-1}\|_2^2 \leq \|X^{-1}\|_1 \|X_\nu^{-1}\|_\infty$ it additionally follows that $\|X^{-1}\|_2 \leq 2s^*$.

4. Examples

We present two examples for our method, where we approximate the Lyapunov function from (2) at the grid points with some appropriately chosen $T, \delta, p > 0$. Then we interpolate and verify the negativity of the orbital derivative of the interpolation as in [1], but use the sharper error estimate (15) in the LP program. Note that the orbital derivative of the Lyapunov functions computed by our method is not guarantied to be negative very close to the equilibrium. This is a known feature of the method, that can, however, be easily accounted for by using a local Lyapunov function for the linearized system at the equilibirum to assert its local stability.

We compare our results with the Massera construction from [1], i.e. where the Lyapunov function is approximated using (3) at the vertices, and to two other approaches suggested in the literature. The computations were programmed in C++ and run on a PC with an i9-7900X processor.

4.1. Example 1

The first example is a planar system from [6, Ex. 6],

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \quad \text{with} \quad \mathbf{f}(x,y) = \begin{pmatrix} -x+y \\ 0.1x - 2y - x^2 - 0.1x^3 \end{pmatrix}. \tag{16}$$

We assign in the LP problem (notation from Remark 1 in Section 3)

$$B^\nu_{1,1} = 2 + 0.6 \max_{(x,y)\in\mathfrak{S}_\nu} |x| \quad \text{and} \quad B^\nu_{1,2} = B^\nu_{2,1} = B^\nu_{2,2} = 0.$$

We set $T = 20$ for (3) and (2) and for the latter we set $\delta = 0.6$, and $p = 0.6$. The grid used for the vertices of the simplices was 2001×2001 with 4,004,001 points and 8,000,000 simplices/triangles. This corresponds to using the simplices $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$ for $\mathbf{z} \in \{0 : 999\}^2$, $\mathcal{J} \in \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, and $\sigma \in \{(1,2),(2,1)\}$ in the notation of Section 3. The computation of the Lyapunov function using (3) was done on the rectangle $[-20, 20]^2$, i.e. the mapping $\mathbf{PS}$ from Section 3 is given by $\mathbf{PS}(\mathbf{x}) = 0.02\mathbf{x}$ (because $0.02 \cdot 1000 = 20$). The computation took 43.6 s and the verification of the negativity of the orbital derivative took 0.45s. In 11.96% of the triangles/simplices the orbital derivative was not negative. For the computation using (2) on the rectangle $[-20, 20] \times [-40, 40]$, i.e. $\mathbf{PS}(x, y) = (0.02x, 0.04y)^\top$, the corresponding runtimes were 51.8 s and 0.45 s. In 10.05% of the triangles/simplices the orbital derivative was not negative. In Figure 1 the Lyapunov functions using formulas (3) and (2) respectively are plotted. In Figure 2 the level sets $\{\mathbf{x} \in \mathbb{R}^2 : V(\mathbf{x}) \leq 33\}$ and $\{\mathbf{x} \in \mathbb{R}^2 : V(\mathbf{x}) \leq 9\}$ for these functions respectively are plotted. These level sets are chosen such that they do not intersect with the areas where the orbital derivative is nonnegative and thus give lower bounds on the basin of attraction.

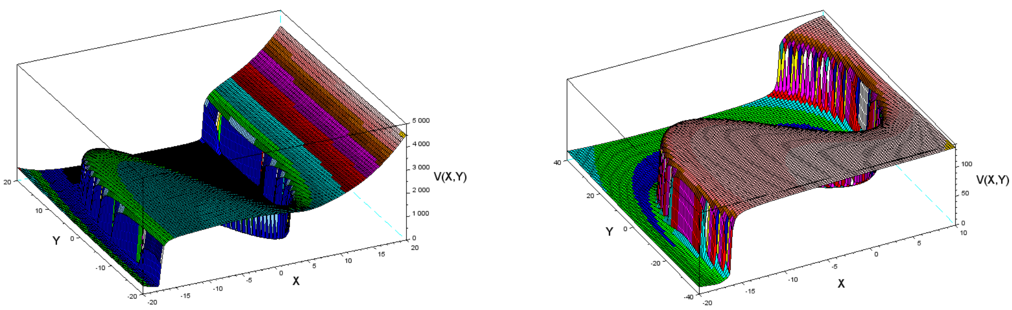

Figure 1. The Lyapunov functions computed for system (16) using formula (3) [left] and formula (2) [right].

In Figure 3 we compare our results with the approach from [17] as implemented in [15], where a rational Lyapunov function is computed for the same system, and to the method presented in [3], where Lyapunov functions that are sums of squared polynomials (SOS) are computed. The software SMRSOFT from [3] was downloaded and used for the computations. We computed 4th, 6th, and 8th order polynomial Lyapunov functions, but only draw the level set for the 4th order one, because it delivered the least conservative estimate. It is notable, that even though this method delivers a much smaller estimate of the basin of

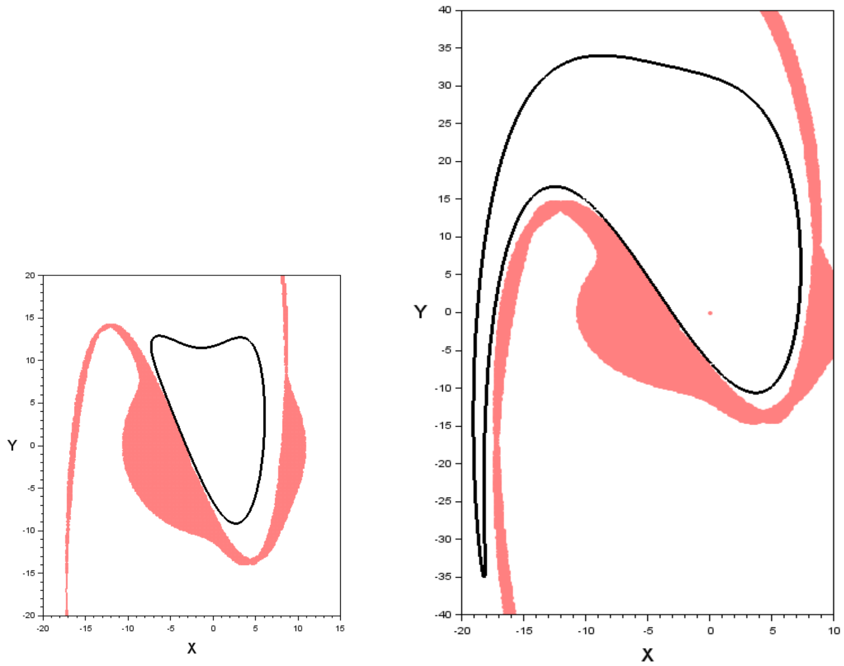

Figure 2. Level-sets of the Lyapunov functions computed for the system (16) using formula (3) [left] and (2) [right]. The area where the orbital derivative is not negative is drawn in red. Since the level-sets do not intersect the area where the orbital derivative is nonnegative they are lower bounds on the basin of attraction of the equilibrium at the origin.

attraction, it is not a proper subset of our estimates.

4.2. Example 2

The second example is a planar system from [2, Ex. 1],

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \quad \text{with} \quad \mathbf{f}(x,y) = \begin{pmatrix} -x + y + \frac{1}{2}(e^x - 1) \\ -x - y + xy + x\cos(x) \end{pmatrix}. \tag{17}$$

We assign

$$B^{\nu}_{1,1} = \max_{(x,y)\in\mathfrak{S}_\nu} \max(e^x/2, 2|\sin(x)| + |x\cos(x)|), \quad B^{\nu}_{1,2} = B^{\nu}_{2,1} = 1, \quad \text{and} \quad B^{\nu}_{2,2} = 0.$$

Further, we set $T = 20$ for (3) and (2) and for latter we set $\delta = 0.4$, and $p = 0.3$. As in Example 1 the grid was 2001×2001 with 4,004,001 points and 8,000,000 simplices/triangles. The computation of the Lyapunov function using (3) was done on the rectangle $[-8, 4] \times$

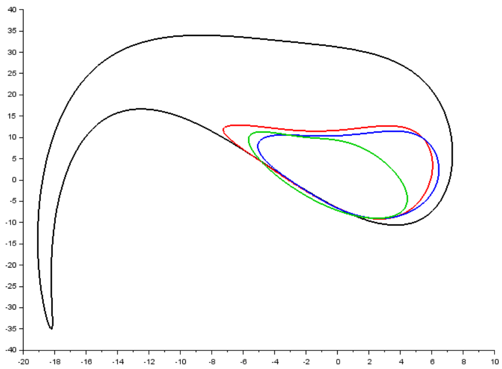

Figure 3. Level-sets of the Lyapunov functions computed for the system (16) using formula (2) (outermost, black), (3) (red), the method from [15, 17] (blue), and using the software SMRSOFT [3] (green).

$[-8, 8]$ and took 35.6 s and the verification of the negativity of the orbital derivative took 0.4 s. In 27.9% of the triangles/simplices the orbital derivative was not negative. In most of the area where the orbital derivative was not negative the Lyapunov function was not defined because the initial-value problems diverge too fast on the interval $[0, T]$ for the numerical solver.

For the computation using (2) on the rectangle $[-8, 3] \times [-10, 10]$ the corresponding numbers were 45.2 s and 0.4 s. In 23.4% of the triangles/simplices the orbital derivative was not negative, also mostly because the numerical solver was not able to assign values to the Lyapunov function at the grid points.

In Figure 4 the Lyapunov functions using formulas (3) and (2) respectively are plotted. In Figure 5 the level sets $\{\mathbf{x} \in \mathbb{R}^2 : V(\mathbf{x}) \leq 8\}$ and $\{\mathbf{x} \in \mathbb{R}^2 : V(\mathbf{x}) \leq 5.9\}$ for these functions are plotted. These level sets are chosen such that they do not intersect with the ares where the orbital derivative is nonnegative and thus give lower bounds on the basin of attraction.

In Figure 6 we compare our results with the approach from [17] as implemented in [15], where a rational Lyapunov function is computed for the same system. We also compared it with the method from [2], but the level sets obtained are very close to the ones from [15] and we omit drawing them.

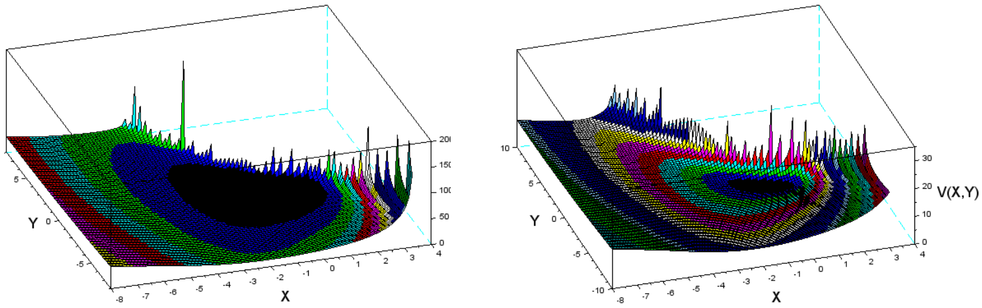

Figure 4. The Lyapunov functions computed for system (17) using formula (3) [left] and formula (2) [right].

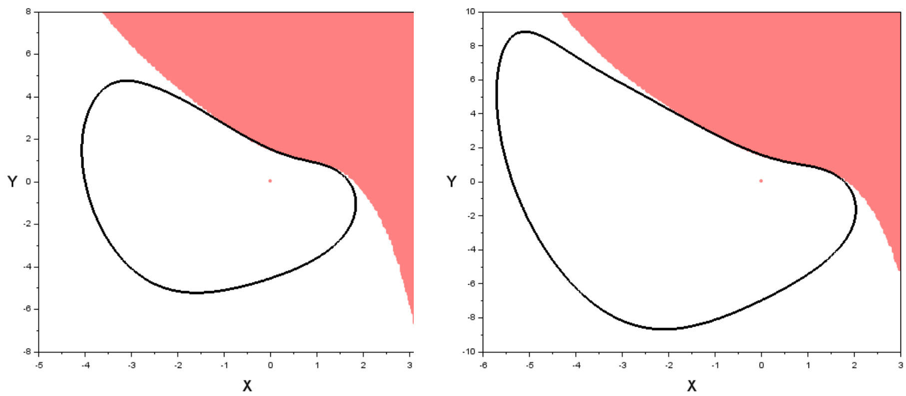

Figure 5. Level-sets of the Lyapunov functions computed for the system (17) using formula (3) [left] and (2) [right]. The area where the orbital derivative is not negative is drawn in red. Since the level-sets do not intersect the area where the orbital derivative is nonnegative they are lower bounds on the basin of attraction of the equilibrium at the origin.

5. Conclusions

We presented an improved method to estimate the basin of attraction for equilibria of dynamical systems. The method is based on approximating the values of Lyapunov functions from converse theorems and assign these values to the variables of a linear programming problem. The linear constraints of the problem are then verified and in simplices, of which they are fulfilled at all vertices, the function defined by interpolating these values over the simplex has a negative orbital derivative along the solutions of the system. Our method is an advancement of the method presented in [1], but with sharper error estimates and thus less

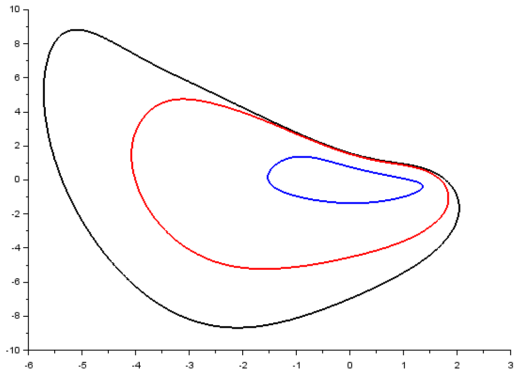

Figure 6. Level-sets of the Lyapunov functions computed for the system (17) using formula (2) (outermost, black), (3) (middle, red), and by using the method from [15, 17] (innermost, blue). In [2] results very close to the ones from [15,17] are obtained using SOS programming.

conservative linear constraints and a more general positive definite function of the solution under the integral in the Massera construction. We compared our novel method for two systems with the method from [1] and two other approaches from the literature; one using rational Lyapunov functions [15,17] and another using sum-of-squares programming [2,3]. In all cases our method delivered considerably larger inner estimates of the basins of attraction.

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