Existence of piecewise affine Lyapunov functions in two dimensions

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Abstract

In [10] a method to compute Lyapunov functions for systems with asymptotically stable equilibria was presented. The method uses finite differences on finite elements to generate a linear programming problem for the system in question, of which every feasible solution parameterises a piecewise affine Lyapunov function. In [2] it was proved that the method always succeeds in generating a Lyapunov function for systems with an exponentially stable equilibrium. However, the proof could not guarantee that the generated function has negative orbital derivative locally in a small neighborhood of the equilibrium. In this article we give an example of a system, where no piecewise affine Lyapunov function with the proposed triangulation scheme exists. This failure is due to the triangulation of the method being too coarse at the equilibrium, and we suggest a fan-like triangulation around the equilibrium. We show that for any two-dimensional system with an exponentially stable equilibrium there is a local triangulation scheme such that the system possesses a piecewise affine Lyapunov function. Hence, the method might eventually be equipped with an improved triangulation scheme that does not have deficits locally at the equilibrium.

Keywords: exponentially stable equilibrium; piecewise linear Lyapunov function; triangulation

MSC 2010: 37C10; 37B25

1 Introduction

Consider the autonomous system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, and assume that the origin is an exponentially stable equilibrium of the system. Denote its basin of

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attraction by $\mathcal{A}$. The standard method to obtain a local Lyapunov function and thus a subset of the basin of attraction is to solve the Lyapunov equation, i.e. to find a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ that is a solution to $A^TQ + QA = -P$, where $A = Df(\mathbf{0})$ is the Jacobian of f at the origin and $P \in \mathbb{R}^{n\times n}$ is an arbitrary positive definite matrix. Then the quadratic function $\mathbf{x} \mapsto \mathbf{x}^TQ\mathbf{x}$ is a local Lyapunov function for the system $\dot{\mathbf{x}} = f(\mathbf{x})$, i.e. it is a Lyapunov function for the system in some neighborhood of the origin. The size of this neighborhood is a priori not known and is, except for linear f, in general a poor estimate of $\mathcal{A}$ (see, for example, [2] for more details).

In the last decades there have been several proposals of how to numerically construct Lyapunov functions. To name a few, Johansson and Rantzer proposed a construction method in [7] for piecewise quadratic Lyapunov functions for piecewise affine autonomous systems. In [6] Johansen uses linear programming to parameterise Lyapunov functions for autonomous nonlinear systems. His results are, however, only valid within an approximation error, which is difficult to determine. Giesl proposed in [1] a method to construct Lyapunov functions for autonomous systems with an exponentially stable equilibrium by solving numerically a generalised Zubov equation, cf. [11],

$$\nabla V(\mathbf{x}) \cdot f(\mathbf{x}) = -p(\mathbf{x}), \tag{1.1}$$

where usually $p(\mathbf{x}) = \|\mathbf{x}\|_2$. A solution to the partial differential equation (1.1) is a Lyapunov function for the system. He uses radial basis functions to find a numerical solution to (1.1) and there are error estimates given.

In [10] Hafstein (alias Marinosson) presented a method to compute piecewise affine Lyapunov function for autonomous systems with an asymptotically stable equilibrium. This is the method we will be considering in this paper. In this method one first triangulates a compact neighborhood $\mathcal{C} \subset \mathcal{A}$ of the origin and then constructs a linear programming problem with the property, that a continuous Lyapunov function V, affine on any triangle of the triangulation, can be constructed from any feasible solution to it. In [2] it was proved that for exponentially stable equilibria this method is always capable of generating a Lyapunov function $V : \mathcal{C} \setminus \mathcal{N} \longrightarrow \mathbb{R}$, where $\mathcal{N} \subset \mathcal{C}$ is an arbitrary small, a priori determined neighborhood of the origin. In [3] these results were generalised to asymptotically stable systems and in [4] to asymptotically stable, arbitrary switched, non-autonomous systems.

Since the existence of a piecewise affine Lyapunov function in a neighborhood of an equilibrium implies its exponential stability, one must necessarily cut out some neighborhood $\mathcal{N}$ of the origin, if the equilibrium is not exponentially stable. However, it has not been clear if the same holds for exponentially stable equilibria. In this paper we will show, that due to the triangulation scheme used in [10, 2, 3, 4], it is necessary to cut out $\mathcal{N}$ from the domain where the Lyapunov function will be constructed. We show this by giving an example of a two-dimensional system with an exponentially stable equilibrium at the origin, which cannot possess a Lyapunov function that is affine on the eight triangles of the triangulation scheme sharing the origin as a central vertex. This raises the question, whether this is because there are no piecewise affine Lyapunov functions around the origin, or if this is just the result of the suboptimal triangulation scheme used by the method.

In two dimensions we can give a definite answer to this question. The failure of the method to construct a piecewise affine Lyapunov function locally at the equilibrium for certain systems, is not an intrinsic property of the method, but is caused by a triangulation scheme which is too coarse. Further, we show that for any system with an exponentially stable equilibrium, there is a local triangulation scheme, which generates triangles with the equilibrium as a central vertex such that the system possesses a Lyapunov function, affine on all of the triangles. Further, we explain why the proof for two dimensions cannot be generalised to higher dimensions in a straight-forward way.

Because of these results, the authors are optimistic, that the method in [10, 2, 3, 4] to construct piecewise affine Lyapunov functions, can be equipped with a more advanced triangulation scheme such that it can compute a piecewise affine Lyapunov function $V : \mathcal{C} \longrightarrow \mathbb{R}$ for any system with an exponentially stable equilibrium. Note, that in contrast to the standard method to generate local quadratic Lyapunov functions as described above, this would lead to an algorithm that kills two birds with one stone: For every system with an exponentially stable equilibrium, it could generate a both local and global Lyapunov function establishing exponential stability and giving a reasonable estimate on the size of the basin of attraction.

This paper is organised as follows: In Section 2 we discuss triangulations and piecewise affine Lyapunov functions. Section 3 deals with a counterexample, a two-dimensional system such that there is no piecewise affine Lyapunov function for any triangulation as in [10]. Section 4 contains the main result, Theorem 4.3, which shows the existence of a piecewise affine Lyapunov function for any two-dimensional system with an exponentially stable equilibrium. The paper closes with a summary and outlook in Section 5.

2 Piecewise affine Lyapunov functions

In the following, $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^T\mathbf{y} = \sum_{i=1}^n x_i y_i$ denotes the usual dot product and $\|\mathbf{x}\|_2 := \sqrt{\mathbf{x}\cdot\mathbf{x}}$ denotes the Euclidean norm for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Moreover, $\|Q\|_2 := \sup_{\|\mathbf{x}\|_2=1} \|Q\mathbf{x}\|_2$ denotes the induced matrix norm of a matrix $Q \in \mathbb{R}^{n\times n}$. The convex hull of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ is denoted by

$$\mathrm{co}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} := \left\{ \sum_{i=1}^k \lambda_i \mathbf{x}_i \mid \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^k \lambda_i = 1 \right\}.$$

Triangulation

In $\mathbb{R}^n$ we triangulate an area containing the origin into a finite number of closed simplices $\mathcal{T} = \{T_\nu \mid \nu = 1, 2, \ldots, N\}$, such that $\mathcal{C} := \bigcup_{T_\nu \in \mathcal{T}} T_\nu$ is simply connected and $\mathbf{0} \in \overset{\circ}{\mathcal{C}}$. Further, we demand that if $\mathbf{0} \in T_\nu$, then $\mathbf{0}$ is a vertex of T_ν. Recall, that a triangulation in $\mathbb{R}^n$ is defined as a subdivision of $\mathbb{R}^n$ into n-simplices (n-dimensional objects), such that any two different simplices intersect in a common face or not at all.

For the two-dimensional case $n = 2$, the simplices in $\mathcal{T}$ are triangles and this condition reads for $\mu \neq \nu$,

$$T_\mu \cap T_\nu = \begin{cases} \emptyset, & \text{or,} \\ \{\mathbf{y}\}, & \text{where } \mathbf{y} \text{ is a vertex common to } T_\mu \text{ and } T_\nu, \text{ or} \\ \mathrm{co}\{\mathbf{y}, \mathbf{z}\}, & \text{where } \mathbf{y} \text{ and } \mathbf{z} \text{ are vertices common to } T_\mu \text{ and } T_\nu. \end{cases}$$

This is necessary because we want a function $V : \mathcal{C} \longrightarrow \mathbb{R}$ to be uniquely defined by its values on the vertices of the simplices in $\mathcal{T}$ such that

- $V : \mathcal{C} \longrightarrow \mathbb{R}$ is continuous and
- the restriction of V to any simplex $T_\nu \in \mathcal{T}$ is affine, i.e. there are $\mathbf{w}_\nu \in \mathbb{R}^n$ and $a_\nu \in \mathbb{R}$ such that $V(\mathbf{x}) = \mathbf{w}_\nu \cdot \mathbf{x} + a_\nu$ for every $\mathbf{x} \in T_\nu$.

If we fix $\nu = 1, 2, \ldots, N$ and define the restriction of V to the simplex T_ν by $V_\nu(\mathbf{x}) := V\big|_{T_\nu}(\mathbf{x})$, then the gradient satisfies $\nabla V_\nu(\mathbf{x}) = \mathbf{w}_\nu$ for $\mathbf{x} \in T_\nu$. We will call such a set $\mathcal{C}$ a **triangulated domain** and such a function V **piecewise affine with respect to $\mathcal{C}$**. Later we will use a triangulation where every triangle has $\mathbf{0}$ as a vertex and the function V satisfies $V(\mathbf{0}) = 0$. Hence, the function V will even be piecewise linear.

Lyapunov functions

We consider the autonomous ordinary differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{2.1}$$

where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, and we denote the solution at time t with initial value $\boldsymbol{\xi}$ at time 0 by $\boldsymbol{\phi}(t, \boldsymbol{\xi})$. Moreover, we assume that the origin $\mathbf{x}_0 = \mathbf{0}$ is an exponentially stable equilibrium. Then the basin of attraction $\mathcal{A} := \{\boldsymbol{\xi} \in \mathbb{R}^n \mid \lim_{t\to\infty} \boldsymbol{\phi}(t, \boldsymbol{\xi}) = \mathbf{0}\}$ is an open and non-empty set.

A method to determine compact subsets of the basin of attraction is to use sublevel sets of a Lyapunov function. A smooth (strict) Lyapunov function for the system (2.1) is a function $V \in C^1(U, \mathbb{R})$, where $U \subset \mathbb{R}^n$ is an open neighborhood of $\mathbf{0}$, such that for all $\mathbf{x} \in U$ we have

1. $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ and $V(\mathbf{0}) = 0$.
2. $V'(\mathbf{x}) < 0$ for $\mathbf{x} \neq \mathbf{0}$, where $V'(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot f(\mathbf{x})$ denotes the orbital derivative, i.e. the derivative along solutions of $\dot{\mathbf{x}} = f(\mathbf{x})$.

Then any compact sublevel set $K = \{\mathbf{x} \in \mathbb{R}^n \mid V(\mathbf{x}) \leq R\}$ with $R > 0$ and $K \subset U$ is a subset of the basin of attraction of $\mathbf{0}$.

Now we consider the case that the function V is not C^1, but only continuous. In this case, the same consequences hold if the orbital derivative in 2. is replaced by the Dini derivative $D^+[V(\mathbf{x})]$ of V along the trajectories of the system.

Definition 2.1 (Dini derivative) *For a continuous function $V \in C^0(\mathbb{R}^n, \mathbb{R})$ and the ODE (2.1) we define the Dini derivative*

$$D^+[V(\boldsymbol{\phi}(t, \boldsymbol{\xi}))] := \limsup_{h \to 0^+} \frac{V(\boldsymbol{\phi}(t+h, \boldsymbol{\xi})) - V(\boldsymbol{\phi}(t, \boldsymbol{\xi}))}{h}.$$

Proposition 2.2 *Let $V \in C^0(U, \mathbb{R})$, where $U \subset \mathbb{R}^n$ is an open neighborhood of $\mathbf{0}$, be a (strict) Lyapunov function for (2.1), i.e. for all $\mathbf{x} \in U$ we have*

1. *$V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ and $V(\mathbf{0}) = 0$.*
2. *$D^+[V(\mathbf{x})] < 0$ for $\mathbf{x} \neq \mathbf{0}$, cf. Definition 2.1.*

Then any compact sublevel set $K = \{\mathbf{x} \in \mathbb{R}^n \mid V(\mathbf{x}) \leq R\}$ with $R > 0$ and $K \subset U$ is a subset of the basin of attraction of $\mathbf{0}$.

For a proof cf. [9, Theorem 1.16].

In the following Theorem 2.3 we derive a different way of calculating the derivative, using small variations not along the solution but along the vector field f. This theorem is stated in [5, p. 196] without any restrictions on V, but there is no proof or references given. In our theorem, we assume that V is locally Lipschitz.

Theorem 2.3 *Let $U \subset \mathbb{R}^n$ be a domain, $f \in C^1(U, \mathbb{R}^n)$, and assume that $V \in C^0(U, \mathbb{R})$ is locally Lipschitz on U. Let $\boldsymbol{\phi}(t, \boldsymbol{\xi})$ be the solution of the initial value problem*

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad \mathbf{x}(0) = \boldsymbol{\xi}.$$

Then

$$D^+[V(\boldsymbol{\phi}(t, \boldsymbol{\xi}))] = \limsup_{h \to 0^+} \frac{V(\boldsymbol{\phi}(t, \boldsymbol{\xi}) + hf(\boldsymbol{\phi}(t, \boldsymbol{\xi}))) - V(\boldsymbol{\phi}(t, \boldsymbol{\xi}))}{h}$$

for every t in the domain of $\boldsymbol{\phi}(t, \boldsymbol{\xi})$ with $\boldsymbol{\phi}(t, \boldsymbol{\xi}) \in U$.

Proof: We denote $\mathbf{x}(t) := \boldsymbol{\phi}(t, \boldsymbol{\xi})$. By Taylor's theorem there is a constant $\vartheta_h \in$

$(0,1)$ for any h small enough, such that

$$\begin{aligned}
&\limsup_{h\to 0+}\frac{V(\mathbf{x}(t+h))-V(\mathbf{x}(t))}{h}\\
&\quad=\limsup_{h\to 0+}\frac{V(\mathbf{x}(t)+h\dot{\mathbf{x}}(t+h\vartheta_h))-V(\mathbf{x}(t))}{h}\\
&\quad=\limsup_{h\to 0+}\frac{V(\mathbf{x}(t)+hf(\mathbf{x}(t+h\vartheta_h)))-V(\mathbf{x}(t))}{h}\\
&\quad=\limsup_{h\to 0+}\Bigg(\frac{V(\mathbf{x}(t)+hf(\mathbf{x}(t)))-V(\mathbf{x}(t))}{h}\\
&\qquad\qquad+\frac{V(\mathbf{x}(t)+hf(\mathbf{x}(t+h\vartheta_h)))-V(\mathbf{x}(t)+hf(\mathbf{x}(t)))}{h}\Bigg).
\end{aligned}$$

For any real-valued functions a and b such that $\limsup_{x\to y+} a(x)$ and $\lim_{x\to y+} b(x)$ exist, we have

$$\limsup_{x\to y+}(a(x)+b(x))=\limsup_{x\to y+}a(x)+\lim_{x\to y+}b(x).$$

Hence, to finish the proof it suffices to prove that

$$\lim_{h\to 0+}\frac{V(\mathbf{x}(t)+hf(\mathbf{x}(t+h\vartheta_h)))-V(\mathbf{x}(t)+hf(\mathbf{x}(t)))}{h}=0.$$

Let K be a compact neighborhood of $\mathbf{x}(t)$ and L_K be a Lipschitz constant for the restriction of V on K. Then for every h small enough,

$$\begin{aligned}
&\left|\frac{V(\mathbf{x}(t)+hf(\mathbf{x}(t+h\vartheta_h)))-V(\mathbf{x}(t)+hf(\mathbf{x}(t)))}{h}\right|\\
&\qquad\le\frac{L_K}{h}\|hf(\mathbf{x}(t+h\vartheta_h))-hf(\mathbf{x}(t))\|\\
&\qquad= L_K\|f(\mathbf{x}(t+h\vartheta_h))-f(\mathbf{x}(t))\|
\end{aligned}$$

and the continuity of f and $\mathbf{x}$ imply the vanishing of the limit above. □

Corollary 2.4 *Let $\mathcal{C}\subset\mathbb{R}^n$ be a triangulated domain, $f\in C^1(\mathcal{C},\mathbb{R}^n)$, and let $V:\mathcal{C}\longrightarrow\mathbb{R}$ be a continuous, piecewise affine function with respect to $\mathcal{C}$, cf. Section 2.*

Then for each $\mathbf{x}\in\overset{\circ}{\mathcal{C}}$ there is a simplex T_ν and $\delta>0$ such that $\mathbf{x}+hf(\mathbf{x})\in T_\nu$ for all $h\in[0,\delta]$. Since V is affine in T_ν, the orbital derivative $V'(\mathbf{x})$ exists in T_ν and we have

$$D^+[V(\mathbf{x})]=V'(\mathbf{x}).$$

Proof: The first statement follows from the fact that simplices are convex sets. Since V is affine on T_ν, $V(\mathbf{x})=\mathbf{w}_\nu\cdot\mathbf{x}+a_\nu$ for $\mathbf{x}\in T_\nu$. With $\mathbf{x}=\boldsymbol{\phi}(t,\boldsymbol{\xi})$, Theorem

2.3 implies that

$$
\begin{aligned}
D^+[V(\mathbf{x})] &= \limsup_{h\to 0+} \frac{V(\mathbf{x}+hf(\mathbf{x}))-V(\mathbf{x})}{h} \\
&= \limsup_{h\to 0+} \frac{\mathbf{w}_\nu\cdot[\mathbf{x}+hf(\mathbf{x})]+a_\nu-(\mathbf{w}_\nu\cdot\mathbf{x}+a_\nu)}{h} \\
&= \mathbf{w}_\nu\cdot f(\mathbf{x}) \\
&= \nabla V(\mathbf{x})\cdot f(\mathbf{x}).
\end{aligned}
$$

□

Corollary 2.4 implies that if we wish to show $D^+[V(\mathbf{x})]<0$, then it is sufficient to show $V'(\mathbf{x})<0$ for all simplices.

3 A counterexample

In [10] a method to generate Lyapunov functions was presented, but even for exponentially stable equilibria a small neighborhood $\mathcal{N}$ has to be cut out beforehand, if one wants to be sure that the method succeeds in computing a Lyapunov function for the system. If this is done, then the generated Lyapunov function does not have a negative orbital derivative in this local set $\mathcal{N}$ in general. In $\mathbb{R}^2$ this method uses a triangulation around the equilibrium in eight triangles, cf. Figure 1. We will give an example where no piecewise affine Lyapunov function with respect to such a triangulation exists, cf. Lemma 3.1, and therefore the method is bound to fail. In a second lemma, cf. Lemma 3.2, we will show that this problem can be overcome by using a finer triangulation around the equilibrium. Note that a piecewise affine function V with respect to a triangulation where all triangles have $\mathbf{0}$ as a vertex and which satisfies $V(\mathbf{0})=0$, is a piecewise linear function.

Lemma 3.1 *Consider the linear two-dimensional system*

$$
\left\{\begin{array}{rcl}\dot{x} &=& -\epsilon x - y\\ \dot{y} &=& -\epsilon y + x\end{array}\right. \tag{3.1}
$$

For $\epsilon>0$, the origin is exponentially stable and solutions spiral towards it; in polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ the system reads $\dot{r}=-\epsilon r$, $\dot{\theta}=1$.

Let

$$\epsilon \le \sqrt{2}-1.$$

Then, starting with any partition in eight rectangular triangles with vertices on a rectangle around the origin, there cannot exist a continuous Lyapunov function which is linear in each of the triangles.

Proof: Without loss of generality we can restrict ourselves to choosing the vertices $\neq (0,0)$ of the triangles lying on a rectangle parallel to the axes since the system is radially symmetric.

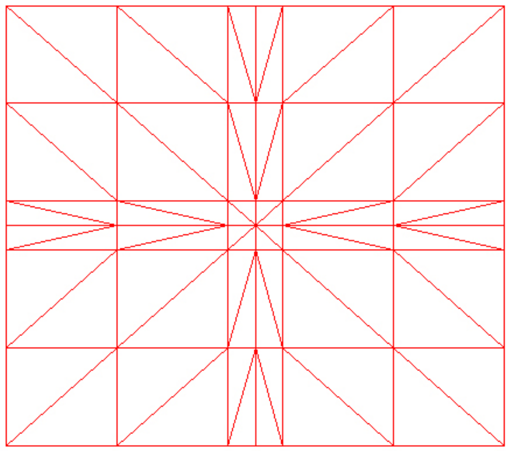


Figure 1: An example of the triangulation scheme proposed in [10]. Note the eight triangles surrounding the origin at the center of the picture. For $\epsilon < \sqrt{2}-1$, there is no continuous Lyapunov function, which is linear on these eight triangles for the system (3.1), no matter how the triangles are scaled (cf. Lemma 3.1).

We choose the vertices of the triangles on the rectangle $(a,b), (-c,b), (-c,-d), (a,-d)$ where $a,b,c,d > 0$. We assume that V is a continuous Lyapunov function which is linear on each triangle. We fix the values of the piecewise linear function V by $V(0,0) = 0$ and

$$V(a,0) = c_1, \qquad V(a,b) = c_2, \qquad V(0,b) = c_3, \qquad V(-c,b) = c_4,$$

$$V(-c,0) = c_5, \qquad V(-c,-d) = c_6, \qquad V(0,-d) = c_7, \qquad V(a,-d) = c_8.$$

Note that we have $c_i > 0$ for all i since V is a Lyapunov function. Due to Corollary 2.4, we calculate the orbital derivative of V on each triangle.

We start with the first triangle with vertices $(0,0)$, $(a,0)$ and (a,b). Points in this triangle fulfill $0 \le x \le a$ and $0 \le \frac{y}{b} \le \frac{x}{a}$. We can write points in this triangle as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} a \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} a \\ b \end{pmatrix} \text{ with } 0 \le \lambda_1,\ 0 \le \lambda_2, \text{ and } \lambda_1 + \lambda_2 \le 1.$$

Then

$$V(x,y) = \lambda_1 V(a,0) + \lambda_2 V(a,b) = c_1 \left(\frac{x}{a} - \frac{y}{b} \right) + c_2 \frac{y}{b}.$$

The orbital derivative of V is given by

$$\begin{aligned} V'(x,y) &= V_x(x,y)\,(-\epsilon x - y) + V_y(x,y)(-\epsilon y + x) \\ &= x \left[-\left(\frac{\epsilon}{a} + \frac{1}{b} \right) c_1 + \frac{1}{b} c_2 \right] + y \left[\left(-\frac{1}{a} + \frac{\epsilon}{b} \right) c_1 - \frac{\epsilon}{b} c_2 \right]. \end{aligned}$$

$V'(x,y) < 0$ holds for all $(x,y) \neq (0,0)$ in the triangle if and only if the equation holds at all points at the boundaries $0 = \frac{y}{b}$ and $\frac{y}{b} = \frac{x}{a}$ other than $(0,0)$ since $V'(x,y)$ is linear:

$$
\begin{aligned}
-\left(\frac{\epsilon}{a} + \frac{1}{b}\right) c_1 + \frac{1}{b} c_2 &< 0 && (3.2)\\
\text{and } -\left(\frac{\epsilon}{a} + \frac{1}{b}\right) c_1 + \frac{1}{b} c_2 + \frac{b}{a}\left[\left(-\frac{1}{a} + \frac{\epsilon}{b}\right) c_1 - \frac{\epsilon}{b} c_2\right] &< 0. && (3.3)
\end{aligned}
$$

We can rewrite (3.2) and (3.3) into

$$
\begin{aligned}
c_1 &> \frac{a}{a + b\epsilon} c_2 && (3.4)\\
\text{and } c_1 &> \frac{a - b\epsilon}{a + b^2/a} c_2. && (3.5)
\end{aligned}
$$

Since $\frac{a}{a+b\epsilon} > \frac{a-b\epsilon}{a+b^2/a}$, (3.4) implies (3.5).

In a similar way we obtain the following condition for the second triangle

$$
c_2 > \frac{a^2 + b^2}{ab\epsilon + b^2} c_3.
$$

For the other six triangles we find similar conditions, which lead to the following condition for $V'(x,y)$ to be negative in all triangles:

$$
\begin{aligned}
c_1 \quad > \quad & \frac{a}{a + b\epsilon} \cdot \frac{a^2 + b^2}{ab\epsilon + b^2} \cdot \frac{b}{b + c\epsilon} \cdot \frac{b^2 + c^2}{bc\epsilon + c^2}\\
& \cdot \frac{c}{c + d\epsilon} \cdot \frac{c^2 + d^2}{cd\epsilon + d^2} \cdot \frac{d}{d + a\epsilon} \cdot \frac{d^2 + a^2}{da\epsilon + a^2} \cdot c_1.
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
F \quad := \quad & \frac{(a + b\epsilon)(a\epsilon + b)}{a^2 + b^2} \cdot \frac{(b + c\epsilon)(b\epsilon + c)}{b^2 + c^2} \cdot \frac{(c + d\epsilon)(c\epsilon + d)}{c^2 + d^2} \cdot \frac{(d + a\epsilon)(d\epsilon + a)}{d^2 + a^2}\\
> \quad & 1. && (3.6)
\end{aligned}
$$

We define $\mu := \frac{1+\epsilon^2}{2\epsilon} \geq 1$, which implies $1 + \epsilon^2 = 2\mu\epsilon$. We consider the term

$$
\begin{aligned}
\frac{(a + b\epsilon)(a\epsilon + b)}{a^2 + b^2} &= \frac{(a^2 + b^2)\epsilon + ab(1 + \epsilon^2)}{a^2 + b^2}\\
&= \epsilon\left(1 + \mu\frac{2ab}{a^2 + b^2}\right)\\
&\leq \epsilon\,(1 + \mu)
\end{aligned}
$$

by the binomial formula; note that equality holds if and only if $a = b$.

By treating the other factors similarly, we obtain

$$
F \leq \epsilon^4 (1 + \mu)^4
$$

with equality if and only if $a = b = c = d$. Now we have

$$\epsilon(1 + \mu) = \epsilon\frac{(1 + \epsilon)^2}{2\epsilon} = \left(\frac{1 + \epsilon}{\sqrt{2}}\right)^2$$

which means

$$F \leq \left(\frac{1 + \epsilon}{\sqrt{2}}\right)^8 \leq 1$$

if $\epsilon \leq \sqrt{2} - 1$. This is a contradiction to (3.6).

Hence, for $\epsilon \leq \sqrt{2} - 1$ the system (3.1) has no piecewise linear Lyapunov function, linear in each of the eight triangles defined above, for any choice of (a, b, c, d). This bound is sharp since the proof shows that the best choice is a square $a = b = c = d$. $\square$

The main idea to overcome this problem is to divide the rectangle into more triangles with zero as a central vertex and to find a piecewise linear Lyapunov function which is linear on each triangle. In the following lemma we show that this is possible for this particular example if we take the unit sphere and divide it into sufficiently many parts. Since this example provides a formula for the number of triangles necessary and sufficient as a function of ϵ, we have an interesting example to be tested with the method from [10], when it has been equipped with a more advanced triangulation scheme.

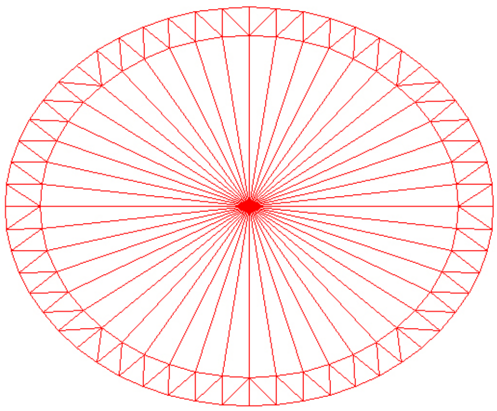


Figure 2: An example of the improved triangulation scheme we propose in two dimensions. Note the triangle-fan at the origin. Additionally, the picture gives an idea of how to extend the triangulation away from the origin. For (3.1) with $\epsilon = 0.1$ and this triangulation in 48 triangles with the origin as a central vertex, there exists a continuous Lyapunov function that is linear on every triangle in the triangle-fan, cf. Lemma 3.2.

Lemma 3.2 *We consider the linear two-dimensional system (3.1). We will investigate the relation between the parameter ϵ and the existence of a continuous Lyapunov function, which is linear on all triangles of the form $(0,0)$, p_l, p_{l+1}, where*

$$p_l := \left(\cos\left(\frac{l}{k}2\pi\right), \sin\left(\frac{l}{k}2\pi\right)\right), \quad l = 0, \ldots, k-1,$$

denotes a point on the unit sphere and $k \in \mathbb{N}$, $k \geq 8$.

Such a piecewise linear Lyapunov function exists if and only if

$$1 < \epsilon \sin\left(\frac{2\pi}{k}\right) + \cos\left(\frac{2\pi}{k}\right).$$

Proof: We consider the triangle with vertices $(0,0)$, p_l and p_{l+1}. We denote $\alpha := \frac{2\pi}{k}$, $\alpha_l := l\alpha = \frac{l}{k}2\pi$ and the values of the Lyapunov function at the vertices by

$$V(p_l) = c_l > 0.$$

For simplicity assume that $\cos\alpha_l > \cos\alpha_{l+1}$ and $\sin\alpha_l < \sin\alpha_{l+1}$, the other cases are similar. Then all points in this triangle fulfill $0 \leq x \leq \cos\alpha_l$ and $x\tan\alpha_l \leq y \leq x\tan\alpha_{l+1}$. We can write points in this triangle as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} \cos\alpha_l \\ \sin\alpha_l \end{pmatrix} + \lambda_2 \begin{pmatrix} \cos\alpha_{l+1} \\ \sin\alpha_{l+1} \end{pmatrix}$$

with $0 \leq \lambda_1$, $0 \leq \lambda_2$, and $\lambda_1 + \lambda_2 \leq 1$. Then

$$\begin{aligned}
\lambda_1 &= \frac{x\sin\alpha_{l+1} - y\cos\alpha_{l+1}}{\sin\alpha}, \\
\lambda_2 &= \frac{y\cos\alpha_l - x\sin\alpha_l}{\sin\alpha} \quad \text{and} \\
V(x,y) &= \lambda_1 V(p_l) + \lambda_2 V(p_{l+1}) \\
&= c_l \frac{x\sin\alpha_{l+1} - y\cos\alpha_{l+1}}{\sin\alpha} + c_{l+1}\frac{y\cos\alpha_l - x\sin\alpha_l}{\sin\alpha}.
\end{aligned}$$

The orbital derivative of V is given by

$$\begin{aligned}
V'(x,y) &= V_x(x,y)(-\epsilon x - y) + V_y(x,y)(-\epsilon y + x) \\
&= \frac{x}{\sin\alpha}[(-\cos\alpha_{l+1} - \epsilon\sin\alpha_{l+1})c_l + (\epsilon\sin\alpha_l + \cos\alpha_l)c_{l+1}] \\
&\quad + \frac{y}{\sin\alpha}[(\epsilon\cos\alpha_{l+1} - \sin\alpha_{l+1})c_l + (\sin\alpha_l - \epsilon\cos\alpha_l)c_{l+1}].
\end{aligned}$$

$V'(x,y) < 0$ holds for all $(x,y) \neq (0,0)$ in the triangle if and only if the condition holds at all points at both boundaries, i.e. $y = x\tan\alpha_{l+1}$ and $y = x\tan\alpha_l$, other than $(0,0)$. Hence,

$$\begin{aligned}
0 &> c_l\left(-\cos\alpha_{l+1}-\epsilon\sin\alpha_{l+1}+\epsilon\sin\alpha_{l+1}-\frac{\sin^2\alpha_{l+1}}{\cos\alpha_{l+1}}\right)\\
&\quad +c_{l+1}\left(\epsilon\sin\alpha_l+\cos\alpha_l+\frac{\sin\alpha_l\sin\alpha_{l+1}}{\cos\alpha_{l+1}}-\epsilon\frac{\cos\alpha_l\sin\alpha_{l+1}}{\cos\alpha_{l+1}}\right) \quad \text{and}\\
0 &> c_l\left(-\cos\alpha_{l+1}-\epsilon\sin\alpha_{l+1}+\epsilon\frac{\cos\alpha_{l+1}\sin\alpha_l}{\cos\alpha_l}-\frac{\sin\alpha_{l+1}\sin\alpha_l}{\cos\alpha_l}\right)\\
&\quad +c_{l+1}\left(\epsilon\sin\alpha_l+\cos\alpha_l+\frac{\sin^2\alpha_l}{\cos\alpha_l}-\epsilon\sin\alpha_l\right),
\end{aligned}$$

which simplifies to

$$c_l > (\cos\alpha-\epsilon\sin\alpha)c_{l+1} \quad \text{and} \tag{3.7}$$

$$c_l > \frac{1}{\cos\alpha+\epsilon\sin\alpha}c_{l+1} \tag{3.8}$$

Since $\cos^2\alpha-\epsilon\sin^2\alpha \le \cos^2\alpha < 1$ because of $k \ge 8$, the inequality (3.8) implies (3.7). Define $C := \cos\alpha+\epsilon\sin\alpha$.

This leads to the sequence of inequalities

$$\begin{aligned}
c_0 &> \frac{1}{C}c_1 > \frac{1}{C^2}c_2 > \frac{1}{C^3}c_3 > \ldots > \frac{1}{C^{k-1}}c_{k-1}\\
&> \frac{1}{C^k}c_0.
\end{aligned}$$

Hence, we get the following condition:

$$C = \cos\alpha+\epsilon\sin\alpha > 1$$

which shows the lemma. □

For example, if $\alpha = \frac{2\pi}{8}$, then the condition becomes

$$\frac{1}{\sqrt{2}}+\epsilon\frac{1}{\sqrt{2}} > 1 \Leftrightarrow \epsilon > \sqrt{2}-1.$$

Thus, for eight triangles there exists a piecewise linear Lyapunov function with vertices on the unit circle for the same values of ϵ as for vertices on a rectangle, cf. Lemma 3.1.

4 Main result

In this section we show the main result, which states that a piecewise linear Lyapunov function exists for any two-dimensional ODE with an exponentially stable equilibrium.

First, in Proposition 4.1, we prove that for any system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, with an exponentially stable equilibrium at $\mathbf{x}_0 = \mathbf{0}$, there exists an open neighborhood $\mathcal{N}$ of $\mathbf{0}$ and a Lyapunov function $v \in C^\infty(\mathcal{N} \setminus \{\mathbf{0}\}, \mathbb{R}) \cap C^0(\mathcal{N}, \mathbb{R})$ fulfilling

$$a\|\mathbf{x}\|_2 \leq v(\mathbf{x}) \leq b\|\mathbf{x}\|_2 \quad \text{and} \quad \nabla v(\mathbf{x}) \cdot f(\mathbf{x}) \leq -c\|\mathbf{x}\|_2, \quad \mathbf{x} \neq \mathbf{0}, \tag{4.1}$$

where a, b, and c are some positive constants. In the standard theory of Lyapunov functions for exponentially stable systems one often sees the defining inequalities for Lyapunov functions $V \in C^\infty(\mathcal{N}, \mathbb{R})$ as

$$a\|\mathbf{x}\|_2^2 \leq V(\mathbf{x}) \leq b\|\mathbf{x}\|_2^2 \quad \text{and} \quad \nabla V(\mathbf{x}) \cdot f(\mathbf{x}) \leq -c\|\mathbf{x}\|_2^2. \tag{4.2}$$

Corollary 4.2 tells us that the conditions (4.1) and (4.2) are equivalent in the sense, that both are sufficient and necessary if the equilibrium is exponentially stable.

In Theorem 4.3, we use the Lyapunov function v from Proposition 4.1 to construct a piecewise linear Lyapunov function w. Our construction only works for $n = 2$. We do this by triangulating an area around the origin with a triangle-fan with the origin as a central vertex. Then we fix the values of w at the vertices of these triangles by setting $w(\mathbf{x}) = v(\mathbf{x})$ and show that the function w, continuous and linear on each of the triangles, is a Lyapunov function for the system.

Proposition 4.1 *Consider $\dot{\mathbf{x}} = f(\mathbf{x})$, where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and assume that $\mathbf{x}_0 = \mathbf{0}$ is an exponentially stable equilibrium. Then there is a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and a number $r > 0$, such that the function $v \in C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R}) \cap C^0(\mathbb{R}^n, \mathbb{R})$, given by*

$$v(\mathbf{x}) \;:=\; r\left\|Q^{\frac{1}{2}}\mathbf{x}\right\|_2, \tag{4.3}$$

satisfies

$$\begin{aligned} a\|\mathbf{x}\|_2 \;&\leq\; v(\mathbf{x}) \;\leq\; b\|\mathbf{x}\|_2 \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n, \textit{ and} \\ v'(\mathbf{x}) \;&=\; \nabla v(\mathbf{x}) \cdot f(\mathbf{x}) \;\leq\; -2c\|\mathbf{x}\|_2 \quad \textit{for all } \mathbf{x} \in E_2 \setminus \{\mathbf{0}\}, \end{aligned}$$

where

$$a := \frac{r}{\left\|Q^{-\frac{1}{2}}\right\|_2}, \quad b := r\|Q^{\frac{1}{2}}\|_2, \quad c := \frac{r}{8\left\|Q^{\frac{1}{2}}\right\|_2}, \quad \textit{and } E_2 := \left\{\mathbf{x} \in \mathbb{R}^n \mid \|Q^{\frac{1}{2}}\mathbf{x}\|_2 \leq r\right\}.$$

PROOF: We consider the linearised system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = Df(\mathbf{0})$ and $f(\mathbf{x}) = A\mathbf{x} + \psi(\mathbf{x})$ with $\lim_{\|\mathbf{x}\|_2 \to 0} \frac{\|\psi(\mathbf{x})\|_2}{\|\mathbf{x}\|_2} = 0$. Let Q be the positive definite solution to the equation

$$A^T Q + QA = -I. \tag{4.4}$$

Since Q is positive definite, the square-root $Q^{\frac{1}{2}}$ is well defined and itself symmetric and positive definite. We choose $r > 0$ so small that

$$\|\psi(\mathbf{x})\|_2 \quad \leq \quad \frac{1}{4\|Q\|_2}\|\mathbf{x}\|_2 \text{ holds for all } \mathbf{x} \text{ with } \|Q^{\frac{1}{2}}\mathbf{x}\|_2 \leq r. \tag{4.5}$$

The first inequality follows from

$$\begin{aligned}
v(\mathbf{x}) &= r\|Q^{\frac{1}{2}}\mathbf{x}\|_2 \leq r\|Q^{\frac{1}{2}}\|_2\|\mathbf{x}\|_2 = b\|\mathbf{x}\|_2 \text{ and because} \\
\|\mathbf{x}\|_2 &= \|Q^{-\frac{1}{2}}Q^{\frac{1}{2}}\mathbf{x}\|_2 \leq \|Q^{-\frac{1}{2}}\|_2 \cdot \|Q^{\frac{1}{2}}\mathbf{x}\|_2 \text{ we have} \\
v(\mathbf{x}) &= r\left\|Q^{\frac{1}{2}}\mathbf{x}\right\|_2 \geq \frac{r}{\left\|Q^{-\frac{1}{2}}\right\|_2}\|\mathbf{x}\|_2 \geq a\|\mathbf{x}\|_2.
\end{aligned}$$

We now show the second inequality: let $\mathbf{x} \in E_2 \setminus \{\mathbf{0}\}$, i.e. $0 < \|Q^{\frac{1}{2}}\mathbf{x}\|_2 \leq r$. We have

$$\begin{aligned}
\nabla v(\mathbf{x}) \cdot f(\mathbf{x}) &= \frac{r}{2\left\|Q^{\frac{1}{2}}\mathbf{x}\right\|_2}\left[\mathbf{x}^T Q f(\mathbf{x}) + f(\mathbf{x})^T Q\mathbf{x}\right] \\
&= \frac{r}{2\left\|Q^{\frac{1}{2}}\mathbf{x}\right\|_2}\left[\mathbf{x}^T QA\mathbf{x} + \mathbf{x}^T Q\psi(\mathbf{x}) + \mathbf{x}^T A^T Q\mathbf{x} + \psi(\mathbf{x})^T Q\mathbf{x}\right] \\
&\leq \frac{r}{2\left\|Q^{\frac{1}{2}}\mathbf{x}\right\|_2}\left(-\|\mathbf{x}\|_2^2 + 2\|\mathbf{x}\|_2 \cdot \|Q\|_2 \cdot \|\psi(\mathbf{x})\|_2\right) \text{ by (4.4)} \\
&\leq -\frac{r}{4\left\|Q^{\frac{1}{2}}\mathbf{x}\right\|_2}\|\mathbf{x}\|_2^2 \text{ by (4.5)} \\
&\leq -\frac{r}{4\left\|Q^{\frac{1}{2}}\right\|_2}\|\mathbf{x}\|_2 \text{ since } \|Q^{\frac{1}{2}}\mathbf{x}\|_2 \leq \|Q^{\frac{1}{2}}\|_2 \cdot \|\mathbf{x}\|_2.
\end{aligned}$$

Hence,

$$\nabla v(\mathbf{x}) \cdot f(\mathbf{x}) \leq -2c\|\mathbf{x}\|_2.$$

□

It is well known that the exponential stability of $\mathbf{x}_0 = \mathbf{0}$ for the system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, is equivalent to the existence of a Lyapunov function fulfilling the conditions in (4.2), cf. [8, Theorem 4.10 and 4.14]. In the next corollary we show that one can just as well use the conditions in (4.1). In this case one sacrifices the smoothness of the Lyapunov function at the origin for the prize of a Lyapunov function much better suited for the construction of a piecewise affine Lyapunov function.

Corollary 4.2 *Consider the system $\dot{\mathbf{x}} = f(\mathbf{x})$, with $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Then $\mathbf{x}_0 = \mathbf{0}$ is an exponentially stable equilibrium of the system, if and only if there exists a neighborhood $\mathcal{N}$ of $\mathbf{0}$ and a function $v \in C^\infty(\mathcal{N} \setminus \{\mathbf{0}\}, \mathbb{R}) \cap C^0(\mathcal{N}, \mathbb{R})$ fulfilling*

$$a\|\mathbf{x}\|_2 \leq v(\mathbf{x}) \leq b\|\mathbf{x}\|_2 \quad \text{and} \quad \nabla v(\mathbf{x}) \cdot f(\mathbf{x}) \leq -c\|\mathbf{x}\|_2 \quad \text{for all} \quad \mathbf{x} \in \mathcal{N} \setminus \{\mathbf{0}\}$$

for some positive constants a, b, and c.

PROOF: The "only if" part follows by Proposition 4.1. The "if" part can be proved by mimicing the proof of [8, Theorem 4.10]. $\square$

The following theorem shows that locally there exists a piecewise linear Lyapunov function w for any two-dimensional system with an exponentially stable equilibrium. In the proof, the system $\dot{\mathbf{x}} = f(\mathbf{x})$ in $\mathbf{x}$ is transformed to a system in $\mathbf{y} = Q^{\frac{1}{2}}\mathbf{x}$ and the vertices $\mathbf{z}_i$ of the triangulation in the $\mathbf{x}$-space, which lie on a rectangle, are transformed to points $\mathbf{y}_i$ on a sphere with radius r. Note that this result uses that the phase space is two-dimensional, in particular in Step 5, where we express the vertices $\mathbf{y}_i$ on the sphere by polar coordinates. The main step is the estimate (4.29), where the difference between the gradient of the piecewise linear Lyapunov function (w_1, w_2) and the gradient of the quadratic Lyapunov function in the $\mathbf{y}$-space $\mathbf{y}_i$ is expressed by $r\boldsymbol{\xi}$, locally on any triangle. We can derive the estimate $\|\boldsymbol{\xi}\|_2 \leq \alpha C_0$ on the difference vector in (4.30), where α is an upper bound on the angles between the vertices $\mathbf{y}_i$ and $\mathbf{y}_{i+1}$. Although large parts of the proof work also in higher dimensions, it is not immediately clear how to generalise this crucial step to higher dimensions.

Theorem 4.3 *Consider $\dot{\mathbf{x}} = f(\mathbf{x})$, $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, and assume that $\mathbf{x}_0 = \mathbf{0}$ is an exponentially stable equilibrium. Let $v(\mathbf{x}) := r\left\|Q^{\frac{1}{2}}\mathbf{x}\right\|_2$ be the Lyapunov function from Proposition 4.1 that can be constructed for the system. Then for any rectangle $R = \{(x, y) \in \mathbb{R}^2 \mid -a \leq x \leq a, -b \leq y \leq b\}$ with $a, b > 0$ small enough and any triangulation*

$$\mathcal{T} = \{\mathrm{co}\{\mathbf{z}_i, \mathbf{z}_{i+1}, \mathbf{0}\}\big|\mathbf{z}_i, \mathbf{z}_{i+1} \in \partial R,\ i = 1, 2, \ldots, N\}$$

of R fine enough, i.e. $\|\mathbf{z}_i - \mathbf{z}_{i+1}\|_2$ small, the function $w : R \longrightarrow \mathbb{R}$, defined by

- *$w(\mathbf{0}) = v(\mathbf{0}) = 0$ and $w(\mathbf{z}_i) = v(\mathbf{z}_i)$ for $i = 1, 2, \ldots, N$ and*
- *w is linear on every triangle $\mathrm{co}\{\mathbf{z}_i, \mathbf{z}_{i+1}, \mathbf{0}\}$ for $i = 1, 2, \ldots, N$,*

is a continuous Lyapunov function for the system. In particular, we have

$$w(\mathbf{x}) \geq C\|\mathbf{x}\|_2 \text{ for all } \mathbf{x} \in \mathbb{R}^2 \text{ and } D^+[w(\mathbf{x})] \leq -c\|\mathbf{x}\|_2 \text{ for all } \mathbf{x} \in R \setminus \{\mathbf{0}\}$$

with positive constants c and C.

PROOF: **Step 1: Transformation to the y-system**

As in the proof of Proposition 4.1, we consider the linearised system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = Df(\mathbf{0})$ and $f(\mathbf{x}) = A\mathbf{x} + \psi(\mathbf{x})$ with $\lim_{\|\mathbf{x}\|_2 \to 0} \frac{\|\psi(\mathbf{x})\|_2}{\|\mathbf{x}\|_2} = 0$. We define Q as the positive definite symmetric matrix solution of the equation

$$A^T Q + QA = -I. \tag{4.6}$$

Since Q is positive definite, the square-root $Q^{\frac{1}{2}}$ is well defined and itself symmetric and positive definite. We use the transformation $\mathbf{y} = Q^{\frac{1}{2}}\mathbf{x}$. Then the differential equation $\dot{\mathbf{x}} = f(\mathbf{x})$ becomes

$$\begin{aligned} \dot{\mathbf{y}} &= Q^{\frac{1}{2}} f(Q^{-\frac{1}{2}}\mathbf{y}) =: g(\mathbf{y}) \\ &= Q^{\frac{1}{2}} A Q^{-\frac{1}{2}}\mathbf{y} + \underbrace{Q^{\frac{1}{2}}\psi(Q^{-\frac{1}{2}}\mathbf{y})}_{=:\varphi(\mathbf{y})}. \end{aligned} \tag{4.7}$$

The function $\varphi(\mathbf{y})$ is the nonlinear part of $g(\mathbf{y})$, i.e. $g(\mathbf{y}) = Q^{\frac{1}{2}}AQ^{-\frac{1}{2}}\mathbf{y} + \varphi(\mathbf{y})$ with $\lim_{\|\mathbf{y}\|_2\to 0} \frac{\|\varphi(\mathbf{y})\|_2}{\|\mathbf{y}\|_2} = 0$. We also consider the linearised equation

$$\dot{\mathbf{y}} = Q^{\frac{1}{2}}AQ^{-\frac{1}{2}}\mathbf{y}. \tag{4.8}$$

Step 2: Definitions and constants

Define the constants

$$C_0 := \frac{4}{7}\left(\sqrt{2} + \frac{1}{2}\right), \tag{4.9}$$

$$\alpha := \frac{1}{2}\min\left(\frac{1}{2C_0\|Q^{\frac{1}{2}}\|_2^2\|A\|_2}, 1\right) \in \left(0, \frac{1}{2}\right], \tag{4.10}$$

$$\nu := \frac{1}{8\left\|Q^{\frac{1}{2}}\right\|_2 (1+\alpha C_0)}, \tag{4.11}$$

$$\rho := \sqrt{2(1-\cos\alpha)} > 0. \tag{4.12}$$

Since $\lim_{\|\mathbf{y}\|_2\to 0} \frac{\|\varphi(\mathbf{y})\|_2}{\|\mathbf{y}\|_2} = 0$, we can choose $r > 0$ so small that, additionally to (4.5) in Proposition 4.1, we have

$$r \leq \frac{1}{\left\|Q^{-\frac{1}{2}}\right\|_2}, \quad \text{and} \tag{4.13}$$

$$\|\varphi(\mathbf{y})\|_2 \leq \nu\|Q^{-\frac{1}{2}}\mathbf{y}\|_2 \text{ holds for all } \mathbf{y} \text{ with } \|\mathbf{y}\|_2 \leq r. \tag{4.14}$$

Note that with (4.11) and $c := \frac{r}{8\left\|Q^{\frac{1}{2}}\right\|_2}$ as in Proposition 4.1 we have

$$\nu = \frac{c}{r(1+\alpha C_0)}. \tag{4.15}$$

Choose a and b in the definition of the rectangle R so small that

$$\left\|Q^{\frac{1}{2}}\begin{pmatrix} a \\ 0 \end{pmatrix}\right\|_2 + \left\|Q^{\frac{1}{2}}\begin{pmatrix} 0 \\ b \end{pmatrix}\right\|_2 \leq \frac{r}{2}.$$

This implies $R \subset E_1 := \left\{\mathbf{x} \in \mathbb{R}^2 \mid \|Q^{\frac{1}{2}}\mathbf{x}\|_2 \leq \frac{r}{2}\right\}$, since for $\mathbf{x} \in R$ we have $\mathbf{x} = \lambda_1 \begin{pmatrix} a \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ b \end{pmatrix}$ with $\lambda_1, \lambda_2 \in [-1, 1]$ and thus

$$\left\|Q^{\frac{1}{2}}\mathbf{x}\right\|_2 \leq |\lambda_1| \left\|Q^{\frac{1}{2}} \begin{pmatrix} a \\ 0 \end{pmatrix}\right\|_2 + |\lambda_2| \left\|Q^{\frac{1}{2}} \begin{pmatrix} 0 \\ b \end{pmatrix}\right\|_2 \leq \frac{r}{2}.$$

Using the function v of Proposition 4.1, we also define the function

$$\tilde{v}(\mathbf{y}) \quad := \quad v(Q^{-\frac{1}{2}}\mathbf{y}) = r\|\mathbf{y}\|_2. \tag{4.16}$$

Choose the points $\mathbf{z}_i$ on ∂R such that

$$\mathcal{T} = \{\text{co}\{\mathbf{z}_i, \mathbf{z}_{i+1}, \mathbf{0}\} \big| \mathbf{z}_i, \mathbf{z}_{i+1} \in \partial R,\ i = 1, 2, \ldots, N\}$$

is a triangulation of R and

$$\|\mathbf{z}_{i+1} - \mathbf{z}_i\|_2 \quad \leq \quad \rho \min(a, b) \frac{1}{2\|Q^{\frac{1}{2}}\| \cdot \|Q^{-\frac{1}{2}}\|} \tag{4.17}$$

holds for all i. In particular, we have $\mathbf{z}_{N+1} = \mathbf{z}_1$ and the four corner points of R are vertices in the triangulation. Define the ellipse $E_2 := \left\{\mathbf{x} \in \mathbb{R}^2 \mid \|Q^{\frac{1}{2}}\mathbf{x}\|_2 \leq r\right\}$ and define by $\mathbf{x}_i$ the projections of $\mathbf{z}_i$ on ∂E_2, i.e.

$$\mathbf{x}_i \quad := \quad \frac{r}{\left\|Q^{\frac{1}{2}}\mathbf{z}_i\right\|_2} \mathbf{z}_i \tag{4.18}$$

$$\text{and let } \mathbf{y}_i \quad := \quad Q^{\frac{1}{2}}\mathbf{x}_i. \tag{4.19}$$

Then $\|\mathbf{y}_i\|_2 = r$, i.e. the points $\mathbf{y}_i$ lie on a circle of radius r. Hence, we can write

$$\mathbf{y}_i \quad = \quad r \begin{pmatrix} \cos(\alpha_i) \\ \sin(\alpha_i) \end{pmatrix}. \tag{4.20}$$

Step 3: α is an upper bound on the angles between $\mathbf{y}_i$ and $\mathbf{y}_{i+1}$

We show that $|\alpha_{i+1} - \alpha_i| \leq \alpha$, where α_i and α_{i+1} are defined by (4.20) up to a multiple of 2π. Then we can, if needed, change the order of the $\mathbf{y}_i$ so that $0 \leq \alpha_{i+1} - \alpha_i \leq \alpha$. Indeed, using equation (4.21) which we show below,

$$\begin{aligned} 2r^2(1 - \cos(|\alpha_{i+1} - \alpha_i|)) &= \|\mathbf{y}_{i+1} - \mathbf{y}_i\|_2^2 \\ &= r^2 \left\| \frac{Q^{\frac{1}{2}}\mathbf{z}_{i+1}}{\left\|Q^{\frac{1}{2}}\mathbf{z}_{i+1}\right\|_2} - \frac{Q^{\frac{1}{2}}\mathbf{z}_i}{\left\|Q^{\frac{1}{2}}\mathbf{z}_i\right\|_2} \right\|_2^2 \text{ by (4.18) and (4.19)} \\ &\leq r^2\rho^2 \text{ by (4.21)} \\ &= 2r^2(1 - \cos\alpha) \text{ by (4.12).} \end{aligned}$$

This implies $\cos\alpha \le \cos(\alpha_{i+1}-\alpha_i)$ and thus $|\alpha_{i+1}-\alpha_i| \le \alpha$.

We show (4.21): Note that $\mathbf{z}_i + \lambda(\mathbf{z}_{i+1}-\mathbf{z}_i)$ lies on the boundary of the rectangle for all i and $\lambda \in [0,1]$, since the corners are points $\mathbf{z}_i$. We use the mean value theorem for $g(\mathbf{z}) = \frac{Q^{\frac{1}{2}}\mathbf{z}}{\left\|Q^{\frac{1}{2}}\mathbf{z}\right\|_2}$, where $Dg(\mathbf{z}) = \frac{1}{\left\|Q^{\frac{1}{2}}\mathbf{z}\right\|_2} Q^{\frac{1}{2}} - \frac{1}{\left\|Q^{\frac{1}{2}}\mathbf{z}\right\|_2^3} Q^{\frac{1}{2}}\mathbf{z}(Q^{\frac{1}{2}}\mathbf{z})^T Q^{\frac{1}{2}}$

$$\begin{aligned}
\left\| \frac{Q^{\frac{1}{2}}\mathbf{z}_{i+1}}{\left\|Q^{\frac{1}{2}}\mathbf{z}_{i+1}\right\|_2} - \frac{Q^{\frac{1}{2}}\mathbf{z}_i}{\left\|Q^{\frac{1}{2}}\mathbf{z}_i\right\|_2} \right\|_2 &\le \int_0^1 \|Dg(\mathbf{z}_i + \lambda(\mathbf{z}_{i+1}-\mathbf{z}_i))\|\, d\lambda \cdot \|\mathbf{z}_{i+1}-\mathbf{z}_i\|_2 \\
&\le 2\max_{\mathbf{z}\in\partial R} \frac{1}{\left\|Q^{\frac{1}{2}}\mathbf{z}\right\|_2} \|Q^{\frac{1}{2}}\|_2 \cdot \|\mathbf{z}_{i+1}-\mathbf{z}_i\|_2.
\end{aligned}$$

Using $\|\mathbf{z}\|_2 = \left\|Q^{-\frac{1}{2}}Q^{\frac{1}{2}}\mathbf{z}\right\|_2 \le \left\|Q^{-\frac{1}{2}}\right\|_2 \cdot \left\|Q^{\frac{1}{2}}\mathbf{z}\right\|_2$ we conclude with (4.17)

$$\begin{aligned}
&\left\| \frac{Q^{\frac{1}{2}}\mathbf{z}_{i+1}}{\left\|Q^{\frac{1}{2}}\mathbf{z}_{i+1}\right\|_2} - \frac{Q^{\frac{1}{2}}\mathbf{z}_i}{\left\|Q^{\frac{1}{2}}\mathbf{z}_i\right\|_2} \right\|_2 \\
&\qquad\le 2\max_{\mathbf{z}\in\partial R} \frac{\|Q^{-\frac{1}{2}}\|_2}{\|\mathbf{z}\|_2} \|Q^{\frac{1}{2}}\|_2 \cdot \rho\min(a,b) \frac{1}{2\|Q^{\frac{1}{2}}\|_2 \cdot \|Q^{-\frac{1}{2}}\|_2} \\
&\qquad\le \rho. \qquad (4.21)
\end{aligned}$$

Step 4: equivalent definitions of $w(\mathbf{x})$ and $\tilde{w}(\mathbf{y})$

We define the continuous function w by

$$w(\mathbf{0}) = 0, \quad w(\mathbf{z}_i) := v(\mathbf{z}_i) = r\|Q^{\frac{1}{2}}\mathbf{z}_i\|_2 \text{ for every vertex } \mathbf{z}_i,$$

and w is linear on every triangle $\mathrm{co}\{\mathbf{z}_i, \mathbf{z}_{i+1}, \mathbf{0}\}$ and beyond on the whole sector

$$\mathrm{sect}\{\mathbf{z}_i, \mathbf{z}_{i+1}, \mathbf{0}\} := \{\mathbf{y} = \lambda_1\mathbf{z}_i + \lambda_2\mathbf{z}_{i+1}, \quad \lambda_1, \lambda_2 \ge 0\}.$$

Another way to define w is by specifying its values at the $\mathbf{x}_i$ instead of the $\mathbf{z}_i$.

$$w(\mathbf{0}) = 0, \quad w(\mathbf{x}_i) \;:=\; v(\mathbf{x}_i) = r\|Q^{\frac{1}{2}}\mathbf{x}_i\|_2 \text{ for every vertex } \mathbf{x}_i, \qquad (4.22)$$

and w is linear on every sector $\mathrm{sect}\{\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{0}\}$. Indeed, define w by the values on $\mathbf{z}_i$ and let $\mathbf{x}_i = \frac{r}{\left\|Q^{\frac{1}{2}}\mathbf{z}_i\right\|_2}\mathbf{z}_i$, cf. (4.18). Then, since the piecewise linear function w satisfies

$$w(\mathbf{x}_i) = \frac{r}{\left\|Q^{\frac{1}{2}}\mathbf{z}_i\right\|_2} w(\mathbf{z}_i) = \frac{r}{\left\|Q^{\frac{1}{2}}\mathbf{z}_i\right\|_2} r\|Q^{\frac{1}{2}}\mathbf{z}_i\|_2 = r\|Q^{\frac{1}{2}}\mathbf{x}_i\|_2.$$

The other direction is shown in a similar way.

Moreover, we define the continuous, piecewise linear function $\tilde{w}$ by

$$\tilde{w}(\mathbf{y}) := w(Q^{-\frac{1}{2}}\mathbf{y}). \qquad (4.23)$$

Note that we can equivalently define the continuous function $\tilde{w}$ by its values on the vertices $\mathbf{y}_i$:

$$\tilde{w}(\mathbf{0}) = 0, \quad \tilde{w}(\mathbf{y}_i) \;:=\; \tilde{v}(\mathbf{y}_i) = r\|\mathbf{y}_i\|_2 \text{ for every vertex } \mathbf{y}_i, \tag{4.24}$$

cf. (4.16), and linear on each sector $\mathrm{sect}\{\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{0}\}$. We show that the definition (4.24) gives the function in (4.23). Indeed, for the point $\mathbf{y}_i$ we have

$$\begin{aligned} w(Q^{-\frac{1}{2}}\mathbf{y}_i) &= w(\mathbf{x}_i) \text{ by (4.19)} \\ &= r\|Q^{\frac{1}{2}}\mathbf{x}_i\|_2 \text{ by (4.22)} \\ &= r\|\mathbf{y}_i\|_2. \end{aligned}$$

Now let $\mathbf{y} = \sum_{j=0}^{1} \lambda_j \mathbf{y}_{i+j}$ with $0 \le \lambda_j$. Then the piecewise linear function $\tilde{w}$ defined by the values on $\mathbf{y}_i$, cf. (4.24), satisfies

$$\begin{aligned} \tilde{w}(\mathbf{y}) &= \sum_{j=0}^{1} \lambda_j \tilde{w}(\mathbf{y}_{i+j}) \\ &= \sum_{j=0}^{1} \lambda_j r^2 \\ &= \sum_{j=0}^{1} \lambda_j r \|\mathbf{y}_{i+j}\|_2 \;=\; \sum_{j=0}^{1} \lambda_j w(Q^{-\frac{1}{2}}\mathbf{y}_{i+j}) \text{ as shown above} \\ &= \sum_{j=0}^{1} \lambda_j w(\mathbf{x}_{i+j}) \;=\; w\left(\sum_{j=0}^{1} \lambda_j \mathbf{x}_{i+j}\right) \;=\; w\left(Q^{-\frac{1}{2}} \sum_{j=0}^{1} \lambda_j \mathbf{y}_{i+j}\right) \\ &= w(Q^{-\frac{1}{2}}\mathbf{y}) \end{aligned} \tag{4.25}$$

as in the first definition (4.23).

If $\tilde{w}$ is defined by (4.23), then the values on $\mathbf{y}_i$ are given by the formula (4.24). Since w is continuous and linear on the sector given by $\mathrm{sect}\{\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{0}\}$, $\tilde{w}$ is continuous and linear on each sector given by $\mathrm{sect}\{Q^{\frac{1}{2}}\mathbf{x}_i, Q^{\frac{1}{2}}\mathbf{x}_{i+1}, \mathbf{0}\} = \mathrm{sect}\{\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{0}\}$. Hence, it coincides with the function defined by (4.24).

Now we show the first inequality of the theorem: Let $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ since for $\mathbf{x} = \mathbf{0}$ the inequality holds trivially. Then there is an i with $\mathbf{x} \in \mathrm{sect}\{\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{0}\}$, i.e. there are $\lambda_0, \lambda_1 \ge 0$ with $\mathbf{x} = \sum_{j=0}^{1} \lambda_j \mathbf{x}_{i+j}$. Since $\mathbf{y} = Q^{\frac{1}{2}}\mathbf{x} = \sum_{j=0}^{1} \lambda_j Q^{\frac{1}{2}}\mathbf{x}_{i+j} = \sum_{j=0}^{1} \lambda_j \mathbf{y}_{i+j}$, we have with (4.25) and $\|\mathbf{x}_{i+j}\|_2 = \|Q^{-\frac{1}{2}}\mathbf{y}_{i+j}\|_2 \le \|Q^{-\frac{1}{2}}\|_2 \cdot r$:

$$\begin{aligned} w(\mathbf{x}) &= w(Q^{-\frac{1}{2}}\mathbf{y}) \;=\; \sum_{j=0}^{1} \lambda_j r^2 \text{ by (4.25)} \\ &\ge \frac{r}{\|Q^{-\frac{1}{2}}\|_2} \sum_{j=0}^{1} \lambda_j \|\mathbf{x}_{i+j}\|_2 \;\ge\; \frac{r}{\|Q^{-\frac{1}{2}}\|_2} \left\| \sum_{j=0}^{1} \lambda_j \mathbf{x}_{i+j} \right\|_2 \\ &= \frac{r}{\|Q^{-\frac{1}{2}}\|_2} \|\mathbf{x}\|_2 \end{aligned}$$

which shows the first inequality of the theorem with $C := \frac{r}{\left\|Q^{-\frac{1}{2}}\right\|_2}$.

Step 5: $\tilde{w}(\mathbf{y})$ has negative orbital derivative at the vertices $\mathbf{y}_i$

Consider two adjacent vertices

$$\mathbf{y}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} = r \begin{pmatrix} \cos(\alpha_i) \\ \sin(\alpha_i) \end{pmatrix} \text{ and} \tag{4.26}$$

$$\mathbf{y}_{i+1} = \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = r \begin{pmatrix} \cos(\alpha_{i+1}) \\ \sin(\alpha_{i+1}) \end{pmatrix}. \tag{4.27}$$

On each triangle $T_i = \operatorname{co}\left\{\begin{pmatrix} x_i \\ y_i \end{pmatrix}, \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$ we have defined the piecewise linear function $\tilde{w}_i(\mathbf{y}) = (w_1, w_2)\mathbf{y}$ by $\tilde{v}(\mathbf{y}) = r\|\mathbf{y}_i\|_2 = r^2$ on the vertices, cf. (4.24). We fix an i and write $\tilde{w}(\mathbf{y})$.

We seek to show that $\tilde{w}(\mathbf{y})$ is a Lyapunov function for (4.8). We will first check that the orbital derivative with respect to the linearised system (4.8), namely $\nabla_{\mathbf{y}}\tilde{w}(\mathbf{y})^T Q^{\frac{1}{2}} A Q^{-\frac{1}{2}}\mathbf{y}$ is negative at the vertices $\mathbf{y_i} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ and $\mathbf{y_{i+1}} = \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix}$. We deal with the other points of the triangle in Step 6.

First we determine (w_1, w_2). Since $\tilde{w}(\mathbf{y}) = (w_1, w_2)\,\mathbf{y}$, we have $(w_1, w_2)\begin{pmatrix} x_i \\ y_i \end{pmatrix} = r^2$ and $(w_1, w_2)\begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = r^2$. This leads to the system of linear equations

$$(w_1, w_2)\begin{pmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{pmatrix} = \left(r^2, r^2\right).$$

The solution is

$$(w_1, w_2) = r^2\,(1,1)\begin{pmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{pmatrix}^{-1}.$$

Using the special form (4.26) and (4.27) we obtain

$$\begin{aligned}
&\begin{pmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{pmatrix}^{-1} \\
&= \frac{1}{x_i y_{i+1} - x_{i+1} y_i}\begin{pmatrix} y_{i+1} & -x_{i+1} \\ -y_i & x_i \end{pmatrix} \\
&= \frac{1}{r}\frac{1}{\cos(\alpha_i)\sin(\alpha_{i+1}) - \cos(\alpha_{i+1})\sin(\alpha_i)}\begin{pmatrix} \sin(\alpha_{i+1}) & -\cos(\alpha_{i+1}) \\ -\sin(\alpha_i) & \cos(\alpha_i) \end{pmatrix} \\
&= \frac{1}{r}\frac{1}{\sin(\alpha_{i+1} - \alpha_i)}\begin{pmatrix} \sin(\alpha_{i+1}) & -\cos(\alpha_{i+1}) \\ -\sin(\alpha_i) & \cos(\alpha_i) \end{pmatrix}.
\end{aligned}$$

Hence, we obtain

$$(w_1, w_2) = \frac{r}{\sin(\alpha_{i+1} - \alpha_i)}\left(\sin(\alpha_{i+1}) - \sin(\alpha_i), -\cos(\alpha_{i+1}) + \cos(\alpha_i)\right).$$

Now we use Taylor's Theorem to obtain

$$\begin{aligned}\sin(\alpha_{i+1})-\sin(\alpha_i) &= [\cos(\alpha_i)+\epsilon_1](\alpha_{i+1}-\alpha_i)\\ -\cos(\alpha_{i+1})+\cos(\alpha_i) &= [\sin(\alpha_i)+\epsilon_2](\alpha_{i+1}-\alpha_i)\\ \sin(\alpha_{i+1}-\alpha_i) &= \sin 0+\cos(\tilde\alpha)(\alpha_{i+1}-\alpha_i)\end{aligned}$$

where $|\epsilon_j|\le\frac{1}{2}|\alpha_{i+1}-\alpha_i|\le\frac{\alpha}{2}$ for $j=1,2$. Note that $\tilde\alpha\in[0,\alpha_{i+1}-\alpha_i]\subset[0,\alpha]$ and thus $\cos\tilde\alpha\in[\cos\alpha,1]$ since $\alpha\in\left(0,\frac{1}{2}\right]$. Thus,

$$\frac{1}{\cos\tilde\alpha}\ \le\ \frac{1}{\cos\alpha}\le\frac{1}{1-\frac{\alpha^2}{2}}=\frac{2}{2-\alpha^2}. \tag{4.28}$$

We obtain

$$\begin{aligned}\begin{pmatrix} w_1\\ w_2\end{pmatrix} &= \mathbf{y}_i+r\boldsymbol{\xi} \qquad (4.29)\\ \text{where } \boldsymbol{\xi} &= \frac{1}{\cos(\tilde\alpha)}\begin{pmatrix}\epsilon_1\\ \epsilon_2\end{pmatrix}+\left(\frac{1}{\cos(\tilde\alpha)}-1\right)\begin{pmatrix}\cos(\alpha_i)\\ \sin(\alpha_i)\end{pmatrix}\\ \text{so that } \|\boldsymbol{\xi}\|_2 &\le \frac{\sqrt{2}\,\alpha}{2-\alpha^2}+\frac{\alpha^2}{2-\alpha^2}\\ &= \frac{\alpha}{2-\alpha^2}\left(\sqrt{2}+\alpha\right)\\ &\le \frac{4\alpha}{7}\left(\sqrt{2}+\frac{1}{2}\right)\\ &= \alpha\, C_0 \qquad (4.30)\end{aligned}$$

using (4.9), (4.10) and (4.28). Hence,

$$\left\|\begin{pmatrix} w_1\\ w_2\end{pmatrix}\right\|_2\ \le\ r+r\alpha C_0. \tag{4.31}$$

Note that $\nabla_{\mathbf{y}}\tilde w(\mathbf{y})^T=(w_1,w_2)$, since $\tilde w$ is linear. We can conclude that the orbital derivative of $\tilde w$ with respect to the linearised system (4.8) in the vertex $\mathbf{y_i}$ is

$$\begin{aligned}&\nabla_{\mathbf{y}}\tilde w(\mathbf{y}_i)^TQ^{\frac12}AQ^{-\frac12}\ \mathbf{y}_i\\ =\ & \mathbf{y}_i^TQ^{\frac12}AQ^{-\frac12}\mathbf{y}_i+r\boldsymbol{\xi}^TQ^{\frac12}AQ^{-\frac12}\mathbf{y}_i \text{ by (4.29)}\\ \le\ & \frac{1}{2}\mathbf{y}_i^TQ^{\frac12}AQ^{-\frac12}\mathbf{y}_i+\frac{1}{2}[Q^{\frac12}AQ^{-\frac12}\mathbf{y}_i]^T\mathbf{y}_i\\ & +r\cdot\|\boldsymbol{\xi}\|_2\cdot\|Q^{\frac12}\|_2\cdot\|A\|_2\cdot\|Q^{-\frac12}\mathbf{y}_i\|_2\\ \le\ & \frac{1}{2}[Q^{-\frac12}\mathbf{y}_i]^T[QA+A^TQ]Q^{-\frac12}\mathbf{y}_i+\alpha C_0\|Q^{\frac12}\|_2^2\cdot\|A\|_2\cdot\|Q^{-\frac12}\mathbf{y}_i\|_2^2 \text{ by (4.30)}\\ \le\ & \left(-\frac{1}{2}+\frac{1}{4}\right)\|Q^{-\frac12}\mathbf{y}_i\|_2^2\\ \le\ & -\frac{1}{4}\frac{r}{\|Q^{\frac12}\|_2}\|Q^{-\frac12}\mathbf{y}_i\|_2\end{aligned}$$

by (4.6) and (4.10); we have also used

$$r = \|\mathbf{y}_i\|_2 = \|Q^{\frac{1}{2}}Q^{-\frac{1}{2}}\mathbf{y}_i\|_2 \leq \|Q^{\frac{1}{2}}\| \cdot \|Q^{-\frac{1}{2}}\mathbf{y}_i\|_2.$$

A similar argument holds for the other vertex $\mathbf{y}_{i+1}$.

Step 6: $\tilde{w}(\mathbf{y})$ has negative orbital derivative on the triangle

In Step 5 we have shown that

$$\nabla_{\mathbf{y}}\tilde{w}(\mathbf{y}_i)^T Q^{\frac{1}{2}}AQ^{-\frac{1}{4}}\ \mathbf{y}_i \quad \leq \quad -2c\|Q^{-\frac{1}{2}}\mathbf{y}_i\|_2, \text{ where } c = \frac{r}{8\|Q^{\frac{1}{2}}\|_2} \tag{4.32}$$

holds for the vertices $\mathbf{y}_i$. Now choose any $\mathbf{y}$ with $\|\mathbf{y}\|_2 \leq r$. Then there is an i such that $\mathbf{y} \in \mathrm{sect}\{\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{0}\}$, i.e. $\mathbf{y} = \sum_{j=0}^{1} \lambda_j \mathbf{y}_{i+j}$ with $\lambda_0, \lambda_1 \geq 0$. The orbital derivative with respect to the nonlinear system (4.7) is given by

$$\begin{aligned}
\tilde{w}'(\mathbf{y}) &= \nabla_{\mathbf{y}}\tilde{w}(\mathbf{y})^T g(\mathbf{y}) \\
&= (w_1, w_2)(Q^{\frac{1}{2}}AQ^{-\frac{1}{2}}\mathbf{y} + \varphi(\mathbf{y})) \\
&= \sum_{j=0}^{1} \lambda_j (w_1, w_2) Q^{\frac{1}{2}}AQ^{-\frac{1}{2}}\mathbf{y}_{i+j} + (w_1, w_2)\varphi(\mathbf{y}) \\
&\leq -2c\sum_{j=0}^{1} \lambda_j \|Q^{-\frac{1}{2}}\mathbf{y}_{i+j}\|_2 + \left\| \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\|_2 \cdot \|\varphi(\mathbf{y})\|_2
\end{aligned}$$

using (4.32). By (4.14), $\|\varphi(\mathbf{y})\|_2 \leq \nu\|Q^{-\frac{1}{2}}\mathbf{y}\|_2$ holds for all $\|\mathbf{y}\|_2 \leq r$. This means,

$$\|\varphi(\mathbf{y})\|_2 \quad \leq \quad \nu\|Q^{-\frac{1}{2}}\mathbf{y}\|_2 \;=\; \nu \left\| \sum_{j=0}^{1} \lambda_j Q^{-\frac{1}{2}}\mathbf{y}_{i+j} \right\|_2 \;\leq\; \nu \sum_{j=0}^{1} \lambda_j \|Q^{-\frac{1}{2}}\mathbf{y}_{i+j}\|_2.$$

Moreover, using $r = \|\mathbf{y}_{i+j}\|_2 \leq \|Q^{\frac{1}{2}}\|_2 \cdot \|Q^{-\frac{1}{2}}\mathbf{y}_{i+j}\|_2$ and (4.31) we have

$$\begin{aligned}
\tilde{w}'(\mathbf{y}) &\leq -2c\sum_{j=0}^{1} \lambda_j \|Q^{-\frac{1}{2}}\mathbf{y}_{i+j}\|_2 + r(1+\alpha C_0)\nu \sum_{j=0}^{1} \lambda_j \|Q^{-\frac{1}{2}}\mathbf{y}_{i+j}\|_2 \\
&= (-2c + c)\sum_{j=0}^{1} \lambda_j \|Q^{-\frac{1}{2}}\mathbf{y}_{i+j}\|_2 \text{ by (4.15).}
\end{aligned}$$

This means,

$$\tilde{w}'(\mathbf{y}) \quad = \quad \nabla_{\mathbf{y}}\tilde{w}(\mathbf{y}) \cdot g(\mathbf{y}) \;\leq\; -c\|Q^{-\frac{1}{2}}\mathbf{y}\|_2 \tag{4.33}$$

for all $\mathbf{y}$ with $\|\mathbf{y}\|_2 \leq r$.

Step 7: $w(\mathbf{x})$ has negative orbital derivative

With (4.23) we can conclude

$$w(\mathbf{x}) = \tilde{w}(Q^{\frac{1}{2}}\mathbf{x}).$$

Let $\mathbf{x} \in E_2 \setminus \{\mathbf{0}\}$, i.e. with $\mathbf{y} = Q^{\frac{1}{2}}\mathbf{x}$ we have $\|\mathbf{y}\|_2 \leq r$ and thus

$$\begin{aligned} w'(\mathbf{x}) &= \nabla_{\mathbf{x}} w(\mathbf{x})^T f(\mathbf{x}) \\ &= \nabla_{\mathbf{y}} \tilde{w}(Q^{\frac{1}{2}}\mathbf{x})^T Q^{\frac{1}{2}} f(\mathbf{x}) \\ &= \nabla_{\mathbf{y}} \tilde{w}(\mathbf{y})^T Q^{\frac{1}{2}} f(Q^{-\frac{1}{2}}\mathbf{y}) \\ &= \nabla_{\mathbf{y}} \tilde{w}(\mathbf{y})^T g(\mathbf{y}) \\ &\leq -c\|Q^{-\frac{1}{2}}\mathbf{y}\|_2 \text{ by (4.33)} \\ &= -c\|\mathbf{x}\|_2. \end{aligned}$$

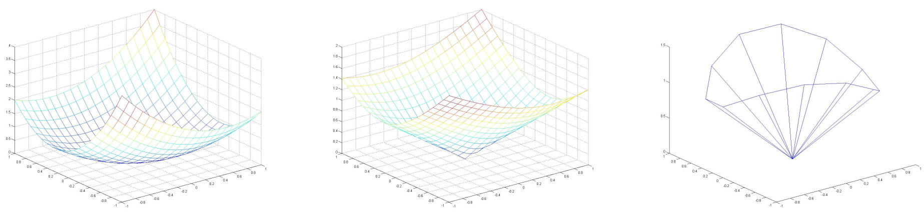
This holds for all triangles in $\mathcal{T}' = \{\mathrm{co}\{\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{0}\} \big| \; i = 1, 2, \ldots, N\}$ because $\mathrm{co}\{\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{0}\} \subset E_2$ for all i. Since $\mathrm{co}\{\mathbf{z}_i, \mathbf{z}_{i+1}, \mathbf{0}\} \subsetneq \mathrm{co}\{\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{0}\}$ for every i, Corollary 2.4 implies that the above inequality also holds for the Dini derivative $D^+[w(\mathbf{x})]$ for every $\mathbf{x} \in R \setminus \{\mathbf{0}\}$; this shows the second inequality and finishes the proof. □

5 Summary and Outlook

In [2] it was proved that the method in [10] can always generate a piecewise affine Lyapunov function for a system with an exponentially stable equilibrium, except for a small neighborhood of the equilibrium. In this article we have given an example of a system, where no piecewise affine Lyapunov function with the proposed triangulation scheme exists. Hence, there are indeed examples where the method in [10, 2, 3, 4] must fail to generate a Lyapunov function locally.

Further, we have shown that this is not an intrinsic disadvantage of the method, but it is caused by its triangulation scheme which is too coarse. We have shown that for any two-dimensional system with an exponentially stable equilibrium, there is a local triangulation scheme, which generates triangles with the equilibrium as a central vertex, such that the system possesses a piecewise affine Lyapunov function in a neighborhood of the equilibrium, cf. Figure 3.

Natural further questions are how to adapt the method to an improved triangulation scheme and to show that it always succeeds in generating a Lyapunov function if a proper triangulation is chosen. Furthermore, our conjecture is that a generalisation of Theorem 4.3 to higher dimensions holds true. Although some parts of the proof hold for arbitrary dimensions, other parts use tools specific to the two-dimensional case. However, we are optimistic that these tools can be generalised to higher dimensions.



(a) $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$.

(b) $\sqrt{V(\mathbf{x})} = \|P^{\frac{1}{2}}\mathbf{x}\|_2$.

(c) Piecewise linear approximation to $\sqrt{V(\mathbf{x})}$.

Figure 3: A system with an exponentially stable equilibrium possesses a local quadratic Lyapunov function (a). The square root of this function is also a Lyapunov function (b) and this Lyapunov function can be approximated by a piecewise linear Lyapunov function (c), cf. Theorem 4.3.

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