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Computation of Continuous and Piecewise Affine Lyapunov Functions for Discrete-Time Systems

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In this paper, we present a new approach for computing Lyapunov functions for nonlinear discrete-time systems with an asymptotically stable equilibrium at the origin. Given a suitable triangulation of a compact neighbourhood of the origin, a continuous and piecewise affine function can be parameterised by the values at the vertices of the triangulation. If these vertex values satisfy system-dependent linear inequalities, the parameterised function is a Lyapunov function for the system. We propose calculating these vertex values using constructions from two classical converse Lyapunov theorems originally due to Yoshizawa and Massera. Numerical examples are presented to illustrate the proposed approach.

Keywords: Lyapunov Theory, Nonlinear Systems, Converse Theorems, Computational Methods

1. Introduction

The Lyapunov function is among the most useful tools for stability analysis of dynamic systems since it allows one to conclude (asymptotic) stability of an equilibrium without knowledge of the explicit solution of the dynamic system. This utility has motivated the search for Lyapunov functions for dynamic systems for many years. In the present article, we focus on computing Lyapunov functions for discrete-time dynamic systems. Such systems are widely used to study practical phenomena in many fields such as engineering, finance, and biology.

Several methods have been proposed for computing Lyapunov functions for discrete-time dynamic systems. For instance, collocation methods were presented in [5] and [6], graph algorithms are used to compute complete Lyapunov functions in [3] and [14], and continuous piecewise affine (CPA) Lyapunov functions for discrete-time systems were computed as the solution of a linear programming problem in [7].

The CPA method is of particular interest since it directly delivers a true, rather than approximate, Lyapunov function for discrete-time dynamic systems on a compact subset of the state space. No additional a posteriori analysis of the computed function is needed. The reason for this property of the CPA method is that the interpolation errors are included. This method relies on a partitioning of the state space

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into simplices, called a triangulation. Values are defined at each vertex of the triangulation and a continuous and piecewise affine function is then defined via a convex interpolation of these values. In [7], the vertex values are obtained by solving a linear programming problem that incorporates error estimates.

A similar CPA method has been developed for continuous-time systems [18] (see also [8]) where the need to solve a linear programming problem can result in long computation times. As an alternative to solving a linear programming problem, in [10] we proposed a new approach to compute a CPA Lyapunov function for continuous-time systems using a function in a converse Lyapunov theorem originally introduced by Yoshizawa in [22] (see also [23]). In [4], rather than using the function proposed in Yoshizawa's converse Lyapunov theorem, we used an approximation to the function proposed in Massera's converse Lyapunov theorem [19].

In this paper, we present a similar approach for the discrete-time nonlinear system described by

$$x^+ = g(x), \tag{1}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous, and $g(\mathbf{0}) = \mathbf{0}$. We observe that deriving such discrete-time results from their continuous-time counterparts is non-trivial due to the fact that solutions in the discrete-time setting are sequences of points rather than absolutely continuous functions as in the continuous-time setting. We further note that CPA Lyapunov functions are particularly well-suited to the discrete-time setting since they are, by definition, not differentiable. Differentiability is a desirable property in the continuous-time setting since the classical decrease condition is in terms of the gradient of the Lyapunov function, but is of less importance in the discrete-time setting since the Lyapunov function decrease condition is given in terms of a finite difference rather than a differential.

The paper is organised as follows: in Section 2 we present the theory required for CPA Lyapunov functions for discrete-time nonlinear dynamical systems. In Section 3 we formulate the required stability estimate and in Sections 3.1 and 3.2 we define the discrete-time Massera and Yoshizawa functions, respectively. We also present the procedure to compute a CPA Lyapunov function for system (1) using either of these constructions. In Section 4, we present three representative examples to demonstrate the proposed method. In Section 5, we provide a brief summary.

2. Continuous and Piecewise Affine (CPA) Lyapunov Functions

We denote the nonnegative integers by $\mathbb{N}^0$ and the integers by $\mathbb{Z}$. Let $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$ denote intervals $(0, +\infty)$, $[0, +\infty)$ respectively. Given a subset $\Omega \subset \mathbb{R}^n$, we denote the interior, the closure, the boundary, the complement, and the closed convex hull of Ω by Ω°, $\overline{\Omega}$, $\partial\Omega$, Ω^c, and $\operatorname{co}\Omega$ respectively. We denote the origin in $\mathbb{R}^n$ by $\mathbf{0} \in \mathbb{R}^n$ and we denote the Euclidian length of a vector $x \in \mathbb{R}^n$ by $|x|$. For $\delta \in \mathbb{R}_{>0}$, let $\mathcal{B}_\delta := \{x \in \mathbb{R}^n : |x| < \delta\}$ denote the open ball of radius $\delta > 0$ centered at $\mathbf{0}$. Let $\lceil a \rceil := \min\{q \in \mathbb{N}^0 : q \geq a\}$ denote the smallest integer which is not less than $a \in \mathbb{R}_{\geq 0}$. The k^{th} element of the solution sequence of (1) with initial condition $x \in \Omega$ is denoted by $\phi(k, x)$ for all $k \in \mathbb{N}^0$ with $\phi(0, x) = x$.

A continuous function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be *positive definite* if $g(\mathbf{0}) = 0$ and

$g(x) > 0$ for all $x \neq \mathbf{0}$. In what follows, we will make use of the common function classes $\mathcal{K}_\infty$ and $\mathcal{KL}$. A function $\alpha \in \mathcal{K}_\infty$ if $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, zero at zero, strictly increasing, and approaches infinity as its argument approaches infinity. A function $\beta \in \mathcal{KL}$ if $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous and is zero at zero and strictly increasing in its first argument and strictly decreasing to zero in its second argument. For details of these functions, their properties, and their general usage in systems theory, we refer to [12, 15].

In order to define CPA functions, we recall the definition of a suitable triangulation of a compact set from [10]. A suitable triangulation is defined based on the following basic concepts:

Definition 2.1 *The* convex hull *of vectors $x_0, \dots, x_n \in \mathbb{R}^n$ is given by*

$$\operatorname{co}(x_0, \dots, x_n) = \left\{ \sum_{i=0}^{n} \lambda_i x_i \,:\, 0 \leq \lambda_i \leq 1, \sum_{i=0}^{n} \lambda_i = 1 \right\}. \tag{2}$$

Definition 2.2 *A set of vectors $x_0, \dots, x_n \in \mathbb{R}^n$ is called* affine independent *if $\sum_{i=1}^{n} \lambda_i (x_i - x_0) = 0$ implies $\lambda_i = 0$ for $i = 1, \dots, n$.*

Note that the above definition is independent of choice of reference node; i.e., the numbering of the vectors is unimportant.

Definition 2.3 *For the ordered set of vectors $x_i \in \mathbb{R}^n$, $i = 0, 1, \dots, n$, where $x_1, \dots, x_n \in \mathbb{R}^n$ are affine independent, an n-*simplex *is defined by*

$$\mathcal{S} := \operatorname{co}(x_0, \dots, x_n) = \left\{ \sum_{i=0}^{n} \lambda_i x_i \,:\, 0 \leq \lambda_i \leq 1, \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

The point defined by the vector x_i is called a vertex. The face of the n-simplex is defined as the convex hull of any nonempty subset of the $n+1$ vertices.

Definition 2.4 *We call a finite collection $\mathcal{T} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_N\}$ of n-simplices in $\mathbb{R}^n$ a* suitable triangulation *if:*

i) Simplices $\mathcal{S}_\nu, \mathcal{S}_\mu \in \mathcal{T}$, $\nu \neq \mu$, intersect in a common face or not at all.
ii) For $\mathcal{D}_\mathcal{T} := \cup_\nu \mathcal{S}_\nu$, the set $\mathcal{D}_\mathcal{T}^\circ$ is a connected neighbourhood of the origin.
iii) If $\mathbf{0} \in \mathcal{S}_\nu$, then $\mathbf{0}$ is a vertex of $\mathcal{S}_\nu$.

Definition 2.5 *Given a suitable triangulation $\mathcal{T}$, we say that a set $\mathcal{O} \subset \mathcal{D}_\mathcal{T} \subset \mathbb{R}^n$ satisfies* Property A *if:*

(i) $\mathcal{O}$ is a compact and connected neighbourhood of the origin.
(ii) There exists no simplex $\mathcal{S}_\nu$ with $x, y \in \mathcal{S}_\nu$ satisfying $x \in \mathcal{O}^\circ, y \in \mathcal{D}_\mathcal{T} \setminus \mathcal{O}$.
(iii) If $x \in \mathcal{O}$, then $g(x) \in \mathcal{D}_\mathcal{T}$.

Condition (i) requires that the origin be in the domain of interest, while (ii) implies that simplices cannot include elements of $\partial\mathcal{O}$ in their interior. Condition (iii) implies that system (1) is forward complete in $\mathcal{D}_\mathcal{T}$.

For a suitable triangulation $\mathcal{T}$, we define CPA$[\mathcal{T}]$ as the set of continuous functions

$V : \mathcal{D}_\mathcal{T} \to \mathbb{R}$ which are linearly affine on each simplex $\mathcal{S}_\nu$; i.e.,

$$V(x) = w_\nu^\top x + a_\nu, \quad x \in \mathcal{S}_\nu, \tag{3}$$

where $w_\nu \in \mathbb{R}^n$ and $a_\nu \in \mathbb{R}$.

In the interior of any simplex, a function $V \in \mathrm{CPA}[\mathcal{T}]$ is differentiable and we denote the gradient of a function $V \in \mathrm{CPA}[\mathcal{T}]$ in the interior of simplex $\mathcal{S}_\nu$ by ∇V_ν. In other words, with (2), for each $x \in \mathcal{S}_\nu^\circ$ we have

$$\nabla V_\nu := \nabla V(x) = w_\nu. \tag{4}$$

In the following, we present the definition of a CPA Lyapunov function for system (1) on a compact, connected set $\mathcal{O} \subset \mathcal{D}_\mathcal{T} \subset \mathbb{R}^n$ minus an arbitrary fixed small neighbourhood of the origin.

Definition 2.6 *Let $\mathcal{T}$ be a suitable triangulation, $V \in \mathrm{CPA}[\mathcal{T}]$ be a positive definite function, and $\mathcal{O} \subset \mathcal{D}_\mathcal{T}$ satisfy Property A. Define the constant*

$$q^* = \inf\{q \in \mathbb{R}_{\geq 0} : |g(x)| \leq q|x|, \forall x \in \mathcal{O}\} \tag{5}$$

where $q^* < \infty$ since g is locally Lipschitz. Let $\varepsilon > 0$ be such that

$$\begin{cases} \max\limits_{|x| \leq q^*\varepsilon} V(x) < \min\limits_{x \in \partial\mathcal{O}} V(x), \\ & \text{for } q^* > 1, \\ \mathcal{B}_{q^*\varepsilon} \subset \mathcal{O}. \end{cases} \tag{6}$$

or

$$\max_{|x| \leq \varepsilon} V(x) < \min_{x \in \partial\mathcal{O}} V(x), \qquad \text{for } q^* \leq 1. \tag{7}$$

If

$$V(g(x)) - V(x) < 0 \tag{8}$$

holds for all $x \in \mathcal{O} \setminus \mathcal{B}_\varepsilon$, then V is called a $\mathrm{CPA}[\mathcal{T}]$ Lyapunov function *for* (1) *on $\mathcal{O} \setminus \mathcal{B}_\varepsilon$.*

By a slight abuse of notation we denote the set of solutions of (1) at time $k \in \mathbb{N}^0$ from a compact set $\mathcal{C} \subset \mathbb{R}^n$ by $\phi(k, \mathcal{C}) := \bigcup_{x \in \mathcal{C}} \phi(k, x)$. Denote the sublevel sets of V by

$$\mathcal{L}_{V,c} := \{x \in \mathcal{D}_\mathcal{T} : V(x) \leq c\}, \quad c \in \mathbb{R}_{>0}. \tag{9}$$

Theorem 2.7. *Let $\mathcal{T}$ be a suitable triangulation, $\mathcal{O} \subset \mathcal{D}_\mathcal{T}$ satisfy Property A, and let $V : \mathcal{D}_\mathcal{T} \to \mathbb{R}_{\geq 0}$ be a $\mathrm{CPA}[\mathcal{T}]$ Lyapunov function for* (1) *on $\mathcal{O} \setminus \mathcal{B}_\varepsilon$ with appropriate $q^*, \varepsilon \in \mathbb{R}_{>0}$ as in Definition 2.6. Define*

$$m := \begin{cases} \max_{|x| \leq q^*\varepsilon} V(x), & \text{if } \quad q^* > 1, \\ \max_{|x| \leq \varepsilon} V(x), & \text{if } \quad q^* \leq 1, \end{cases} \tag{10}$$

and $M := \min_{x \in \partial\mathcal{O}} V(x)$. If for certain $c \in [m, M)$, $\mathcal{L}_{V,c}$ and $\mathcal{L}_{V,m}$ are connected, then $\mathcal{B}_\varepsilon \subset \mathcal{L}_{V,c} \subset \mathcal{O}$ and there exists a $K_c \in \mathbb{N}^0$ such that $\phi(k, \mathcal{L}_{V,c}) \subset \mathcal{L}_{V,m}$ for all $k \geq K_c$.

Proof. According to (6), (7), and the definitions of m and M, we get that $m < M$. It follows directly by the definitions of m and M and the continuity of V that $\mathcal{B}_\varepsilon \subset \mathcal{L}_{V,c} \subset \mathcal{O}$.

For $x \in \mathcal{L}_{V,c} \setminus \mathcal{B}_\varepsilon$, (8) implies that $g(x) \in \mathcal{L}_{V,c}$. For $x \in \mathcal{B}_\varepsilon$ we get by (5) that $|g(x)| < q^*\varepsilon$. Hence, by the definition of m we get $g(x) \in \mathcal{L}_{V,m} \subset \mathcal{L}_{V,c}$. Thus $\mathcal{L}_{V,c}$

is positively invariant. The last assertion of the theorem now follows from (8) with $\alpha^* := -\max_{x\in\mathcal{O}\setminus\mathcal{B}_\varepsilon}[V(g(x)) - V(x)] > 0$ and with $K_c \geq (c-m)/\alpha^*$. □

Remark 1. The conditions of Theorem 2.7 are more restrictive than the conditions in [10, Theorem 2.3] for continuous-time systems. These more restrictive conditions are required because the solution of (1) is a sequence of points rather than an absolutely continuous function. Similar to the continuous time result in [10, Theorem 2.3], Theorem 2.7 provides an estimate of the domain of attraction for the positively invariant set $\mathcal{L}_{V,m}$. In particular, in order to ensure $\mathcal{B}_{q^*\varepsilon} \subset \mathcal{O}$ and define the smallest forward invariant set covering $\mathcal{B}_\varepsilon$, it is necessary to distinguish between the cases $q^* > 1$ and $q^* \leq 1$ in (6), (7), and (10).

For $q^* \leq 1$, $g(x) \in \mathcal{B}_\varepsilon$ for $x \in \mathcal{B}_\varepsilon$. Hence $\mathcal{B}_\varepsilon$ is forward invariant. The level set $\mathcal{L}_{V,m}$ defined by (9)-(10) is the smallest forward invariant set covering $\mathcal{B}_\varepsilon$ when $q^* \leq 1$.

On the other hand, for $q^* > 1$, it is possible that $g(x) \notin \mathcal{B}_\varepsilon$ for $x \in \mathcal{B}_\varepsilon$. In order to guarantee $g(x) \in \mathcal{O}$ for $x \in \mathcal{B}_\varepsilon$, $\mathcal{B}_{q^*\varepsilon} \subset \mathcal{O}$ should be required. Then we have $V(g(g(x))) < V(g(x))$ for $x \in \mathcal{B}_\varepsilon$, $g(x) \in \mathcal{O} \setminus \mathcal{B}_\varepsilon$. Thus the level set $\mathcal{L}_{V,m}$ defined in (10) is the smallest forward invariant set covering $\mathcal{B}_\varepsilon$ when $q^* > 1$.

We state the criteria for verifying that a given CPA function is a CPA Lyapunov function in Theorem 2.8 and Corollary 2.9. We denote the diameter of a simplex $\mathcal{S} \subset \mathbb{R}^n$ by $\operatorname{diam}(\mathcal{S}) := \max\{|x-y| : x, y \in \mathcal{S}\}$.

THEOREM 2.8. *Let $\mathcal{T}$ be a suitable triangulation, $\mathcal{O} \subset \mathcal{D}_\mathcal{T}$ satisfy Property A and $\varepsilon > 0$ be a constant. Let $V \in \mathrm{CPA}[\mathcal{T}]$ and C, $G_\nu \in \mathbb{R}_{\geq 0}$ satisfy*

$$|g(x) - g(y)| \leq G_\nu |x-y|, \text{ for all } x, y \in \mathcal{S}_\nu, \tag{11}$$

$$\max_{\nu=1,2,\ldots,N} |\nabla V_\nu| \leq C. \tag{12}$$

If for each simplex $\mathcal{S}_\nu \subset \mathcal{O}$ satisfying $\mathcal{S}_\nu \cap \mathcal{B}_\varepsilon^c \neq \emptyset$ the inequalities

$$V(g(x_i^\nu)) - V(x_i^\nu) + CG_\nu \operatorname{diam}(\mathcal{S}_\nu) < 0 \tag{13}$$

hold for all $i = 0, 1, \ldots, n$, then

$$V(g(x)) - V(x) < 0$$

for all $x \in \mathcal{O} \setminus \mathcal{B}_\varepsilon$.

Proof. Let $x \in \mathcal{O} \setminus \mathcal{B}_\varepsilon$ be arbitrary. Then there exists an $\mathcal{S}_\nu \subset \mathcal{O}$ such that $x \in \mathcal{S}_\nu$; i.e., $x = \sum_{i=0}^n \lambda_i x_i^\nu$ where $\sum_{i=0}^n \lambda_i = 1$. We note that (12) implies that $V \in \mathrm{CPA}[\mathcal{T}]$ is Lipschitz on $\mathcal{D}_\mathcal{T}$ with Lipschitz constant C. Therefore,

$$\begin{aligned} V(g(x)) - V(x) &= V(g(x)) - \sum_{i=0}^n \lambda_i V(g(x_i^\nu)) + \sum_{i=0}^n \lambda_i V(g(x_i^\nu)) - \sum_{i=0}^n \lambda_i V(x_i^\nu) \\ &\leq \sum_{i=0}^n \lambda_i [V(g(x_i^\nu)) - V(x_i^\nu) + CG_\nu \operatorname{diam}(\mathcal{S}_\nu)]. \end{aligned}$$

Based on (13), we conclude that $V(g(x)) - V(x) < 0$ for all $x \in \mathcal{O} \setminus \mathcal{B}_\varepsilon$. □

Remark 2. For x_i^ν close to the origin, the (positive) interpolation error term $CG_\nu \operatorname{diam}(\mathcal{S}_\nu)$ may become the dominant factor on the left hand side of (13) and, in this case, (13) fails to hold. For a fixed distance from the origin, one can overcome this by choosing a finer triangulation, i.e., reducing the size of $\operatorname{diam}(\mathcal{S}_\nu)$. For simplices with the origin as a vertex, however, this problem requires a much more involved solution that only works if the origin is exponentially stable, cf. [7].

With the result of Theorem 2.8, the following Corollary is immediate.

COROLLARY 2.9. *Let* $V \in \mathrm{CPA}[\mathcal{T}]$ *from Theorem 2.8 be positive definite and the constant* $\varepsilon \in \mathbb{R}_{>0}$ *satisfy* (6) *or* (7) *as appropriate. If the inequalities (13) are satisfied for all* $\mathcal{S}_\nu \subset \mathcal{O}$ *with* $\mathcal{S}_\nu \cap \mathcal{B}_\varepsilon^c \neq \emptyset$, *then* V *is a CPA Lyapunov function for* (1) *on* $\mathcal{O} \setminus \mathcal{B}_\varepsilon$.

Remark 3. From Theorem 2.8 and Corollary 2.9, for a candidate Lyapunov function $V \in \mathrm{CPA}[\mathcal{T}]$, the verification that V is a $\mathrm{CPA}[\mathcal{T}]$ Lyapunov function for system (1) can be done by checking that V is positive definite and that the inequality (13) holds for each vertex of each simplex. The problem then is to find a candidate $\mathrm{CPA}[\mathcal{T}]$ Lyapunov function which, in essence, requires defining function values at each vertex in the triangulation. In [7], this is done by constructing and solving a linear programming problem. In what follows, we will calculate vertex values using two constructions from classical converse Lyapunov theorems and then verify inequality (13) for each vertex in the triangulation.

In the following we recall the definition of CPA approximations to functions as stated in [10, Definition 2.6].

Definition 2.10 *Let* $\mathcal{D} \subset \mathbb{R}^n$ *be a domain,* $W : \mathcal{D} \to \mathbb{R}$ *be a function, and* $\mathcal{T}$ *be a triangulation such that* $\mathcal{D}_\mathcal{T} \subset \mathcal{D}$. *The* $\mathrm{CPA}[\mathcal{T}]$ approximation V to W on $\mathcal{D}_\mathcal{T}$ *is the function* $V \in \mathrm{CPA}[\mathcal{T}]$ *defined by* $V(x) = W(x)$ *for all vertices* x *of all simplices in* $\mathcal{T}$.

Given a triangulation $\mathcal{T}$, and simplex $\mathcal{S}_\nu := \mathrm{co}(x_0^\nu, x_1^\nu, \ldots, x_n^\nu) \in \mathcal{T}$, the *shape-matrix* X_ν of $\mathcal{S}_\nu$ is defined by writing the vectors $x_1^\nu - x_0^\nu, x_2^\nu - x_0^\nu, \ldots, x_n^\nu - x_0^\nu$ in its rows subsequently; i.e.,

$$X_\nu = \begin{bmatrix} (x_1^\nu - x_0^\nu)^\top \\ (x_2^\nu - x_0^\nu)^\top \\ \vdots \\ (x_n^\nu - x_0^\nu)^\top \end{bmatrix}. \tag{14}$$

Note that, as we have defined a simplex by an *ordered* set of vertices (Definition 2.3), the shape-matrix is well defined. In order to make sure that the simplex $\mathcal{S}_\nu$ is not degenerate, the value $\mathrm{diam}(\mathcal{S}_\nu)|X_\nu^{-1}|$ should be bounded. Here $|X_\nu^{-1}|$ is the spectral norm of the inverse of X_ν (see part (ii) in the proof of [2, Theorem 4.6]). Hence, we see that degeneracy of a simplex corresponds to a linear dependence in the rows of the shape matrix, indicating a decrease in dimension, or “flattening”, of the simplex which results in a large value for $|X_\nu^{-1}|$. However, since it may be necessary to refine a triangulation to obtain smaller simplices, resulting in vertices that may be close together without coming closer to decreasing the dimension of the simplex, our measure of degeneracy is scaled by the diameter of the simplex, $\mathrm{diam}(\mathcal{S}_\nu)$. For example, this measure of degeneracy is bounded in the proof of Theorem 2.12 in (20) below.

We now state conditions under which the $\mathrm{CPA}[\mathcal{T}]$ approximation V to a Lyapunov function $W : \mathcal{D} \to \mathbb{R}_{\geq 0}$ is a CPA[$\mathcal{T}$] Lyapunov function on $\mathcal{O} \setminus \mathcal{B}_\varepsilon \to \mathbb{R}_{\geq 0}$ for a compact inner approximation $\mathcal{O}$ of $\mathcal{D}$ and a small $\varepsilon > 0$. We first define a Lyapunov function.

Definition 2.11 *Given a domain $\mathcal{D} \subset \mathbb{R}^n$, $\mathbf{0} \in \mathcal{D}$, a continuous function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lyapunov function for* (1) *on $\mathcal{D}$ if $W(\mathbf{0}) = 0$ and there exist positive definite functions $\alpha, \alpha_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for every $x \in \mathcal{D}$,*

$$\alpha_1(|x|) \leq W(x), \quad \textit{and}$$
$$W(g(x)) - W(x) \leq -\alpha(|x|).$$

Theorem 2.12. *Assume $W \in C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ is a Lyapunov function for system* (1) *on domain $\mathcal{D}$. Let $\mathcal{C}_I, \mathcal{C}_O \subset \mathcal{D}$ ($\mathcal{C}$ "Inner" and "Outer") be simply connected compact neighbourhoods of the origin such that*

$$\mathcal{C}_I \subset \mathcal{C}_O^\circ \subset \mathcal{C}_O \subset \mathcal{D}.$$

Then, for every $\varepsilon \in \mathbb{R}_{>0}$ small enough, we can construct a triangulation $\mathcal{T}$ and a set $\mathcal{O}$, $\mathcal{C}_I \subset \mathcal{O} \subset \mathcal{C}_O$, fulfilling Property A, such that the CPA[$\mathcal{T}$] *approximation V to W is a* CPA[$\mathcal{T}$] *Lyapunov function for system* (1) *on $\mathcal{O} \setminus \mathcal{B}_\varepsilon$.*

Proof. Since we must prove that, among other things, $V(g(x)) - V(x) < 0$ for all $x \in \mathcal{O} \setminus \mathcal{B}_\varepsilon$, V must be defined at least on $\mathcal{O} \cup g(\mathcal{O})$. To simplify the proof we define V on a larger set. For this it is convenient to introduce the sets

$$\mathcal{D}_I := \mathrm{co}[\mathcal{C}_O \cup g(\mathcal{C}_O)] \ \text{ and } \ \mathcal{D}_O := \mathcal{D}_I + \overline{\mathcal{B}}_1,$$

where the addition in the latter term denotes the Minkowski sum. Note that both $\mathcal{D}_I$ and $\mathcal{D}_O$ are compact.

Set

$$L := \max_{x \in \mathcal{D}_O} |\nabla W(x)| \ \text{ and } \ G := \sup_{x,y \in \mathcal{D}_O} \frac{|g(x) - g(y)|}{|x - y|}. \tag{15}$$

Clearly $L < \infty$ because $W \in C^1(\mathbb{R}^n)$ and $G < \infty$ because g is locally Lipschitz. Since $W(x)$ is a Lyapunov function for (1) on $\mathcal{D}$, we get that there exists a positive definite function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$W(g(x)) - W(x) \leq -\alpha(|x|), \text{ for all } x \in \mathcal{D}\ . \tag{16}$$

Set $q^* = \inf\{q \in \mathbb{R}_{\geq 0} : |g(x)| \leq q|x|, \forall x \in \mathcal{C}_O\}$ and choose $0 < \varepsilon < 1$ so small that

$$\begin{cases} \max\limits_{|x| \leq (G+1)\varepsilon} W(x) < \min\limits_{x \in \mathcal{C}_O \setminus \mathcal{C}_I^\circ} W(x) \\ \\ \mathcal{B}_{G\varepsilon} \subset \mathcal{C}_I \end{cases}, \quad \text{for } q^* > 1, \tag{17}$$

or

$$\max_{|x| \leq 2\varepsilon} W(x) < \min_{x \in \mathcal{C}_O \setminus \mathcal{C}_I^\circ} W(x), \qquad \text{for } q^* \leq 1. \tag{18}$$

Fix an $R > 0$ and choose a $\delta_R > 0$ to satisfy

$$\delta_R < \min\left\{\frac{\varepsilon}{2}, \frac{\alpha(\varepsilon/2)}{L(1 + RG)}, \min\{|x - y| \, : \, x \in \mathcal{C}_O^c, y \in \mathcal{C}_I\}\right\}, \tag{19}$$

where α comes from (16).

Let $\mathcal{T}$ be any triangulation such that $\mathcal{D}_I \subset \mathcal{D}_\mathcal{T}$,

$$\max_{\mathcal{S}_\nu \in \mathcal{T}} \operatorname{diam}(\mathcal{S}_\nu) \le \delta_R \ , \ \text{ and } \ \max_{\mathcal{S}_\nu \in \mathcal{T}} \operatorname{diam}(\mathcal{S}_\nu)|X_\nu^{-1}| \le R. \tag{20}$$

Note that the second expression provides a bound on how close any simplex can be to degeneracy as per the discussion following (14). Such a triangulation exists and is not difficult to construct; e.g. $\mathcal{T}_{K,b}^{\mathcal{D}_I}$ from [8, Def. 13] with $K = 0$ and $b = \delta_R\sqrt{n}$. It can be verified that for $\mathcal{T}_{0,\delta_R\sqrt{n}}^{\mathcal{D}_I}$ it is sufficient to take $R = 4$ and obviously $\delta_R > 0$ can be chosen as small as one wishes. Further, as $\delta_R < \varepsilon/2 < 1/2$, we have $\mathcal{D}_\mathcal{T} \subset \mathcal{D}_O$ by the definition of $\mathcal{D}_O$.

Let $\mathcal{O}$ be the union of those simplices in $\mathcal{T}$ that have a nonempty intersection with the interior of $\mathcal{C}_I$, i.e.

$$\mathcal{O} := \bigcup_{\mathcal{S}_\nu \cap \mathcal{C}_I^\circ \neq \emptyset} \mathcal{S}_\nu.$$

By (20) and (19) clearly $\mathcal{O} \subset \mathcal{C}_O$ and as shown in [8, Lemma 2] $\mathcal{O}$ is compact, connected, and $\overline{\mathcal{O}^\circ} = \mathcal{O}$, i.e. $\mathcal{O}$ fulfills Property A.

We define V as the CPA[$\mathcal{T}$] approximation to W on $\mathcal{D}_\mathcal{T}$, i.e. for every vertex x_i^ν of every simplex $\mathcal{S}_\nu \in \mathcal{T}$ we fix $V(x_i^\nu) := W(x_i^\nu)$. It is obvious that V is a positive definite function.

By the definition of q^* in (5) and G in (15) and because $\mathcal{O} \subset \mathcal{D}_O$ we have $q^* \le G$. Based on (17) and because V is defined as the interpolation of the vertex values of W we obtain for $q^* \ge 1$ that

$$\max_{|x|\le q^*\varepsilon} V(x) \le \max_{|x|\le q^*\varepsilon+\delta_R} W(x) \le \max_{|x|\le (G+1)\varepsilon} W(x) < \min_{x\in\mathcal{C}_O\setminus\mathcal{C}_I^\circ} W(x) \le \min_{x\in\partial\mathcal{O}} V(x)$$

and

$$\mathcal{B}_{q^*\varepsilon} \subset \mathcal{B}_{G\varepsilon} \subset \mathcal{C}_I \subset \mathcal{O}.$$

If $q^* < 1$ we get similarly by (18) that

$$\max_{|x|\le\varepsilon} V(x) \le \max_{|x|\le\varepsilon+\delta_R} W(x) \le \max_{|x|\le 2\varepsilon} W(x) < \min_{x\in\mathcal{C}_O\setminus\mathcal{C}_I^\circ} W(x) \le \min_{x\in\partial\mathcal{O}} V(x).$$

For every simplex $\mathcal{S}_\nu = \operatorname{co}(x_0^\nu, x_1^\nu, \cdots, x_n^\nu) \in \mathcal{T}$ define

$$W_\nu := \begin{pmatrix} W(x_1^\nu) - W(x_0^\nu) \\ W(x_2^\nu) - W(x_0^\nu) \\ \vdots \\ W(x_n^\nu) - W(x_0^\nu) \end{pmatrix}. \tag{21}$$

Choose one $\mathcal{S}_\nu = \operatorname{co}(x_0^\nu, x_1^\nu, \cdots, x_n^\nu) \in \mathcal{T}$ and let $y = x_0^\nu$ and $x \in \mathcal{S}_\nu$. Since $V \in$ CPA[$\mathcal{T}$], $V(x) = V(y) + \nabla V_\nu^\top(x-y)$. Then taking $x = x_i^\nu \in \mathcal{S}_\nu$ for all $i \in \{1, 2, \cdots, n\}$, using the fact that $V(x_i^\nu) = W(x_i^\nu)$, and the definitions W_ν, (21), and X_ν, (14), we get

$$\nabla V_\nu = X_\nu^{-1} W_\nu. \tag{22}$$

Hence,

$$V(x) = V(y) + W_\nu^\top (X_\nu^\top)^{-1}(x-y), \tag{23}$$

and by (15) and (20) we have

$$|\nabla V_\nu| = |X_\nu^{-1} W_\nu| \le |X_\nu^{-1}| \max_{z\in\mathcal{S}_\nu} |\nabla W(z)| \delta_R \le R \max_{z\in\mathcal{D}_O} |\nabla W(z)| = LR =: C \tag{24}$$

holds uniformly in ν. Thus (12) holds with this C.

Let x_i^ν be an arbitrary vertex of an arbitrary simplex $\mathcal{S}_\nu \subset \mathcal{O}$. Since $g(x) \in \mathcal{D}_\mathcal{T}$, there exists an $\mathcal{S}_\mu := \mathrm{co}(y_0^\mu, y_1^\mu, \ldots, y_n^\mu) \in \mathcal{T}$ such that $g(x_i^\nu) = \sum_{j=0}^{n} \mu_j y_j^\mu \in \mathcal{S}_\mu$ with $0 \le \mu_j \le 1$ for $j = 0, 1, \ldots, n$ and $\sum_{j=0}^{n} \mu_j = 1$. We have assigned $V(x) = W(x)$ for all vertices x of all simplices $\mathcal{S}_\nu$. Hence

$$\begin{aligned} V(g(x_i^\nu)) - V(x_i^\nu) &= \sum_{j=0}^{n} \mu_j W(y_j^\mu) - W(x_i^\nu) \\ &= \sum_{j=0}^{n} \mu_j W(y_j^\mu) - W\left(\sum_{j=0}^{n} \mu_j y_j^\mu\right) + W\underbrace{\left(\sum_{j=0}^{n} \mu_j y_j^\mu\right)}_{g(x_i^\nu)} - W(x_i^\nu). \end{aligned} \tag{25}$$

It follows that

$$V(g(x_i^\nu)) - V(x_i^\nu) \le L\delta_R - \alpha(|x_i^\nu|). \tag{26}$$

For a simplex $\mathcal{S}_\nu \subset \mathcal{O}$ such that $\mathcal{S}_\nu \cap \mathcal{B}_\varepsilon^c \neq \emptyset$ it follows, because $\delta_R < \varepsilon/2$, i.e. $|x_i^\nu| \ge \varepsilon/2$, and by

$$\delta_R < \frac{\alpha(\varepsilon/2)}{L(1+RG)} \iff L\delta_R - \alpha(\varepsilon/2) < -LRG\delta_R$$

that

$$V(g(x_i^\nu)) - V(x_i^\nu) \le L\delta_R - \alpha(\varepsilon/2) \le -LRG\delta_R \le -CG_\nu \,\mathrm{diam}(\mathcal{S}_\nu), \tag{27}$$

where G_ν and C are as in Theorem 2.8. Therefore the linear constraints (13) are fulfilled for all vertices x_i^ν of all simplices $\mathcal{S}_\nu \subset \mathcal{O}$ such that $\mathcal{S}_\nu \cap \mathcal{B}_\varepsilon^c \neq \emptyset$.

By Corollary 2.9 V is a CPA[$\mathcal{T}$] Lyapunov function for (1) on $\mathcal{O} \setminus \mathcal{B}_\varepsilon$. □

3. Converse Theorem Construction of Lyapunov Functions

We now address the problem of how to calculate the vertex values for each simplex. In [7] the vertex values are obtained by constructing and solving a linear program. This linear program is quite large with at least twice as many variables, and three times as many constraints, as vertices in the triangulation. Computationally, solving this linear program can be quite slow and, for a given triangulation, a feasible solution is not guaranteed to exist. Note that in [7] it is shown that by a process of refining the triangulation a feasible solution will be found eventually. However, this process of refinement increases the number of vertices and, hence, the size of the linear program.

Converse Lyapunov theorems such as [1, Theorem 5.12.5], [9], [13], [16], and [21, Theorem 1.7.6] state that if system (1) is asymptotically stable, then there exists a Lyapunov function. These results are, in general, not constructive as the value of the Lyapunov function at each point is given by a function of the system solution forward in time from that point. However, as we shall describe in the sequel, it is possible to either exactly or, with arbitrary precision, approximately calculate these Lyapunov functions using knowledge of solutions of (1) on a finite-time horizon. Furthermore, leveraging the results on CPA functions in the previous section, we only need to perform these calculations at a finite number of points; namely at the vertices of a triangulation.

We will investigate using two different constructions of the Lyapunov function for the purposes of defining the vertex values of a CPA function. The first of these constructions was originally proposed in [19] in continuous-time with a general discrete-time result provided in [13]. The second of these constructions was originally proposed in [22] in continuous-time and extended to the discrete-time case in [16].

Given an open set $\mathcal{D} \subseteq \mathbb{R}^n$ with $\mathbf{0} \in \mathcal{D}$, system (1) is said to be $\mathcal{KL}$-stable on $\mathcal{D}$ if for all $x \in \mathcal{D}$, $g(x) \in \mathcal{D}$ and there exists a $\beta \in \mathcal{KL}$ such that

$$|\phi(k,x)| \leq \beta(|x|,k), \quad \forall x \in \mathcal{D},\ k \in \mathbb{N}^0. \tag{28}$$

It has been proved in [17, Proposition 2.2] that the concept of $\mathcal{KL}$-stability is equivalent to the concept of asymptotic stability of the origin for system (1), given $\mathcal{D}$ is a subset of the domain of attraction. If $\mathcal{D} = \mathbb{R}^n$, then $\mathcal{KL}$-stability is equivalent to global asymptotic stability of the origin for system (1). The function $\beta \in \mathcal{KL}$ of (28) is called a stability estimate.

In order to define our candidate Lyapunov functions, we use a version of Sontag's lemma on $\mathcal{KL}$-estimates [20, Proposition 7] ([15, Lemma 7]).

Lemma 3.1. *For every $\mu \in (0,1)$ and $\beta \in \mathcal{KL}$ there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, with $\alpha_1(\cdot)$ Lipschitz on its domain and smooth on $\mathbb{R}_{>0}$ such that*

$$\alpha_1(\beta(s,k)) \leq \alpha_2(s)\mu^{2k},\ \forall s \in \mathbb{R}_{\geq 0},\ \forall\ k \in \mathbb{N}^0. \tag{29}$$

Proof. Sontag's lemma on $\mathcal{KL}$-estimates [15, Lemma 7] states that, for any $\lambda > 0$ and $\beta \in \mathcal{KL}$ there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ where α_1 is Lipschitz on its domain and smooth on $\mathbb{R}_{>0}$ and so that

$$\alpha_1(\beta(s,k)) \leq \alpha_2(s)e^{-\lambda k},\ \forall s \in \mathbb{R}_{\geq 0},\ \forall\ k \in \mathbb{N}^0.$$

Given $\mu \in (0,1)$, let $\lambda = -2\log\mu$ which satisfies $\lambda > 0$. Applying [15, Lemma 7] with this $\lambda > 0$ then yields (29). □

3.1 The Massera Construction

In continuous-time, the general construction of Lyapunov functions using the integral of a scaled function of the norm of solutions was originally proposed by Massera [19]. With the integral replaced by a sum in the discrete-time setting, we refer to such a function as a discrete-time Massera function.

Definition 3.2 *Assume system (1) is $\mathcal{KL}$-stable on $\mathcal{D}$. Given $\mu \in (0,1)$ and a stability estimate $\beta \in \mathcal{KL}$ with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ from Lemma 3.1 the function*

$$V(x) := \sum_{k=0}^{\infty} \alpha_1(|\phi(k,x)|), \quad x \in \mathcal{D} \subseteq \mathbb{R}^n \tag{30}$$

is called the discrete-time Massera function.

That such a function is indeed a continuous discrete-time Lyapunov function for (1) is a special case of [13, Theorem 1].

Theorem 3.3. *If the solution $\phi(k,x)$ of system (1) for all $x \in \mathcal{D}$ satisfies (28), then the discrete-time Massera function is continuous on $\mathcal{D}$ and satisfies the bounds*

$$\alpha_1(|x|) \leq V(x) \leq \tfrac{1}{1-\mu^2}\,\alpha_2(|x|) \tag{31}$$

and the decrease condition

$$V(\phi(1,x)) - V(x) \leq -\alpha_1(|x|). \tag{32}$$

Proof. The lower bound can be calculated as

$$\sum_{k=0}^{0} \alpha_1(|\phi(k,x)|) = \alpha_1(|x|) \leq V(x)$$

while the upper bound can be calculated as

$$V(x) \leq \sum_{k=0}^{\infty} \alpha_1(\beta(|x|,k)) \leq \sum_{k=0}^{\infty} \alpha_2(|x|)\mu^{2k} \leq \frac{1}{1-\mu^2}\alpha_2(|x|).$$

The decrease condition follows from

$$\begin{aligned} V(\phi(1,x)) - V(x) &= \sum_{j=0}^{\infty} \alpha_1(|\phi(j,\phi(1,x))|) - \sum_{k=0}^{\infty} \alpha_1(|\phi(k,x)|) \\ &= \sum_{k=1}^{\infty} \alpha_1(|\phi(k,x)|) - \sum_{k=0}^{\infty} \alpha_1(|\phi(k,x)|) = -\alpha_1(|x|). \end{aligned} \tag{33}$$

The continuity of V follows as in the proof of [13, Theorem 1]. □

For general systems and stability estimates we will not be able to obtain closed form solutions for the infinite sum of the discrete-time Massera function (30). In order to be able to calculate approximate values of the discrete-time Massera function for use as vertex values to define a CPA Lyapunov function, we propose truncating the infinite sum at some finite value $N \in \mathbb{N}^0$ which may depend on $x \in \mathcal{D}$; i.e., for $N : \mathcal{D} \to \mathbb{N}^0$

$$V(x) := \sum_{k=0}^{N(x)} \alpha_1(|\phi(k,x)|), \quad x \in \mathcal{D} \subseteq \mathbb{R}^n. \tag{34}$$

We observe that the upper and lower bounds (31) still hold if the sum defining the discrete-time Massera function is truncated at any finite upper limit. For a finite upper limit $N(x)$, the decrease condition (32) becomes

$$\begin{aligned} V(\phi(1,x)) - V(x) &= \sum_{j=0}^{N(x)} \alpha_1(|\phi(j,\phi(1,x))|) - \sum_{k=0}^{N(x)} \alpha_1(|\phi(k,x)|) \\ &= \sum_{k=1}^{N(x)+1} \alpha_1(|\phi(k,x)|) - \sum_{k=0}^{N(x)} \alpha_1(|\phi(k,x)|) \\ &= \alpha_1(|\phi(N(x)+1,x)|) - \alpha_1(|x|) \\ &\leq \alpha_2(|x|)\mu^{(N(x)+1)} - \alpha_1(|x|) \end{aligned} \tag{35}$$

so that we can make the positive term arbitrarily small by choosing $N(x)$ sufficiently large.

For a fixed $\varepsilon > 0$, in order to obtain a decrease condition satisfying

$$V(\phi(1,x)) - V(x) \leq -(1-\mu)\alpha_1(|x|), \quad \forall x \in \mathcal{D}_{\mathcal{T}} \backslash \mathcal{B}_\varepsilon, \tag{36}$$

define $\lambda = \mu^{-1}$ and choose

$$N(x) := \left\lceil \log_\lambda \left(\frac{\alpha_2(|x|)}{\alpha_1(|x|)} \right) \right\rceil \tag{37}$$

which follows by rearranging the inequality

$$\alpha_2(|x|)\mu^{(N(x)+1)} \leq \mu\alpha_1(|x|).$$

We observe that if the truncation limit is dependent on the point $x \in \mathcal{D}$, then the obtained function (34) will not be continuous. However, since we are interested in defining a Lyapunov function on a compact set $\mathcal{D}_{\mathcal{T}} \backslash \mathcal{B}_\varepsilon$, we can choose

$$N := \max_{x \in \mathcal{D}_{\mathcal{T}} \backslash \mathcal{B}_\varepsilon} N(x) \tag{38}$$

in order to have an upper limit on the sum that is independent of the point $x \in \mathcal{D}_{\mathcal{T}} \backslash \mathcal{B}_\varepsilon$ while maintaining the decrease condition (36).

Finally, using the uniform truncation (38) on $\mathcal{D}_{\mathcal{T}} \backslash \mathcal{B}_\varepsilon$, we observe that the truncated Massera construction (34) inherits the regularity of the vector field defining (1) since α_1 is smooth on $\mathcal{D}_{\mathcal{T}} \backslash \mathcal{B}_\varepsilon$. Consequently, Theorem 2.12 implies that for any system (1) where the right-hand side is continuously differentiable (i.e., $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$) and which has an asymptotically stable equilibrium at the origin, it is always possible to find a triangulation such that the CPA[$\mathcal{T}$] approximation to a truncated Massera construction is a CPA[$\mathcal{T}$] Lyapunov function.

3.2 The Yoshizawa Construction

Different to the construction used by Massera [19], Yoshizawa [22] constructed a Lyapunov function based on the supremum of solutions over time. We will refer to such a function as the discrete-time Yoshizawa function.

Definition 3.4 *Assume system (1) is $\mathcal{KL}$-stable on $\mathcal{D}$. Given $\mu \in (0,1)$ and $\beta \in \mathcal{KL}$ with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ from Lemma 3.1, the function*

$$V(x) := \sup_{k \in \mathbb{N}^0} \alpha_1(|\phi(k,x)|)\mu^{-k}, \quad x \in \mathcal{D} \subseteq \mathbb{R}^n \tag{39}$$

is called the discrete-time Yoshizawa function.

Based on the results of [17], we summarise some properties of the discrete-time Yoshizawa function in the next theorem.

Theorem 3.5. *If the solution $\phi(k,x)$ of system (1) with $x \in \mathcal{D}$ satisfies (28), and with $\mu \in (0,1)$ and $\alpha_1 \in \mathcal{K}_\infty$ from (39) and $\alpha_1 \in \mathcal{K}_\infty$ locally Lipschitz continuous, then the discrete-time Yoshizawa function is continuous and satisfies the bounds*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{40}$$

and the decrease condition

$$V(\phi(1,x)) \leq V(x)\mu. \tag{41}$$

Further, for each $x \in \mathcal{D}$ there exists a positive integer $K(x)$ such that

$$V(x) = \max_{k \in \{0,\cdots,K(x)\}} \alpha_1(|\phi(k,x)|)\mu^{-k}. \tag{42}$$

That $V(x)$ is continuous, bounded, and satisfies the decrease condition has been proved in [17].

With $\lambda = \mu^{-1}$, the integer $K(x)$ is calculated explicitly in [17, Claim 7] as

$$K(x) = \left\lceil -\log_{\lambda}\left(\frac{V(x)}{\alpha_2(|x|)}\right)\right\rceil + 1, \quad x \neq \mathbf{0}. \tag{43}$$

The upper and lower bounds on V given by (40) yield

$$0 \leq K(x) \leq \left\lceil -\log_{\lambda}\left(\frac{\alpha_1(|x|)}{\alpha_2(|x|)}\right)\right\rceil + 1 \tag{44}$$

$$= \left\lceil \log_{\lambda}\left(\frac{\alpha_2(|x|)}{\alpha_1(|x|)}\right)\right\rceil + 1 := \overline{K(x)}. \tag{45}$$

In the computations in the following section, we use $\overline{K(x)}$ instead of $K(x)$ in the computation of (42). This is done since we do not need to know $V(x)$ to calculate $\overline{K(x)}$ and, from the relations (39) and (42), taking a longer time horizon in (42) will not change the value of $V(x)$. Examples of the use of Sontag's lemma on $\mathcal{KL}$-estimates to define the Yoshizawa function can be found in [10, Examples 1 and 2].

Remark 4. If the origin is exponentially stable for (1) then there exist constants $r \in [0, 1)$, $M \geq 1$ such that

$$|\phi(k, x)| \leq M r^k |x|, \quad \forall x \in \mathbb{R}^n. \tag{46}$$

Then (29) is satisfied with $\alpha_1(s) = s$, $\alpha_2(s) = \lambda s$, and $\mu = \sqrt{r}$. Consequently, both the discrete-time Massera and Yoshizawa functions take a particularly simple form as there is no need to scale the norm of solutions. Furthermore, the window bounds (37) and (45) simplify to constants

$$N(x) = \lceil \log_{\lambda} M \rceil \quad \text{and} \quad \overline{K(x)} = N(x) + 1,$$

respectively, with $\lambda = 1/\sqrt{r}$.

3.3 Computational Procedure

For system (1), we propose the following procedure for constructing and verifying a CPA Lyapunov function:

1: Obtain a stability estimate $\beta \in \mathcal{KL}$ so that (28) holds.
2: Fix $\mu \in (0, 1)$ and find $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying inequality (29).
3: Define a suitable triangulation, $\mathcal{T}$, on a subset of the state space containing the equilibrium.
4: Calculate the vertex values of each simplex via either the truncated Massera function (34) or the truncated Yoshizawa function (42).
5: Construct a CPA[$\mathcal{T}$] function via convex (affine) interpolation of the vertex values of each simplex.
6: Check inequality (13) for each vertex.

Steps 3-6 are straightforward and of low computational complexity. If the CPA[$\mathcal{T}$] function computed does not satisfy the inequality (13) at some of the vertices of a simplex we can refine the triangulation and repeat Steps 3-6. It might be difficult to perform Steps 1 and 2 for some systems and in these cases one might simply perform Steps 3-6 for different $\mu \in (0,1)$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and then perform Step 6 to determine if the procedure has computed a CPA[$\mathcal{T}$] Lyapunov function.

Note that, consistent with the result of Theorem 2.12, we compute a Lyapunov function on $\mathcal{O} \setminus \mathcal{B}_\varepsilon$ for a small $\varepsilon > 0$. Note that we compute $\varepsilon > 0$ *a posteriori* as follows. In Step 6, for points near the origin where (13) is not satisfied, we calculate the norm of each such point and then take $\varepsilon > 0$ as the maximum of these norms. If the $\varepsilon > 0$ so obtained is unsatisfactorily large, one can refine the triangulation to obtain a smaller $\varepsilon > 0$.

From Theorem 2.8 and Corollary 2.9, if (13) holds for each vertex, then the constructed CPA[$\mathcal{T}$] function is a Lyapunov function. If the truncated discrete-time Massera or Yoshizawa functions (34) or (42), respectively, are differentiable with a bounded gradient on $\mathcal{D}_\mathcal{T}$, then Theorem 2.12 implies that our proposed procedure always yields a CPA[$\mathcal{T}$] Lyapunov function. While it is possible to apply smoothing techniques to the discrete-time Yoshizawa function to obtain a smooth Lyapunov function [17], it remains an open question as to whether the discrete-time truncated Massera or Yoshizawa functions can be directly shown to be differentiable with a bounded gradient. Note that, in continuous-time, an example shows that the (continuous-time) Yoshizawa function does not yield the required regularity property (continuous differentiability) [11, Example 3]. However, as previously remarked, if the vector field defining (1) satisfies $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, then the Lyapunov function from the uniformly truncated discrete-time Massera construction, i.e., (34) with (38), satisfies the conditions of Theorem 2.12.

4. Numerical Examples

In this section we present three numerical examples to demonstrate the effectiveness of the method proposed in Section 3.3. In these cases, we compute CPA approximations to both the Massera and Yoshizawa functions. In addition, we compare these with the results of the linear programming approach in [7]. To this end, we briefly summarise a simplified linear programming approach adapted from [7]. In particular, [7] use a particular fan-shaped triangulation at the origin and rely on exponential stability of the origin to obtain a CPA Lyapunov function on the entirety of $\mathcal{D}_\mathcal{T}$. Here, we exclude a neighbourhood of the origin as in the computational procedure of Section 3.3.

4.1 Linear programming based algorithm for computation of CPA Lyapunov functions

We describe a linear optimisation problem to compute a CPA Lyapunov function for system (1).

It is known that CPA functions are determined by values at vertices. In order to make sure $V \in$ CPA[$\mathcal{T}$] is a Lyapunov function, we have to impose certain conditions on the values of V at the vertices x_i of $\mathcal{S}_\nu$.

In order to make sure $V(x)$ is positive definite, we require that $x = 0$ is a vertex in the triangulation, $V(0) = 0$, $V \in \text{CPA}[\mathcal{T}]$, and

$$V(x_i) \geq |x_i|, \tag{47}$$

for each vertex $x_i \in \mathcal{S}_\nu \in \mathcal{T}$.

In order to ensure $V(x)$ satisfies

$$V(g(x)) - V(x) < 0 \tag{48}$$

for all $x \in \mathcal{O} \setminus \mathcal{B}_\varepsilon$, $x \neq 0$, we take care that $g(\mathcal{O}) \subset \mathcal{D}_\mathcal{T}$ and we incorporate interpolation errors into the inequalities. To this end, we impose the constraint

$$V(g(x_i)) - V(x_i) + \sqrt{n}\,\overline{C} G_\nu \operatorname{diam} \mathcal{S}_\nu < 0 \tag{49}$$

for all $x_i \in \mathcal{O} \setminus \mathcal{B}_\varepsilon$, where $\overline{C}$ is chosen so that $|\nabla V_{\nu,l}| \leq \overline{C}$ for $l = 1, \ldots, n$ and $\mathcal{S}_\nu \in \mathcal{T}$. Thus $|\nabla V_\nu| \leq \sqrt{n}\,\overline{C}$ for all $\mathcal{S}_\nu$, cf. (12) and the term $G_\nu \operatorname{diam} \mathcal{S}_\nu$ represents the interpolation error of g in the points $x \in \mathcal{S}_\nu$, $x_i \neq x$.

Let

$$\mathcal{T}^\varepsilon := \{\mathcal{S}_\nu \,:\, \mathcal{S}_\nu \cap \mathcal{B}_\varepsilon^c \neq \emptyset\} \subset \mathcal{T}, \qquad \mathcal{O}^\varepsilon := \bigcup_{\mathcal{S}_\nu \in \mathcal{T}^\varepsilon} (\mathcal{S}_\nu \cap \mathcal{O}). \tag{50}$$

Now we describe the linear programming based algorithm for computing CPA Lyapunov functions for system (1) on $\mathcal{O} \setminus \mathcal{B}_\varepsilon$.

4.1.1 Algorithm

We solve the following linear optimisation problem.

$$\text{Inputs:} \begin{cases} \varepsilon, \\ \text{All vertices } x_i \text{ of all simplices } \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}, \\ \operatorname{diam} \mathcal{S}_\nu \text{ of each simplex } \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}, \\ G_\nu \text{ for each simplex } \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}. \end{cases} \tag{51}$$

$$\text{Optimisation variables:} \begin{cases} V_{x_i} = V(x_i) \text{ for all vertices } x_i \text{ of each simplex } \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}, \\ C_{\nu,l} \text{ for } l = 1, 2, \ldots, n \text{ and every } \mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}, \\ \overline{C} \in \mathbb{R}_{>0}. \end{cases} \tag{52}$$

Optimisation problem: (53)

minimize $\overline{C}$

subject to

$(C1)$: $V_{x_i} \geq |x_i|$ for all vertices x_i of each simplex $\mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}$, and $V_0 = 0$,
$(C2)$: $|\nabla V_{\nu,l}| \leq C_{\nu,l}$ for each simplex $\mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}$, $l = 1, 2, \ldots, n$,
$(C3)$: $C_{\nu,l} \leq \overline{C}$ for each simplex $\mathcal{S}_\nu \subset \mathcal{D}_\mathcal{T}$, $l = 1, 2, \ldots, n$,
$(C4)$: $\max_{x_i} V_{x_i} < \min_{x_j} V_{x_j}$, for all vertices x_i of $\mathcal{S}_\nu$ such that $\mathcal{S}_\nu \cap \mathcal{B}_\varepsilon \neq \emptyset$ and all vertices $x_j \in \partial \mathcal{O}$,

For all vertices x_i of each simplex $\mathcal{S}_\nu \subset \mathcal{O}^\varepsilon$, the condition $(C5)$ is required:

$(C5)$: $V(g(x_i)) - V(x_i) + \sqrt{n}\,\overline{C} G_\nu \operatorname{diam} \mathcal{S}_\nu \leq -|x_i|$.

Remark 5. For $x \in \mathcal{S}_\nu$, $x = \sum_{i=0}^{n} \lambda_i x_i$, $\sum_{i=0}^{n} \lambda_i = 1$, $1 \geq \lambda_i \geq 0$.

(i) Since $V(x) = \sum_{i=0}^{n} \lambda_i V(x_i) \geq \sum_{i=0}^{n} \lambda_i |x_i|$, $V(x) \geq |x|$ for all $x \in \mathcal{D}_{\mathcal{T}}$.

(ii) The condition $(C2)$ defines linear constraints on the optimisation variables V_{x_i}, $C_{\nu,l}$.

(iii) $\overline{C}$ is used in estimating $|V(g(x)) - \sum_{i=0}^{n} \lambda_i V(g(x_i))|$. Constraint $(C3)$ is therefore necessary since $g(x)$ and $g(x_i)$ may not be in the same simplex.

(iv) The condition $(C4)$ ensures that the level set $\{x \in \mathcal{O} : V(x) \leq \max_{x_i \in \partial\mathcal{O}} V_{x_i}\}$ includes the set $\mathcal{B}_{\varepsilon}$.

Remark 6. If the linear optimisation problem (53) has a feasible solution, then the values V_{x_i} from this feasible solution at all vertices x_i of all simplices $\mathcal{S}_\nu \in \mathcal{T}$ and the condition $V \in \mathrm{CPA}[\mathcal{T}]$ uniquely define a continuous, piecewise affine function

$$V : \mathcal{D}_{\mathcal{T}} \to \mathbb{R}. \tag{54}$$

By Theorem 2.8 and Corollary 2.9 V is proved to be a CPA Lyapunov function for system (1).

The triangulations required in our approach can be obtained as described in [11]. In particular, in two dimensions, we define an initial triangulation with vertices at all integer coordinates in a rectangular region of $\mathbb{R}^2$ minus a smaller rectangular region of $\mathbb{R}^2$, both of which contain the origin. For example, in $\mathbb{R}^2$, fix $k, K \in \mathbb{N}$ such that $k < K$ and define vertices of the initial triangulation by $(i,j) \in [-K,K]^2 \backslash (-k,k)^2 \subset \mathbb{Z}^2$. The required 2-dimensional simplices can be defined as described in detail in [11] by defining appropriate edges between these vertices. To obtain the triangulation used in the following examples, the mapping

$$x \mapsto \sigma x, \tag{55}$$

is applied to the vertices of the initial triangulation where $\sigma \in \mathbb{R}_{>0}$ is an appropriately chosen constant. Note that [11] makes use of a nonlinear mapping in place of (55).

In the examples that follow, the linear programs are solved using the GNU Linear Programming Kit (GLPK)[1].

4.2 Example 1 - Linear System

Consider the system

$$x^+ = Ax = \begin{bmatrix} 0.25 & 0.25 \\ -0.125 & -0.25 \end{bmatrix} x \tag{56}$$

Let $x = (x_1, x_2)^\top$. We observe that the origin is globally exponentially stable as the eigenvalues of A are at $\pm\frac{\sqrt{2}}{8}$. We solve the so-called discrete Lyapunov equation (also called the Stein equation),

$$A^\top P A = P - 0.25I \tag{57}$$

for the symmetric positive definite matrix P, where I is the two-by-two identity matrix. Then

$$V(x) = x^\top P x = x^\top \begin{bmatrix} 0.2815 & -0.0235 \\ -0.0235 & 0.2698 \end{bmatrix} x \tag{58}$$

[1] `http://www.gnu.org/software/glpk/`

is a Lyapunov function and is shown in Figure 1 for system (56).

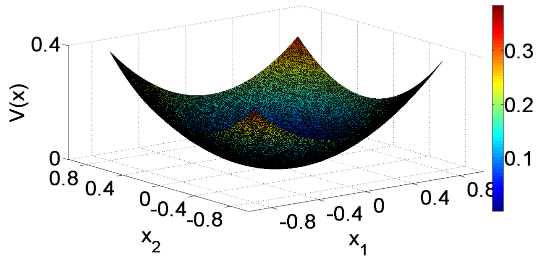

Figure 1.: Quadratic Lyapunov function $V(x)$ defined by (58) for system (56).

We observe that, for all $x \in \mathbb{R}^2$ and $k \in \mathbb{N}^0$,

$$|\phi(k,x)| \le \left(\frac{\sqrt{2}}{8}\right)^k |x| \le \exp(-k)|x| \tag{59}$$

and so (56) has a stability estimate $\beta \in \mathcal{KL}$ given by $\beta(s,k) = s\exp(-k)$ for all $s \in \mathbb{R}_{\ge 0}$ and $k \in \mathbb{N}^0$. With $\alpha_1(s) = \alpha_2(s) = s^2$ and $\mu = \exp(-1)$ we see that $\alpha_1(\beta(s,k)) \le \alpha_2(s)\mu^{2k}$. We can therefore calculate the truncation limits for the Massera (34) and Yoshizawa (42) functions as

$$N(x) = \left\lceil \ln\left(\frac{\alpha_2(|x|)}{\alpha_1(|x|)}\right)\right\rceil = 0 \quad \text{and} \tag{60}$$

$$\overline{K(x)} = \left\lceil \ln\left(\frac{\alpha_2(|x|)}{\alpha_1(|x|)}\right)\right\rceil + 1 = 1, \tag{61}$$

respectively.

Let $\mathcal{D} = [-0.3, 0.3]^2$ and $\mathcal{O} = \mathcal{D}$. The triangulation of $\mathcal{D}$ is obtained as described above with $K = 30, k = 5$, and $\sigma = 0.01$ in (55). Let $\mathcal{B} = (-0.05, 0.05)^2$.

CPA[$\mathcal{T}$] functions $V_{1,M}, V_{1,Y} : \mathcal{D} \setminus \mathcal{B} \to \mathbb{R}_{\ge 0}$ were constructed using the vertex values from the Massera and Yoshizawa constructions, respectively. The inequalities of Theorem 2.8 (equation (13)) can be numerically computed in both cases and shown to hold and hence $V_{1,M}$ and $V_{2,M}$ are CPA[$\mathcal{T}$] Lyapunov functions for (56).

Observe that, for the Yoshizawa function (42), we can explicitly compute

$$V_Y(x) = \max\{\alpha_1(|x|), \alpha_1(|\phi(1,x)|)\exp(1)\} = |x|^2$$

which follows directly from the middle term in (59). Furthermore, $V_M(x) = |x|^2$ and so the CPA[$\mathcal{T}$] Lyapunov functions resulting from the Massera and the Yoshizawa constructions are, in fact, equal. Furthermore, we observe that the resulting Yoshizawa and Massera constructions, V_Y and V_M, are quadratic, similar to the Lyapunov function (58) obtained by solving the discrete-time Lyapunov equation.

Finally, we also compute the CPA[$\mathcal{T}$] Lyapunov function $V_1 : \mathcal{D} \setminus \mathcal{B} \to \mathbb{R}_{\ge 0}$ for system (56) on $\mathcal{O} \setminus \mathcal{B}$ using the linear programming approach of [7] as summarised in Section 4.1. The resulting CPA[$\mathcal{T}$] Lyapunov function is shown in Figure 2, where we also show $V_{1,M}(x) = V_{1,Y}(x)$ for comparison.

Computation of CPA[$\mathcal{T}$] Lyapunov functions using the Massera function, the Yoshizawa function, and the linear programming approach are on the same suitable triangulation with 3641 vertices. Computation times are collected in Table 1.

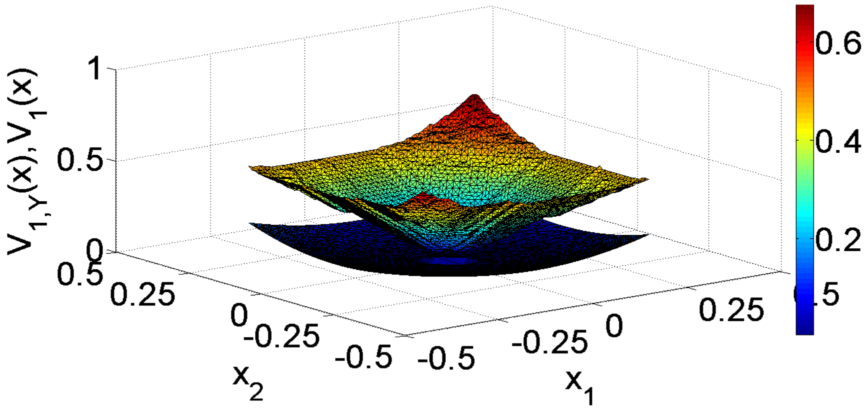

Figure 2.: $V_1(x)$ obtained via linear programming as in [7] (upper curve) and $V_{1,Y}(x) = V_{1,M}(x)$ (lower curve).

4.3 Example 2

Consider the one-dimensional system

$$x^+ = 0.7x^2, \ x \in \mathbb{R}. \tag{62}$$

It is straightforward to see that the origin is a locally asymptotically stable equilibrium point with a basin of attraction $(-0.7^{-1}, 0.7^{-1})$.

In the following we consider system (62) on $\mathcal{D} = [-1.375, 1.375]$. Note that for $x = 1.375$, $\phi(4, x) < 1$. By direct calculation it is possible to show that

$$|\phi(k,x)| \leq \begin{cases} 0.7^k|x| & , |x| \leq 1, \\ 0.7^k|x|^5 & , |x| \in (1, 1.375]. \end{cases} \tag{63}$$

Therefore a stability estimate for (62) is given by

$$\beta(s,k) = \begin{cases} 0.7^k s & , s \leq 1, \\ 0.7^k s^5 & , s > 1. \end{cases} \tag{64}$$

With $\alpha_1(s) = s^2$ and

$$\alpha_2(s) = \begin{cases} s & , s \leq 1, \\ s^{10} & , s > 1 \end{cases} \tag{65}$$

we see that $\alpha_1(\beta(s,k)) \leq \alpha_2(s)0.7^{2k}$.

Let $\mathcal{O} = \mathcal{D} = [-1.375, 1.375]$. The triangulation of $\mathcal{D}$ is obtained as described above with $K = 55$, $k = 5$, and $\sigma = 0.025$. Let $\mathcal{B} = (-0.125, 0.125)$.

4.3.1 Computation of CPA Lyapunov function based on Yoshizawa construction

Let $\mu = 0.7$, $\lambda = \frac{1}{\mu}$. For each $x \in \mathcal{D} \setminus \mathcal{B}$, the required optimisation time horizon (45) is

$$\overline{K(x)} = \left\lceil \log_\lambda \left(\frac{\alpha_2(|x|)}{\alpha_1(|x|)} \right) \right\rceil + 1 = \begin{cases} \left\lceil \log_\lambda \left(|x|^{-1} \right) \right\rceil + 1, & |x| \in [0.125, 1] \\ \left\lceil \log_\lambda \left(|x|^8 \right) \right\rceil + 1, & |x| \in (1, 1.375] \end{cases}$$

so that $\overline{K(x)} \in [1, 9]$ for $x \in \mathcal{D} \setminus \mathcal{B}$.

The Yoshizawa construction (42) for (62) is then given by

$$V_Y(x) = \max_{k \in \{0, \ldots, \overline{K(x)}\}} |\phi(k,x)|^2 0.7^{-k}. \tag{66}$$

Applying the computational procedure of Section 3.3 yields the CPA[$\mathcal{T}$] Lyapunov function $V_{2,Y} : \mathcal{D} \setminus \mathcal{B} \to \mathbb{R}_{\geq 0}$ constructed using the vertex values from the Yoshizawa function V_Y (66) for system (62) on $\mathcal{O} \setminus \mathcal{B}$ shown in Figure 3.

4.3.2 Computation of CPA Lyapunov function based on Massera construction

For each $x \in \mathcal{D} \setminus \mathcal{B}$, the truncated time horizon (37) is

$$N(x) = \left\lceil \log_\lambda \left(\frac{\alpha_2(|x|)}{\alpha_1(|x|)} \right) \right\rceil = \begin{cases} \left\lceil \log_\lambda \left(|x|^{-1} \right) \right\rceil, & |x| \in [0.125, 1] \\ \left\lceil \log_\lambda \left(|x|^{8} \right) \right\rceil, & |x| \in (1, 1.375] \end{cases}$$

and the truncated Massera construction (34) is

$$V_M(x) = \sum_{k=0}^{N(x)} |\phi(k,x)|^2. \tag{67}$$

Using vertex values of the Massera construction V_M defined by (67), a CPA[$\mathcal{T}$] function $V_{2,MS} : \mathcal{D} \setminus \mathcal{B} \to \mathbb{R}_{\geq 0}$ is constructed. The inequalities (13) of Theorem 2.8 are numerically computed and shown to hold. Then the CPA function $V_{2,MS}$ is a CPA Lyapunov function for system (62) on $\mathcal{O} \setminus \mathcal{B}$ shown in Figure 4.

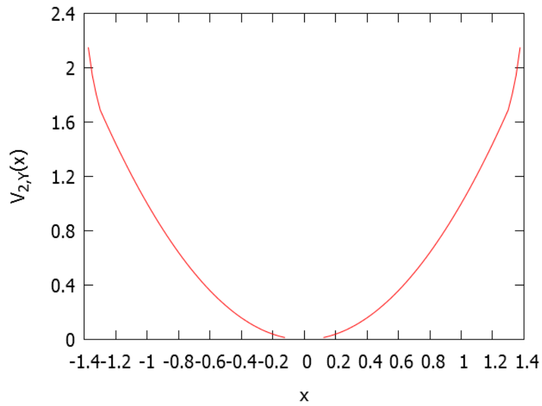

Figure 3.: CPA Lyapunov function $V_{2,Y}(x)$ obtained via the Yoshizawa construction for system (62)on $\mathcal{O} \setminus \mathcal{B}$.

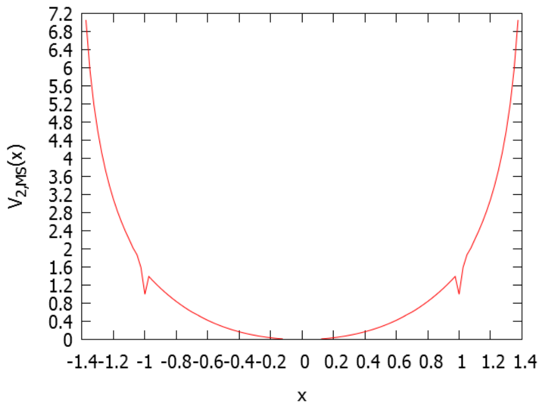

Figure 4.: CPA Lyapunov function $V_{2,MS}(x)$ obtained via the Massera construction for system (62) on $\mathcal{O} \setminus \mathcal{B}$.

Figure 4 of the truncated Massera construction indicates that $V_{2,MS}(x)$ is not smooth. This is not surprising since points close to each other can result in a different number of terms in the summation defining the discrete-time Massera function. In other words, continuity can be lost as a consequence of the possible discontinuity in $N(x)$, (37), caused by the ceiling function.

With

$$N = \max_{x \in \mathcal{D} \setminus \mathcal{B}} N(x) = \left\lceil \log_\lambda \left(1.375^8 \right) \right\rceil = 8,$$

we calculate vertex values of V_M defined by (67) with the upper limit N. Based on these vertex values, a CPA function $V_{2,MC} : \mathcal{D} \setminus \mathcal{B} \to \mathbb{R}_{\geq 0}$ is constructed. It can be verified that the inequalities (13) of Theorem 2.8 are satisfied on $\mathcal{D}_\mathcal{T}$ and hence the CPA[$\mathcal{T}$] function $V_{2,MC}$ is a CPA[$\mathcal{T}$] Lyapunov function for system (62) on $\mathcal{O} \setminus \mathcal{B}$. The constructed CPA[$\mathcal{T}$] Lyapunov function is shown in Figure 5.

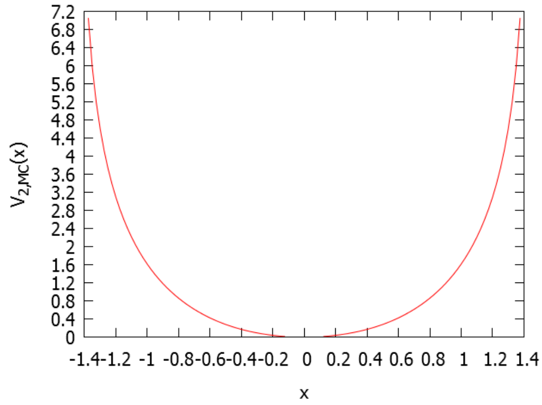

Figure 5.: CPA Lyapunov function $V_{2,MC}(x)$ obtained via the Massera construction for system (62) on $\mathcal{O} \setminus \mathcal{B}$.

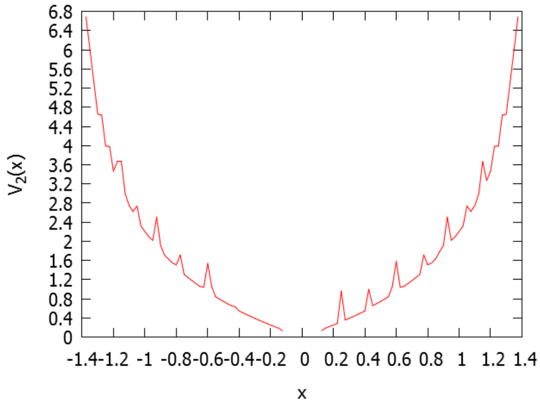

Figure 6.: CPA Lyapunov function $V_2(x)$ from the linear program for system (62) on $\mathcal{O} \setminus \mathcal{B}$.

For comparison with the above two constructions, we also compute a CPA Lyapunov function $V_2(x)$ for (62) on $\mathcal{O} \setminus \mathcal{B}$ using the linear programming approach described in Section 4.1, with the resulting function shown in Figure 6.

The above CPA[$\mathcal{T}$] Lyapunov functions for system (62) are computed on the same grid with 103 vertices. Computation times are shown in Table 1.

4.4 Example 3

Consider the following system studied in [5]

$$x^+ = g(x) = \begin{cases} \frac{1}{2}x_1 + x_1^2 - x_2^2, \\ -\frac{1}{2}x_2 + x_1^2. \end{cases} \tag{68}$$

Let $\mathcal{D} = [-0.675, 0.6] \times [-0.675, 0.625]$.

Observe that $g(\mathbf{0}) = \mathbf{0}$. The linearization of $g(x)$ at $\mathbf{0}$ is given by

$$g(x) \approx \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix} x. \tag{69}$$

Since the eigenvalues of the linearisation lie inside the unit circle, system (68) is locally exponentially stable at $\mathbf{0}$ and, by direct calculation, we have

$$|\phi(k, x)| \leq 0.95^k |x| \tag{70}$$

for all $x \in \mathcal{D}$. Therefore a stability estimate for (68) is given by

$$\beta(s, k) = 0.95^k s \tag{71}$$

and, with $\alpha_1(s) = s^2$ and $\alpha_2(s) = s$, we have

$$\alpha_1(\beta(s, k)) \leq \alpha_2(s) 0.95^{2k}. \tag{72}$$

Let $\mathcal{O} = [-0.65, 0.575] \times [-0.65, 0.6]$. For this example, replacing $[-K, K]^2$ with $[-27, 24] \times [-27, 25]$ we obtain the triangulation of $\mathcal{D}$ as described above with $k = 2$. Let $\mathcal{B} = (-0.25, 0.25)^2$.

4.4.1 Computation of CPA Lyapunov function based on Yoshizawa construction

Let $\mu = 0.95$ and $\lambda = \frac{1}{\mu}$. For each $x \in \mathcal{D} \setminus \mathcal{B}$ the required optimisation time horizon (45) is given by

$$\overline{K(x)} = \left\lceil \log_\lambda \left(\frac{1}{|x|} \right) \right\rceil + 1$$

so that $\overline{K(x)} \leq 60$ for all $x \in \mathcal{D} \setminus \mathcal{B}$.

The Yoshizawa construction (42) for (68) is

$$V_Y(x) = \max_{k \in \{0,\ldots,\overline{K(x)}\}} |\phi(k,x)|^2 0.95^{-k}. \tag{73}$$

A CPA Lyapunov function $V_{3,Y} : \mathcal{D} \setminus \mathcal{B} \to \mathbb{R}_{\geq 0}$ is constructed based on vertex values of V_Y defined by (73). It is shown the inequalities (13) of Theorem 2.8 are satisfied. Thus $V_{3,Y}$ shown in Figure 7 is a CPA Lyapunov function for system (68) on $\mathcal{O} \setminus \mathcal{B}$. The level curves of $V_{3,Y}$ are shown in Figure 9.

4.4.2 Computation of CPA Lyapunov function based on Massera construction

For each $x \in \mathcal{D} \setminus \mathcal{B}$, the uniformly truncated time horizon (37)-(38) is computed by

$$N = \max_{x \in \mathcal{D} \setminus \mathcal{B}} N(x) = \left\lceil \log_\lambda \left(\frac{1}{0.05} \right) \right\rceil = 59,$$

and the uniformly truncated Massera construction (34) is then

$$V_M(x) = \sum_{k=0}^{N} |\phi(k,x)|^2. \tag{74}$$

Using vertex values of V_M defined by (74), a CPA function $V_{3,M} : \mathcal{D} \setminus \mathcal{B} \to \mathbb{R}_{\geq 0}$ is uniquely defined. It is checked that the inequalities (13) of Theorem 2.8 are satisfied. Therefore, $V_{3,M}$ shown in Figure 8 is a CPA Lyapunov function for system (68) on $\mathcal{O} \setminus \mathcal{B}$. The level curves of $V_{3,M}$ are shown in Figure 10.

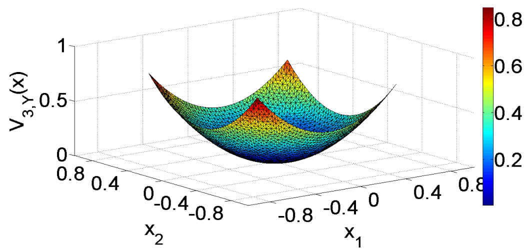

Figure 7.: CPA Lyapunov function $V_{3,Y}(x)$ obtained via the Yoshizawa construction for system (68) on $\mathcal{O} \setminus \mathcal{B}$.

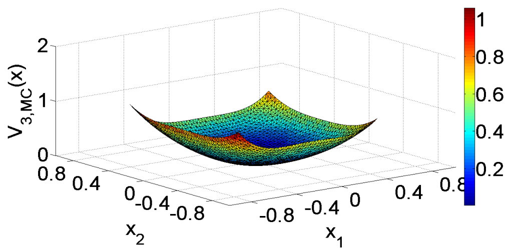

Figure 8.: CPA Lyapunov function $V_{3,M}(x)$ obtained via the Massera construction for system (68) on $\mathcal{O} \setminus \mathcal{B}$.

In order to compare with the above two constructions, a CPA[$\mathcal{T}$] Lyapunov function $V_3(x)$ for (68) on $\mathcal{O} \setminus \mathcal{B}$ is computed using the linear programming approach described in Section 4.1, with the resulting function shown in Figure 11. The level curves of $V_3(x)$ are shown in Figure 12.

The above CPA[$\mathcal{T}$] Lyapunov functions for system (68) are computed on the same grid with 2756 vertices. Computation times for the various methods are shown in Table 1.

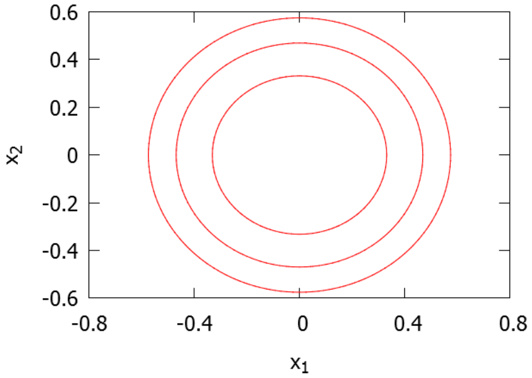

Figure 9.: Level curves of CPA Lyapunov function $V_{3,Y}(x)$ at values $0.11, 0.22, 0.33$.

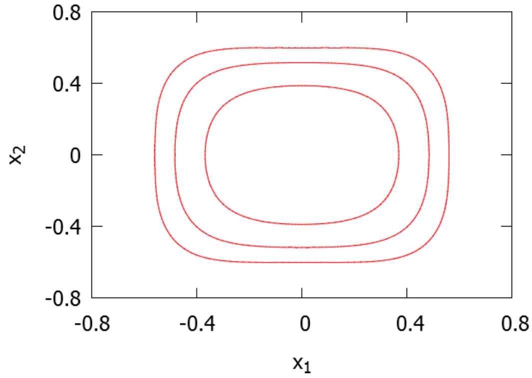

Figure 10.: Level curves of CPA Lyapunov function $V_{3,M}(x)$ at values $0.17, 0.35, 0.52$.

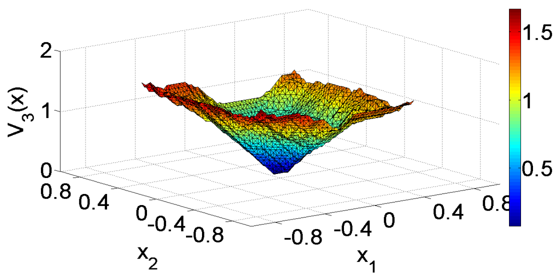

Figure 11.: CPA Lyapunov function $V_3(x)$ from the linear program for system (68) on $\mathcal{O} \setminus \mathcal{B}$.

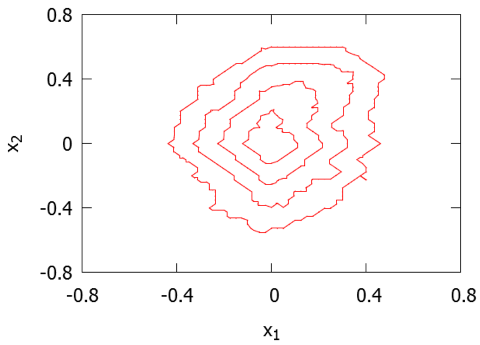

Figure 12.: Level curves of CPA Lyapunov function $V_3(x)$ at values at $0.32, 0.64, 0.95$.

System (Equation No.)	No. Vertices	Yoshizawa	Massera	LP
Example 1 - (58)	3641	229.3s	228.9s	8361.3s
Example 2 - (62)	103	3.4s	3.6s	7.2s
Example 3 - (68)	2756	183.0s	173.8s	5929.3s

Table 1.: Computation times for Examples 1–3 (PC: AMD Athlon II P360 Dual-Core 2.30 GHZ Processor with 2GB memory).

5. Conclusions

In this paper, we proposed a new method of computing a CPA Lyapunov function for discrete-time dynamic system (1). This approach replaces the linear program of [18] and [8] with evaluation of the Yoshizawa function and the Massera function, respectively, followed by a check of the linear inequalities (13) at the vertices of the triangulation. We observe that in the numerical examples presented in Section 4, computing the values of the Yoshizawa construction only requires taking the maximum among easily computable values, making this a very efficient method to obtain vertex values. Furthermore, using the Massera construction to obtain values at the vertices is merely the sum of finite computable values. We also compared the CPA Lyapunov functions computed via the Yoshizawa and Massera constructions with the linear programming approach [18] and [8]. If (1) is $\mathcal{KL}$ stable (i.e., if the origin is asymptotically stable) with a known $\mathcal{KL}$ stability estimate, and if suitable α_1, α_2 satisfying (29) can be found, our approach successfully delivers a CPA Lyapunov function. Furthermore, our proposed approach is significantly more efficient than solving a linear optimisation problem for the presented numerical examples. If the origin is exponentially stable, as per Remark 4, it is easy to find $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that (29) holds. However, for general discrete-time dynamic systems with an asymptotically stable but not exponentially stable origin, there is no explicit procedure to obtain a $\mathcal{KL}$ stability estimate. Even if a $\mathcal{KL}$ stability estimate is known, finding suitable α_1, α_2 satisfying (29) is not necessarily straightforward and the proof of [20, Proposition 7] is not constructive.

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References

[1] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, Marcel Dekker, 2nd ed., 2000.

[2] R. Baier, L. Grüne, and S. Hafstein, *Linear programming based Lyapunov function computation for differential inclusions*, Discrete and Continuous Dynamical Systems Series B, 17 (2012), pp. 33–56.

[3] H. Ban and W. D. Kalies, *A computational approach to Conley's decomposition theroem*, J. Comput. Nonlinear Dynam., 1 (2006), pp. 312–319.

[4] J. Björnsson, P. Giesl, S. Hafstein, C. M. Kellett, and H. Li, *Computation of continuous and piecewise affine Lyapunov functions by numerical approximations of the Massera construction*, in Proceedings of the 53rd IEEE Conference on Decision and Control (to appear), Los Angeles, California, USA, 2014.

[5] P. Giesl, *On the determination of the basin of attraction of discrete dynamical systems*, Journal of Difference Equations and Applications, 13 (2007), pp. 523–546.

[6] P. Giesl, *Construction of a local and global Lyapunov function using radial basis functions*, IMA J. Appl. Math., 73 (2008), pp. 782–802.

[7] P. Giesl and S. Hafstein, *Computation of Lyapunov functions for nonlinear discrete time systems by linear programming*, Journal of Difference Equations and Applications, 20 (2014), pp. 610–640.

[8] ——, *Revised CPA method to compute Lyapunov functions for nonlinear systems*, Journal of Mathematical Analysis and Applications, 410 (2014), pp. 292–306.

[9] S. P. Gordon, *On converses to the stability theorems for difference equations*, SIAM J. Control, 10 (1972), pp. 76–81.

[10] S. Hafstein, C. M. Kellett, and H. Li, *Continuous and piecewise affine Lyapunov functions using the Yoshizawa construction*, in Proc. 2014 American Control Conference, 2014, pp. 548–553 (no. 0158).

[11] S. Hafstein, C. M. Kellett, and H. Li, *A fast method for computing continuous and piecewise affine Lyapunov functions using the Yoshizawa construction*. Submitted to *Automatica*, June 2014.

[12] W. Hahn, *Stability of Motion*, Springer-Verlag, 1967.

[13] Z.-P. Jiang and Y. Wang, *A converse Lyapunov theorem for discrete-time systems with disturbances*, Systems and Control Letters, 45 (2002), pp. 49–58.

[14] W. D. Kalies, K. Mischaikow, and R. C. A. M. VanderVorst, *An algorithmic approach to chain recurrence*, Found. Comput. Math., 5 (2005), pp. 409–449.

[15] C. M. Kellett, *A compendium of comparsion function results*, Mathematics of Control, Signals, and Systems, 26 (2014), pp. 339–374.

[16] C. M. Kellett and A. R. Teel, *Smooth Lyapunov functions and robustness of stability for difference inclusions*, Systems and Control Letters, 52 (2004), pp. 395–405.

[17] ——, *On the robustness of $\mathcal{KL}$-stability for difference inclusions: Smooth discrete-time Lyapunov functions*, SIAM Journal on Control and Optimization, 44 (2005), pp. 777–800.

[18] S. Marinosson, *Lyapunov function construction for ordinary differential equations with linear programming*, Dynamical Systems, 17 (2002), pp. 137–150.

[19] J. L. Massera, *On Liapounoff's conditions of stability*, Annals of Mathematics, 50 (1949), pp. 705–721.

[20] E. D. Sontag, *Comments on integral variants of ISS*, Systems and Control Letters, 34 (1998), pp. 93–100.

[21] A. Stuart and A. Humphries, *Dynamical Systems and Numerical Analysis*, Cambridge University Press, 1996.

[22] T. Yoshizawa, *On the stability of solutions of a system of differential equations*, Memoirs of the College of Science, University of Kyoto, Series A: Mathematics, 29 (1955), pp. 27–33.

[23] ——, *Stability Theory by Liapunov's Second Method*, Mathematical Society of Japan, 1966.