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# Computation of Lyapunov functions for nonlinear discrete time systems by linear programming

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Given an autonomous discrete time system with an equilibrium at the origin and a hypercube $\mathcal{D}$ containing the origin, we state a linear programming problem, of which any feasible solution parameterizes a continuous and piecewise affine (CPA) Lyapunov function $V : \mathcal{D} \to \mathrm{R}$ for the system. The linear programming problem depends on a triangulation of the hypercube. We prove that if the equilibrium at the origin is exponentially stable, the hypercube is a subset of its basin of attraction, and the triangulation fulfills certain properties, then such a linear programming problem possesses a feasible solution. We present an algorithm that generates such linear programming problems for a system, using more and more refined triangulations of the hypercube. In each step the algorithm checks the feasibility of the linear programming problem. This results in an algorithm that is always able to compute a Lyapunov function for a discrete time system with an exponentially stable equilibrium. The domain of the Lyapunov function is only limited by the size of the equilibrium's domain of attraction. The system is assumed to have a $C^2$ right-hand side, but is otherwise arbitrary. Especially, it is not assumed to be of any specific algebraic type like linear, piecewise affine, etc. Our approach is a non-trivial adaption of the CPA method to compute Lyapunov functions for continuous time systems to discrete time systems.

## 1. Introduction

Consider the discrete time dynamical system with an equilibrium at the origin:

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k), \quad \text{where } \mathbf{g} \in C^2(\mathrm{R}^n, \mathrm{R}^n) \text{ and } \mathbf{g}(\mathbf{0}) = \mathbf{0}. \tag{1}$$

Define the mapping $\mathbf{g}^{\circ m} : \mathrm{R}^n \to \mathrm{R}^n$ for all $m \in \mathrm{N}_0$ by induction through $\mathbf{g}^{\circ 0}(\mathbf{x}) := \mathbf{x}$ and $\mathbf{g}^{\circ(m+1)}(\mathbf{x}) := \mathbf{g}(\mathbf{g}^{\circ m}(\mathbf{x}))$ for all $\mathbf{x} \in \mathrm{R}^n$. The origin is said to be an *exponentially stable equilibrium* of the system (1) if there exist constants $\delta, M > 0$ and $0 < \mu < 1$ such that $\|\mathbf{g}^{\circ m}(\mathbf{x})\| \leq \mu^m M \|\mathbf{x}\|$ for all $\|\mathbf{x}\| < \delta$ and all $m \in \mathrm{N}_0$. The set $\mathcal{A} := \{\mathbf{x} \in \mathrm{R}^n : \limsup_{m\to+\infty} \|\mathbf{g}^{\circ m}(\mathbf{x})\| = 0\}$ is called its basin of attraction.

The stability of the equilibrium can be characterized by so-called Lyapunov functions, i.e. continuous functionals on the state-space decreasing along the system trajectories and with a minimum at the equilibrium. Further, Lyapunov functions additionally deliver an inner approximation of the basin of attraction.

For linear time discrete systems there is a well known method, using the discrete Lyapunov equation, to compute a Lyapunov function for the system. If the system is nonlinear one often computes a Lyapunov function for the linearized system, cf. Remark 3. In most cases, however, this gives a very conservative estimate of the basin of

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attraction for the nonlinear system. This is unfortunate, because the size of the basin of attraction is often of great importance. For example in engineering, the system (1) is often a description of some machinery that has to be close to the equilibrium to work as intended. Local stability of the equilibrium translates into "the system can withstand all small enough perturbations" and this property is obviously a necessity if the machinery is to be of any use. However, this property is clearly not sufficient and the robustness of the machinery, i.e. how large perturbations it can withstand, is of central importance. In social sciences or economics, for example, where models and parameters are inherently subject to considerable uncertainty, the robustness of an equilibrium is of even greater importance.

In such cases and many more, a Lyapunov function for the system, defined on a not merely local neighbourhood of an equilibrium, but with a domain that extends over a reasonable subset of the basin of attraction, gives useful and concrete information on the robustness of an equilibrium. Such Lyapunov functions are, however, much more difficult to construct than the local ones. For some general discussion on the stability of equilibrium points of discrete time systems and Lyapunov functions see, for example, chapter 5 in [39] or chapter 5 in [1] and for a more advanced discussion on Lyapunov functions for discrete time systems see [20].

Numerical methods to compute Lyapunov functions for nonlinear discrete time systems have, for example, been presented in [11, 12], where collocation is used to solve numerically a discrete analogue to Zubov's partial differential equation [42] using radial basis functions [8, 41] and in [4, 23], where graph algorithms are used to compute complete Lyapunov functions [9, 36]. For nonlinear systems with a certain structure there are many more approaches in the literature. To name a few, in [35] the parameterization of piecewise-affine Lyapunov functions for linear discrete systems with saturating controls is discussed, [31] is concerned with the computation of Lyapunov functions for (possibly discontinuous) piecewise-affine systems, and in [10] linear matrix inequalities are used to compute piecewise quadratic Lyapunov functions for discrete piecewise-affine systems.

The method in this paper does not require a special structure of the discrete time dynamical system, and includes error estimates within the computations, i.e. it proves that the computed function satisfies all requirements of a Lyapunov function exactly. To the best knowledge of the authors it is the first method that is guaranteed to compute true Lyapunov functions for general nonlinear discrete time systems.

In this paper we adapt the continuous and piecewise-affine (CPA) method to compute Lyapunov functions for continuous time systems, first presented in [21, 22] and in a more refined form delivering true Lyapunov functions in [33, 34], to discrete time systems. Originally the CPA method for continuous time systems was only guaranteed to compute Lyapunov functions for systems with an exponentially stable [17] or an asymptotically stable [18] equilibrium, if an arbitrary small neighbourhood of the equilibrium was cut out from the domain. In [13–16] this restriction was removed by introducing a fan-like triangulation near the equilibrium. A similar approach is used for the discrete time CPA method in this paper. Because a solution trajectory of a discrete time system is a sequence of states rather than an absolutely continuous path, as in the continuous time case, a fundamentally different methodology must be

used to compute a CPA Lyapunov function for a discrete time system.

The CPA method for continuous time systems has been extended to nonautonomous switched systems [19] and to autonomous differential inclusions [2, 3]. The CPA method for discrete time systems can, at least with some limitation, be extended to difference inclusions and we discuss this in Section 6. The details of this extension would, however, go beyond the scope of this paper and are a matter of ongoing research.

In this paper, we state in Definition 2.8 a linear programming feasibility problem with the property, that a solution to the problem parameterizes a Lyapunov function for the system, cf. Theorem 2.10. The domain of the Lyapunov function is only limited by the size of the equilibrium's basin of attraction and not by artificial bounds due to the approach as when linearizing the system. The exponential stability of an equilibrium of the system (1) is equivalent to the existence of a certain type of Lyapunov functions for the system as shown in Lemma 4.1. We use this in Theorem 4.2 to prove that the feasibility problem always possesses a solution if the parameters of the problem are chosen in a certain way. Because there are algorithms, for example the simplex algorithm, that always find a feasible solution to a linear programming problem if one exists, and because we can adequately scan the parameter space algorithmically, cf. Definition 3.1, this delivers an algorithm that is always able to compute a Lyapunov function, of which the domain is only limited by the basin of attraction, for a system of the form (1) possessing an exponentially stable equilibrium.

The structure of the paper is as follows: In Section 2 we define the Lyapunov functions and the triangulations we will be using and then we state our linear programming problem in Definition 2.8. Then, in Theorem 2.10, we prove that a feasible solution to the linear programming problem parameterizes a CPA Lyapunov function for the system. In Section 3 we state an algorithm in Definition 3.1 that systematically generates linear programming problems as in Definition 2.8. In Section 4 we prove the existence of a certain Lyapunov function for systems with an exponentially stable equilibrium in Lemma 4.1 and then use it in Theorem 4.2 to prove that the algorithm from Definition 3.1 will deliver a feasible linear programming problem for any such system. Thus, we can always compute a CPA Lyapunov function for a system with an exponentially stable equilibrium. In Section 5 we give an example of our approach to compute CPA Lyapunov functions and in Section 6 we give some concluding remarks and ideas for future research.

***Notations***

For a vector $\mathbf{x} \in \mathrm{R}^n$ we write $x_i$ or $(\mathbf{x})_i$ for its $i$-th component. For $\mathbf{x} \in \mathrm{R}^n$ and $p \geq 1$ we define the norm $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$. We also define $\|\mathbf{x}\|_\infty = \max_{i\in\{1,2,\ldots,n\}} |x_i|$. We will repeatedly use the Hölder inequality $|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|_p\|\mathbf{y}\|_q$, where $p^{-1}+q^{-1}=1$, and the norm equivalence relations

$$\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq n^{q^{-1}-p^{-1}}\|\mathbf{x}\|_p \quad \text{for } +\infty \geq p > q \geq 1 \text{ and } \mathbf{x} \in \mathrm{R}^n.$$

The *induced matrix norm* $\|\cdot\|_p$ is defined by $\|A\|_p = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p$. Clearly $\|A\mathbf{x}\|_p \leq \|A\|_p\|\mathbf{x}\|_p$. For a symmetric matrix $P \in \mathrm{R}^{n\times n}$ we denote by $\lambda^P_{\min}$ and $\lambda^P_{\max}$

the minimal and maximal eigenvalue of $A$, respectively. Further, if $P$ is additionally positive definite, i.e. its eigenvalues are all strictly larger than zero, we define the energetic norm $\|\mathbf{x}\|_P := \sqrt{\mathbf{x}^T P \mathbf{x}}$. The estimate $\sqrt{\lambda^P_{\min}}\,\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_P \le \sqrt{\lambda^P_{\max}}\,\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathrm{R}^n$ follows immediately from this definition.

Let $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m)$ be an ordered $(m+1)-$tuple of vectors in $\mathrm{R}^n$. The set of all convex combinations of these vectors is denoted by $\mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m) := \left\{\sum_{i=0}^m \lambda_i \mathbf{x}_i : 0 \le \lambda_i \le 1, \sum_{i=0}^m \lambda_i = 1\right\}$. The vectors $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m)$ are called *affinely independent* if $\sum_{i=1}^m \lambda_i(\mathbf{x}_i - \mathbf{x}_0) = \mathbf{0}$ implies $\lambda_i = 0$ for all $i = 1, \ldots, m$. If $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m)$ are affinely independent, then the set $\mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m)$ is called an $m$-simplex and the vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m$ are said to be its vertices.

An inequality such as $\mathbf{x} \le \mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are vectors, is always to be understood componentwise, i.e. $x_i \le y_i$ for all $i$.

The set of $m$-times continuously differentiable functions from an open set $\mathcal{O}$ to a set $\mathcal{P}$ is denoted by $C^m(\mathcal{O}, \mathcal{P})$. We denote the closure of a set $\mathcal{D}$ by $\overline{\mathcal{D}}$, its interior by $\mathcal{D}^\circ$, and its boundary by $\partial\mathcal{D} := \overline{\mathcal{D}} \setminus \mathcal{D}^\circ$. Finally, $\mathcal{B}_\delta$ is defined as the open $\|\cdot\|_2$-ball with center $\mathbf{0}$ and radius $\delta$, i.e. $\mathcal{B}_\delta := \{\mathbf{x} \in \mathrm{R}^n : \|\mathbf{x}\|_2 < \delta\}$.

*Remark 1.* It is unusual to define a simplex as the convex combination of the vectors of an ordered tuple, because the resulting set is obviously independent of the particular order of the vectors. For our purposes their order is, however, important and this definition has several advantages, cf. Definition 2.6 and Remark 5.

## 2. The linear programming problem

In the following definition we define the set $\mathcal{N}$ of certain neighborhoods of the origin that will be used repeatedly in this paper.

**Definition 2.1**. Denote by $\mathcal{N}$ the set of all subsets $\mathcal{D} \subset \mathrm{R}^n$ that fulfill:

i) $\mathcal{D}$ is compact.
ii) The interior $\mathcal{D}^\circ$ of $\mathcal{D}$ is a connected open neighborhood of the origin.
iii) $\mathcal{D} = \overline{\mathcal{D}^\circ}$.

A Lyapunov function for a system is a continuous function $V : \mathcal{D} \to \mathrm{R}$, with a local minimum at the equilibrium at the origin, which is decreasing along system trajectories, i.e. $V(\mathbf{g}(\mathbf{x})) < V(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{0}$. Because the dynamics of a discrete time system are nonlocal, i.e. $\mathbf{g}(\mathbf{x})$ is not necessarily close to $\mathbf{x}$, the property "decreasing along system trajectories" needs some additional consideration compared to the continuous time case, where solution trajectories are absolutely continuous. One must either assume, that $\mathcal{D}$ is forward invariant or, more practically, restrict the demand $V(\mathbf{g}(\mathbf{x})) < V(\mathbf{x})$ to all $\mathbf{x}$ in a subset $\mathcal{O}$ of $\mathcal{D}$, such that $\mathbf{x} \in \mathcal{O}$ implies $\mathbf{g}(\mathbf{x}) \in \mathcal{D}$. We follow the second approach.

**Definition 2.2**. Let $\mathcal{D}, \mathcal{O} \in \mathcal{N}$, $\mathcal{D} \supset \mathcal{O}$, and $\|\cdot\|$ be an arbitrary norm on $\mathrm{R}^n$. A continuous function $V : \mathcal{D} \to \mathrm{R}$ is called a Lyapunov function for the system (1) if it fulfills:

i) $\mathbf{g}(\mathbf{x}) \in \mathcal{D}$ for all $\mathbf{x} \in \mathcal{O}$.
ii) $V(\mathbf{0}) = 0$ and there exist constants $a, b > 0$ such that $a\|\mathbf{x}\| \le V(\mathbf{x}) \le b\|\mathbf{x}\|$ for all

$\mathbf{x} \in \mathcal{D}$.

iii) There exists a constant $c > 0$ such that $V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \leq -c\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathcal{O}$.

*Remark 2.* It is well known, that the origin is an exponentially stable equilibrium of the system (1), if and only if it possesses a Lyapunov function in the sense of Definition 2.2. In this case every connected component of a sublevel set $V^{-1}([0, r])$, $r > 0$, that is compact in $\mathcal{O}^\circ$, is a subset of the equilibrium's basin of attraction.

The sufficiency follows directly by

$$V(\mathbf{g}(\mathbf{x})) \leq -c\|\mathbf{x}\| + V(\mathbf{x}) \leq -\frac{c}{b}V(\mathbf{x}) + V(\mathbf{x}) \leq \left(1 - \frac{c}{b}\right) V(\mathbf{x}),$$

from which

$$a\|\mathbf{x}_k\| \leq V(\mathbf{x}_k) \leq \left(1 - \frac{c}{b}\right)^k V(\mathbf{x}_0) \leq \left(1 - \frac{c}{b}\right)^k b\|\mathbf{x}_0\|$$

follows. The necessity follows by Lemma 4.1 below. The proposition about the sublevel sets follows, for example, by Theorem 2.2 in [11].

This fact implies, that we can only consider discrete time systems possessing an exponentially stable equilibrium in our method, because a CPA Lyapunov function is a Lyapunov function in the sense of Definition 2.2.

*Remark 3.* The classical approach to compute a Lyapunov function for a time discrete system is to solve the so-called discrete Lyapunov equation,

$$A^T P A = P - Q,$$

where $A := D\mathbf{g}(\mathbf{0})$ is the Jacobian matrix of $\mathbf{g}$ at the origin and $Q$ is any positive definite matrix. For discussion of the discrete Lyapunov equation see, for example, Lemma 5.7.19 in [39]. It can be solved numerically in an efficient way [6]. See also [30] and [5].

If the moduli of all eigenvalues of $A$ are less than one then $V(\mathbf{x}) := \sqrt{\mathbf{x}^T P \mathbf{x}}$ is a Lyapunov function for the system in the sense of Definition 2.2 in some neighbourhood of the origin. If $\mathbf{g}$ is linear then $\mathbf{g}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathrm{R}^n$ and this neighborhood is $\mathrm{R}^n$, but for general nonlinear systems this neighbourhood will be much smaller than the equilibrium's basin of attraction.

The idea of how to compute a CPA Lyapunov function for the system (1) given a hypercube $\mathcal{D} \in \mathcal{N}$, is to subdivide $\mathcal{D}$ into a set $\mathcal{T} := \{\mathfrak{S}_\nu : \nu = 1, 2, \ldots, N\}$ of $n$-simplices $\mathfrak{S}_\nu$, such that any two simplices in $\mathcal{T}$ intersect in a common face or are disjoint, cf. Definition 2.3. Then we construct a linear programming problem in Definition 2.8, of which every feasible solution parameterizes a CPA function $V$, i.e. a continuous function that is affine on each simplex in $\mathcal{T}$, cf. Definition 2.4. Then we show in Theorem 2.10 that $V$ is a Lyapunov function for the system in the sense of Definition 2.2.

Because we cannot use a linear programming problem to check the conditions $a\|\mathbf{x}\| \leq V(\mathbf{x}) \leq b\|\mathbf{x}\|$ and $V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \leq -c\|\mathbf{x}\|$ for more than finitely many $\mathbf{x}$, the essence of the linear programming problem is how to ensure that this holds for all $\mathbf{x} \in \mathcal{D}$ and all $\mathbf{x} \in \mathcal{O} \subset \mathcal{D}$, respectively, by only using a finite number of points $\mathbf{x}$.

We start by defining general triangulations and CPA functions, then we define the triangulations we use in this paper and derive their basic properties.

**Definition 2.3** (Triangulation) Let $\mathcal{T}$ be a collection of $n$-simplices $\mathfrak{S}_\nu$ in $\mathrm{R}^n$. $\mathcal{T}$

is called a triangulation of the set $\mathcal{D} := \bigcup_{\mathfrak{S}_\nu \in \mathcal{T}} \mathfrak{S}_\nu$ if for every $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$, either $\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \emptyset$ or $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ intersect in a common face. The latter means, with

$$\mathfrak{S}_\nu = \mathrm{co}\,(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu) \quad \text{and} \quad \mathfrak{S}_\mu = \mathrm{co}\left(\mathbf{x}_0^\mu, \mathbf{x}_1^\mu, \ldots, \mathbf{x}_n^\mu\right),$$

that there are permutations $\alpha$ and $\beta$ of the numbers $0, 1, 2, \ldots, n$ such that

$$\mathbf{z}_i := \mathbf{x}_{\alpha(i)}^\nu = \mathbf{x}_{\beta(i)}^\mu, \quad \text{for } i = 0, 1, \ldots, k, \text{ where } 0 \le k \le n,$$

and

$$\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \mathrm{co}\,(\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_k).$$

**Definition 2.4** (CPA function) Let $\mathcal{T}$ be a triangulation of a set $\mathcal{D} \subset \mathrm{R}^n$. Then we can define a continuous, piecewise affine function $P : \mathcal{D} \to \mathrm{R}$ by fixing its values at the vertices of the simplices of the triangulation $\mathcal{T}$. More exactly, assume that for every vertex $\mathbf{x}$ of every simplex $\mathfrak{S}_\nu \in \mathcal{T}$ we are given a unique real number $P_\mathbf{x}$. In particular, if $\mathbf{x}$ is a vertex of $\mathfrak{S}_\nu \in \mathcal{T}$ and $\mathbf{y}$ is a vertex of $\mathfrak{S}_\mu \in \mathcal{T}$ and $\mathbf{x} = \mathbf{y}$, then $P_\mathbf{x} = P_\mathbf{y}$. Then we can uniquely define a function $P : \mathcal{D} \to \mathrm{R}$ through:

i) $P(\mathbf{x}) := P_\mathbf{x}$ for every vertex $\mathbf{x}$ of every simplex $\mathfrak{S}_\nu \in \mathcal{T}$.
ii) $P$ is affine on every simplex $\mathfrak{S}_\nu \in \mathcal{T}$.

The set of such continuous, piecewise affine functions $\mathcal{D} \to \mathrm{R}$ fulfilling i) and ii) is denoted by $\mathrm{CPA}[\mathcal{T}]$.

*Remark 4.* If $P \in \mathrm{CPA}[\mathcal{T}]$ then for every $\mathfrak{S}_\nu \in \mathcal{T}$ there is a unique vector $\mathbf{a}_\nu \in \mathrm{R}^n$ and a unique number $b_\nu \in \mathrm{R}$, such that $P(\mathbf{x}) = \mathbf{a}_\nu^T \mathbf{x} + b_\nu$ for all $\mathbf{x} \in \mathfrak{S}_\nu$. Further, if $\mathbf{x} \in \mathfrak{S}_\nu = \mathrm{co}\,(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu) \in \mathcal{T}$, then $\mathbf{x}$ can be written uniquely as a convex combination $\mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu$, $0 \le \lambda_i \le 1$ for all $i = 0, 1, \ldots, n$, and $\sum_{i=0}^n \lambda_i = 1$, of the vertices of $\mathfrak{S}_\nu$ and

$$P(\mathbf{x}) = P\left(\sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu\right) = \sum_{i=0}^n \lambda_i P(\mathbf{x}_i^\nu) = \sum_{i=0}^n \lambda_i P_{\mathbf{x}_i^\nu}.$$

For the construction of our triangulations we use the set $S_n$ of all permutations of the numbers $1, 2, \ldots, n$, and the standard orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathrm{R}^n$. For a set $\mathcal{J} \subset \{1, 2, \ldots, \}$, we define the characteristic function $\chi_\mathcal{J}(i)$ equal to one if $i \in \mathcal{J}$ and equal to zero if $i \notin \mathcal{J}$. Further, we use the functions $\mathbf{R}^\mathcal{J} : \mathrm{R}^n \to \mathrm{R}^n$, defined by

$$\mathbf{R}^\mathcal{J}(\mathbf{x}) := \sum_{i=1}^n (-1)^{\chi_\mathcal{J}(i)} x_i \mathbf{e}_i.$$

$\mathbf{R}^\mathcal{J}(\mathbf{x})$ puts a minus in front of the coordinate $x_i$ of $\mathbf{x}$ whenever $i \in \mathcal{J}$.

**Definition 2.5**. We define three general triangulations $\mathcal{T}^{\mathrm{std}}$, $\mathcal{T}_K^{\mathrm{std}}$, and $\mathcal{T}_{K,b}^{\mathrm{std}}$ of $\mathrm{R}^n$.

(1) The standard triangulation $\mathcal{T}^{\mathrm{std}}$ consists of the simplices

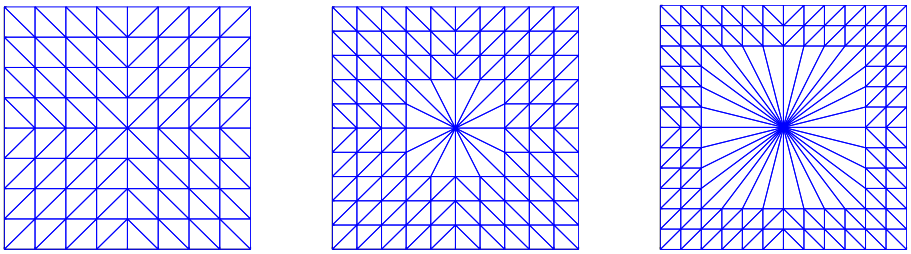
$$\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \mathrm{co}\left(\mathbf{R}^\mathcal{J}\left(\mathbf{z} + \sum_{i=1}^j \mathbf{e}_{\sigma(i)}\right) \,:\, j = 0, 1, 2, \ldots, n\right)$$

for all $\mathbf{z} \in \mathrm{N}_0^n$, all $\mathcal{J} \subset \{1, 2, \ldots, n\}$, and all $\sigma \in S_n$.

(2) Choose a $K \in \mathrm{N}_0$ and consider the intersections of the $n$-simplices $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$ in $\mathcal{T}^{\mathrm{std}}$ and the boundary of $[-2^K, 2^K]^n$. We are only interested in those intersections that are $(n-1)$-simplices, i.e. we take every simplex with vertices $\mathbf{x}_j := \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{i=1}^{j} \mathbf{e}_{\sigma(i)}\right)$, $j \in \{0, 1, \ldots, n\}$, where exactly one vertex satisfies $\|\mathbf{x}_{j^*}\|_\infty \neq 2^K$ and the other $n$ of the $n+1$ vertices satisfy $\|\mathbf{x}_j\|_\infty = 2^K$. Then we replace the vertex $\mathbf{x}_{j^*}$ by $\mathbf{0}$. We collect the thus constructed simplices and remove duplicates, i.e. if $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ are the same subset of $\mathrm{R}^n$ we remove one of them. This collection is a new triangulation of $[-2^K, 2^K]^n$. We denote it $\mathcal{T}_K^{\mathrm{std}}$ and refer to it as simplicial fan.

(3) Now choose a constant $b > 0$ and scale down the triangulation (simplicial fan) $\mathcal{T}_K^{\mathrm{std}}$ of the hypercube $[-2^K, 2^K]^n$ and the triangulation $\mathcal{T}^{\mathrm{std}}$ outside of the hypercube $[-2^K, 2^K]^n$ with the mapping $\mathbf{x} \mapsto \rho\,\mathbf{x}$, where $\rho := 2^{-K} b$. We denote by $\mathcal{T}_{K,b}^{\mathrm{std}}$ the resulting set of $n$-simplices, i.e.

$$\mathcal{T}_{K,b}^{\mathrm{std}} = \rho \mathcal{T}_K^{\mathrm{std}} \cup \rho \left\{ \mathfrak{S} \in \mathcal{T}^{\mathrm{std}} : \mathfrak{S} \cap \left] -2^K, 2^K \right[^n = \emptyset \right\}.$$



(a) $\mathcal{T}^{\mathrm{std}} = \mathcal{T}_{0,b}^{\mathrm{std}}$ (b) $\mathcal{T}_{1,b}^{\mathrm{std}}$ (c) $\mathcal{T}_{2,b}^{\mathrm{std}}$

Figure 1. Schematic pictures in 2D of some of the triangulations used in this paper. The first parameter of the triangulations is a positive integer $K$ that determines the number of simplices in the fan at the origin. The second parameter $b > 0$ is used to fix the size of the regular simplices not at the origin.

The two parameters $b$ and $K$ of the triangulation $\mathcal{T}_{K,b}^{\mathrm{std}}$ in Definition 2.5 refer to the size of the hypercube $[-b, b]^n$ covered by its simplicial fan at the origin and to the fineness of the triangulation, respectively. For schematic pictures of some of these triangulations in 2D see Figure 1. For similar pictures in 3D see Figure 1 in [15].

The triangulations $\mathcal{T}^{\mathrm{std}}$, $\mathcal{T}_K^{\mathrm{std}}$, and $\mathcal{T}_{K,b}^{\mathrm{std}}$ are the same as in [16]. Especially, it is proven in Lemma 3 in that paper that $\mathcal{T}_{K,b}^{\mathrm{std}}$ actually is a triangulation.

The simplicial fan $\mathcal{T}_K^{\mathrm{std}}$ at the origin is needed, for otherwise the triangulation might not support enough structure for a CPA Lyapunov function close to the equilibrium. This is discussed in detail for continuous time systems in [13–16] and the same argumentation applies to the discrete time case.

**Definition 2.6.** For an $n$-simplex $\mathfrak{S} = \mathrm{co}\,(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ we define its **shape matrix** $X_{\mathfrak{S}} \in \mathrm{R}^{n \times n}$ through

$$X_{\mathfrak{S}} := (\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0)^T$$

Thus, the matrix $X_{\mathfrak{S}}$ is defined by writing the entities of the vector $\mathbf{x}_i - \mathbf{x}_0$ in the $i$-th row of $X_{\mathfrak{S}}$ for $i = 1, 2, \ldots, n$.

For a triangulation $\mathcal{T}$ given as a collection of simplices with ordered vertices we refer to the set $\{X_{\mathfrak{S}} : \mathfrak{S} \in \mathcal{T}\}$ as the shape matrices of the triangulation $\mathcal{T}$.

*Remark 5.* Definition 2.6 is the reason why we have defined a simplex as the convex combination of the vectors in an ordered tuple. The resulting simplex is not dependent on the particular order of the vectors, however, the shape matrix is.

Notice, that because $\mathfrak{S}$ is an $n$-simplex, the vectors $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ are affinely independent, i.e. the shape matrix $X_{\mathfrak{S}}$ is nonsingular.

LEMMA 2.7. *The set of the shape matrices of $\mathcal{T}^{std}$ is finite. For any fixed $K \in \mathrm{N}_0$ and $b > 0$ the set of the shape matrices of $\mathcal{T}^{std}_{K,b}$ is finite.*

*Proof.* Notice that $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$ and $\mathfrak{S}_{\mathbf{z}^*\mathcal{J}^*\sigma^*}$ have the same shape matrix if $\mathcal{J} = \mathcal{J}^*$ and $\sigma = \sigma^*$. As there are $2^n$ different subsets $\mathcal{J} \subset \{1, 2, \ldots, n\}$ and $n!$ different permutations $\sigma$ of $\{1, 2, \ldots, n\}$ there can be no more than $2^n n!$ different shape matrices for $\mathcal{T}^{std}$.

The second statement of the lemma now follows immediately, because the simplicial fan at the origin in $\mathcal{T}^{std}_{K,b}$ is finite. □

Now we can formulate our linear programming feasibility problem for the system (1): Let $F > 0$ be a real number and $2 \le N_I < N_O < N_D$ be natural numbers, satisfying some additional assumptions specified below. Define

$$I := N_I \cdot F, \quad O := N_O \cdot F, \quad \text{and} \quad D := N_D \cdot F$$

and the hypercubes

$$\mathcal{D} := [-D, D]^n, \quad \mathcal{O} := [-O, O]^n, \quad \mathcal{I} := [-I, I]^n, \quad \text{and} \quad \mathcal{F} := [-F, F]^n.$$

These hypercubes serve the following purposes: $\mathcal{D}$ is the domain of the Lyapunov function. $\mathcal{O} \subset \mathcal{D}$ (outer set) is the same set as in Definition 2.2. For every $\mathbf{x} \in \mathcal{O}$ we demand $\mathbf{g}(\mathbf{x}) \in \mathcal{D}$ and the decreasing property $V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \le -c\|\mathbf{x}\|$. $\mathcal{I} \subset \mathcal{O}$ (inner set) is defined such that for every $\mathbf{x} \in \mathcal{F}$ we have $\mathbf{g}(\mathbf{x}) \in \mathcal{I}$. Finally, $\mathcal{F} \subset \mathcal{I}$ (fan) is the domain close to the origin where we use the simplicial fan triangulation.

Let the numbers $2 \le N_I < N_O < N_D$ be chosen such that $\mathbf{x} \in \mathcal{F}$ implies $\mathbf{g}(\mathbf{x}) \in \mathcal{I}$ and $\mathbf{x} \in \mathcal{O}$ implies $\mathbf{g}(\mathbf{x}) \in \mathcal{D}$, i.e.

$$\max_{\|\mathbf{x}\|_\infty \le F} \|\mathbf{g}(\mathbf{x})\|_\infty \le I \quad \text{and} \quad \max_{\|\mathbf{x}\|_\infty \le O} \|\mathbf{g}(\mathbf{x})\|_\infty \le D. \tag{2}$$

Clearly $\mathcal{D} \supset \mathcal{O} \supset \mathcal{I} \supset \mathcal{F}$ and $\mathcal{F}$ contains the origin as an inner point.

Let $K \in \mathrm{N}_0$ and consider the triangulation $\mathcal{T}^{\mathrm{std}}_{K,F}$ of $\mathrm{R}^n$ from Definition 2.5. Define

$$\mathcal{T} := \left\{ \mathfrak{S} \in \mathcal{T}^{\mathrm{std}}_{K,F} \,:\, \mathfrak{S} \cap \mathcal{D}^\circ \neq \emptyset \right\}. \tag{3}$$

Then, by the definitions of $\mathcal{T}^{\mathrm{std}}_{K,F}$ and $\mathcal{D}$, clearly $\bigcup_{\mathfrak{S} \in \mathcal{T}} \mathfrak{S} = \mathcal{D}$ and $\mathcal{T}$ is a triangulation of $\mathcal{D}$ in the sense of Definition 2.3. Before we present the linear programming problem we need a few specifications and definitions.

With $A := D\mathbf{g}(\mathbf{0})$ as the Jacobi matrix of $\mathbf{g}$ at the origin and $Q \in \mathrm{R}^{n \times n}$ an arbitrary positive definite matrix, we solve the discrete time Lyapunov equation

$$A^T P A = P - Q \tag{4}$$

for a positive definite $P \in \mathrm{R}^{n\times n}$. We define

$$V_P(\mathbf{x}) := \|\mathbf{x}\|_P, \tag{5}$$

$$\alpha := \frac{1}{8}\sqrt{\lambda^Q_{\min}/\lambda^P_{\max}}, \tag{6}$$

$$H_{\max} := \frac{\lambda^P_{\max}}{\sqrt{\lambda^P_{\min}}}\left(1 + \frac{\lambda^P_{\max}}{\lambda^P_{\min}}\right), \tag{7}$$

for every $\mathfrak{S}_\nu \in \mathcal{T}$ define

$$h_\nu := \max_{\mathbf{x},\mathbf{y}\in\mathfrak{S}_\nu} \|\mathbf{x}-\mathbf{y}\|_2 \tag{8}$$

and let $B_\nu$ and $G_\nu$ be constants fulfilling

$$B_\nu \geq n \cdot \max_{\substack{m,r,s=1,2,\dots,n\\ \mathbf{z}\in\mathfrak{S}_\nu}} \left|\frac{\partial^2 g_m}{\partial x_r \partial x_s}(\mathbf{z})\right| \qquad \text{if } \mathfrak{S}_\nu \subset \mathcal{F} \text{ and} \tag{9}$$

$$G_\nu \geq n \cdot \max_{\substack{i,j=1,2,\dots,n\\ \mathbf{z}\in\mathfrak{S}_\nu}} \left|\frac{\partial g_i}{\partial x_j}(\mathbf{z})\right| \qquad \text{if } \mathfrak{S}_\nu \subset \mathcal{O}. \tag{10}$$

See Remark 6 for an interpretation of the constants $B_\nu$ and $G_\nu$.

We further define

$$h_{\mathcal{I}\setminus\mathcal{F}} := \max\left\{h_\nu \,:\, \mathfrak{S}_\nu \subset \mathcal{I}\setminus\mathcal{F}^\circ\right\}, \tag{11}$$

$$h_{\partial\mathcal{F},P} := \max\left\{\|\mathbf{x}-\mathbf{y}\|_P \,:\, \mathbf{x}\neq\mathbf{0} \text{ and } \mathbf{y}\neq\mathbf{0} \text{ vertices of an } \mathfrak{S}_\nu \subset \mathcal{F}\right\}, \tag{12}$$

$$G_{\mathcal{F}} := \max\left\{G_\nu \,:\, \mathfrak{S}_\nu \subset \mathcal{F}\right\}, \quad \text{and} \tag{13}$$

$$E_{\mathcal{F}} := G_{\mathcal{F}} \cdot \max\left\{H_{\max}\cdot (h_{\mathcal{I}\setminus\mathcal{F}})^2/F, 2h_{\partial\mathcal{F},P}\right\}. \tag{14}$$

Note that all constants $\alpha, H_{\max}, h_\nu, B_\nu, G_\nu, h_{\mathcal{I}\setminus\mathcal{F}}, h_{\partial\mathcal{F},P}, G_{\mathcal{F}}, E_{\mathcal{F}}$ are strictly positive.

We are now ready to state the linear programming problem. The variables of the linear programming problem are $C$ and $V_{\mathbf{x}}$ for all vertices $\mathbf{x}$ of all of the simplices $\mathfrak{S}_1, \mathfrak{S}_2, \dots, \mathfrak{S}_N$ in $\mathcal{T}$. The variable $C$ is an upper bound on the gradient of the function $V : \mathcal{D} \to \mathrm{R}$ and for every vertex $\mathbf{x}$ the variable $V_{\mathbf{x}}$ is its value at $\mathbf{x}$, i.e. $V(\mathbf{x}) = V_{\mathbf{x}}$, cf. Definition 2.4.

**Definition 2.8** (The linear programming feasibility problem) The constraints of the linear programming feasibility problem are:

(I) For every $\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathcal{T}$, $\mathfrak{S}_\nu \subset \mathcal{I}$, we set
$$V_{\mathbf{x}_i} = V_P(\mathbf{x}_i) \quad \text{for } i = 0, 1, \dots, n,$$
where $V_P$ is the local Lyapunov function from (5).

(II) For every $\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathcal{T}$ we demand
$$V_{\mathbf{x}_i} \geq V_P(\mathbf{x}_i) \quad \text{for } i = 0, 1, \dots, n. \tag{15}$$

(III) For every $\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$ we define the vectors
$$\mathbf{w}_\nu := (V_{\mathbf{x}_1} - V_{\mathbf{x}_0}, V_{\mathbf{x}_2} - V_{\mathbf{x}_0}, \dots, V_{\mathbf{x}_n} - V_{\mathbf{x}_0})^T \quad \text{and} \quad \nabla V_\nu := X^{-1}_{\mathfrak{S}_\nu}\mathbf{w}_\nu,$$
where $X_{\mathfrak{S}_\nu}$ is the shape matrix of $\mathfrak{S}_\nu$, cf. Definition 2.6, and we demand
$$\|\nabla V_\nu\|_1 \leq C. \tag{16}$$
These constraints are linear in the variables of the linear programming problem, cf. Remark 7.

(IV) For every $\mathfrak{S}_\nu = \mathrm{co}\,(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{T}$, $\mathfrak{S}_\nu \subset \mathcal{O}$, and every $i = 0, 1, \ldots, n$, there is a simplex $\mathfrak{S}_\mu \in \mathrm{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n) \in \mathcal{T}$ such that $\mathbf{g}(\mathbf{x}_i) \in \mathfrak{S}_\mu$. This means that we can write $\mathbf{g}(\mathbf{x}_i)$ uniquely as a convex combination $\mathbf{g}(\mathbf{x}_i) = \sum_{j=0}^n \mu_j \mathbf{y}_j$ of the vertices of $\mathfrak{S}_\mu$, cf. Remark 8.
If $\mathfrak{S}_\nu \subset \mathcal{O} \setminus \mathcal{F}^\circ$ we demand

$$\sum_{j=0}^{n} \mu_j V_{\mathbf{y}_j} - V_{\mathbf{x}_i} + CG_\nu h_\nu \leq -\alpha \|\mathbf{x}_i\|_Q \quad \text{for } i = 0, 1, \ldots, n. \tag{17}$$

If $\mathfrak{S}_\nu \subset \mathcal{F}$ we demand

$$\sum_{j=0}^{n} \mu_j V_{\mathbf{y}_j} - V_{\mathbf{x}_i} + CB_\nu h_\nu \|\mathbf{x}_i\|_2 + E_{\mathcal{F}} \leq -\alpha \|\mathbf{x}_i\|_Q \quad \text{for } i = 1, \ldots, n. \tag{18}$$

Note, that we do not demand (18) for $i = 0$, cf. Remark 8.

We have several remarks before we prove in Theorem 2.10 that a feasible solution to the linear programming problem in Definition 2.8 parameterizes a CPA Lyapunov function for the system in question. For some of the remarks and for later we need the following results, proved, for example, in Proposition 4.1 and Lemma 4.2 in [3].

PROPOSITION 2.9. *Let* $\mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k) \subset \mathrm{R}^n$ *be a* $k$*-simplex, define* $\mathfrak{S} := \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k)$, $h := \max_{i,j=0,1,\ldots,k} \|\mathbf{x}_i - \mathbf{x}_j\|_2$, *and consider a convex combination* $\sum_{i=0}^k \lambda_i \mathbf{x}_i \in \mathfrak{S}$. *Let* $\mathcal{U} \subset \mathrm{R}^n$ *be an open set with* $\mathfrak{S} \subset \mathcal{U}$.

*a) If* $g : \mathcal{U} \to \mathrm{R}$ *is Lipschitz-continuous with constant* $L$ *on* $\mathcal{U}$, *i.e.* $|g(\mathbf{x}) - g(\mathbf{y})| \leq L\|\mathbf{x} - \mathbf{y}\|_2$ *for all* $\mathbf{x}, \mathbf{y} \in \mathcal{U}$, *then*

$$\left| g\left( \sum_{i=0}^{k} \lambda_i \mathbf{x}_i \right) - \sum_{i=0}^{k} \lambda_i g(\mathbf{x}_i) \right| \leq Lh.$$

*b) If* $g \in C^2(\mathcal{U}, \mathrm{R})$ *and* $B_H := \max_{\mathbf{z} \in \mathfrak{S}} \|H(\mathbf{z})\|_2$, *where* $H(\mathbf{z})$ *is the Hessian of* $g$ *at* $\mathbf{z}$, *then*

$$\begin{aligned}
&\left| g\left( \sum_{i=0}^{k} \lambda_i \mathbf{x}_i \right) - \sum_{i=0}^{k} \lambda_i g(\mathbf{x}_i) \right| \\
&\qquad \leq \frac{1}{2} \sum_{i=0}^{k} \lambda_i B_H \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left( \max_{\mathbf{z} \in \mathfrak{S}} \|\mathbf{z} - \mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2 \right) \\
&\qquad \leq hB_H \sum_{i=0}^{k} \lambda_i \|\mathbf{x}_i - \mathbf{x}_0\|_2 \\
&\qquad \leq B_H h^2,
\end{aligned}$$

*Further useful bounds are obtained by noting that*

$$B_H \leq n \cdot \max_{\substack{r,s=1,2,\ldots,n \\ \mathbf{z} \in \mathfrak{S}}} \left| \frac{\partial^2 g}{\partial x_r \partial x_s}(\mathbf{z}) \right|.$$

*Remark 6.* For every $\mathfrak{S}_\nu = \mathrm{co}\,(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{T}$, $\mathfrak{S}_\nu \subset \mathcal{F}$, we have by Proposition

2.9 with the constants $B_\nu$ in (9) for every convex combination $\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i$:

$$\left\| \mathbf{g}(\mathbf{x}) - \sum_{i=0}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i) \right\|_\infty \leq B_\nu h_\nu^2.$$

Now let $\mathfrak{S}_\nu \subset \mathcal{O}$. For an interpretation of the constants $G_\nu$ in (10), notice that for any $\mathbf{x}, \mathbf{y} \in \mathrm{R}^n$ there is an $i \in \{1, 2, \ldots, n\}$ and a vector $\mathbf{z}_{\mathbf{xy}}$ on the line segment between $\mathbf{x}$ and $\mathbf{y}$ such that

$$\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\|_\infty = |g_i(\mathbf{x}) - g_i(\mathbf{y})| = |\nabla g_i(\mathbf{z}_{\mathbf{xy}}) \cdot (\mathbf{x} - \mathbf{y})| \leq \|\nabla g_i(\mathbf{z}_{\mathbf{xy}})\|_1 \|\mathbf{x} - \mathbf{y}\|_\infty.$$

Hence, we have for $\mathfrak{S}_\nu \subset \mathcal{O}$

$$\sup_{\substack{\mathbf{x},\mathbf{y} \in \mathfrak{S}_\nu \\ \mathbf{x} \neq \mathbf{y}}} \frac{\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\|_\infty}{\|\mathbf{x} - \mathbf{y}\|_\infty} \leq n \cdot \max_{\substack{i,j=1,2,\ldots,n \\ \mathbf{z} \in \mathfrak{S}_\nu}} \left| \frac{\partial g_i}{\partial x_j}(\mathbf{z}) \right| \leq G_\nu. \tag{19}$$

Now let $\mathfrak{S}_\nu \subset \mathcal{F}$. In particular, since $\mathbf{g}(\mathbf{0}) = \mathbf{0}$, we have for every $\mathbf{x} \in \mathfrak{S}_\nu \subset \mathcal{F}$, $\mathbf{x} \neq \mathbf{0}$, that

$$\frac{\|\mathbf{g}(\mathbf{x})\|_\infty}{\|\mathbf{x}\|_\infty} \leq G_\nu \leq G_{\mathcal{F}} \quad \text{and} \quad \|\mathbf{g}(\mathbf{x})\|_\infty \leq G_{\mathcal{F}} F. \tag{20}$$

*Remark 7.* Consider a simplex $\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \cdots, \mathbf{x}_n^\nu)$ in the triangulation $\mathcal{T}$. The components of the vector $\nabla V_\nu$ are linear in the variables $V_{\mathbf{x}_0^\nu}, V_{\mathbf{x}_1^\nu}, \ldots, V_{\mathbf{x}_n^\nu}$ and by introducing the auxiliary variables $C_1^\nu, C_2^\nu, \ldots, C_n^\nu$ it is easily seen that $\|\nabla V_\nu\|_1 \leq C$ can be implemented by the constraints

$$C_1^\nu + C_2^\nu + \ldots + C_n^\nu \leq C \quad \text{and} \quad -C_i^\nu \leq (\nabla V_\nu)_i \leq C_i^\nu \quad \text{for } i = 1, 2, \ldots, n,$$

where $(\nabla V_\nu)_i$ is the $i$-th component of $\nabla V_\nu$.

*Remark 8.* Consider the constraints (IV) in Definition 2.8. Clearly $\mathbf{g}(\mathbf{x}_i)$ can be in more than one simplex of $\mathcal{T}$. However, the representation $\sum_{j=0}^{n} \mu_j V_{\mathbf{y}_j}$ in (17) and (18) does not depend on the particular simplex $\mathfrak{S}_\mu = \mathrm{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n)$ such that $\mathbf{g}(\mathbf{x}_i) = \sum_{j=0}^{n} \mu_j \mathbf{y}_j$ because $\mathcal{T}$ is a triangulation. Further, (18) cannot be fulfilled for $i = 0$ because $E_{\mathcal{F}} > 0$.

We now prove that a feasible solution to the linear programming problem in Definition 2.8 parameterizes a CPA Lyapunov function for the system in question.

THEOREM 2.10. *If the linear programming problem from Definition 2.8 using system (1) has a feasible solution, then the function $V : \mathcal{D} \to \mathrm{R}$, parameterized using the values $V_{\mathbf{x}}$ and the triangulation $\mathcal{T}$ as in Definition 2.4, is a Lyapunov function in the sense of Definition 2.2 for the system (1).*

*Proof.* We will show ii) and iii) in Definition 2.2 with certain norms, namely $\|\cdot\|_P$ and $\|\cdot\|_Q$. As all norms in $\mathrm{R}^n$ are equivalent, this implies the statement with different constants $a, b, c > 0$.

For every $\mathbf{x} \in \mathcal{D}$ there is a $\mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{T}$ such that $\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i$. The convexity of the norm $\|\cdot\|_P$ immediately delivers with (15)

$$V(\mathbf{x}) = V\left( \sum_{i=0}^{n} \lambda_i \mathbf{x}_i \right) = \sum_{i=0}^{n} \lambda_i V_{\mathbf{x}_i} \geq \sum_{i=0}^{n} \lambda_i \|\mathbf{x}_i\|_P \geq \left\| \sum_{i=0}^{n} \lambda_i \mathbf{x}_i \right\|_P = \|\mathbf{x}\|_P$$

and the definition of $V$ as a piecewise affine function such that $V(\mathbf{0}) = 0$ renders the existence of a constant $b > 0$ such that $V(\mathbf{x}) \leq b\|\mathbf{x}\|_P$ for all $\mathbf{x} \in \mathcal{D}$ obvious.

The demanding part of the proof is to show that $V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \leq -\alpha\|\mathbf{x}\|_Q$ for all $\mathbf{x} \in \mathcal{O}$.

To do this we first show the auxiliary result that $|V(\mathbf{z}) - V(\mathbf{y})| \leq C\|\mathbf{z} - \mathbf{y}\|_\infty$ for all $\mathbf{y}, \mathbf{z} \in \mathcal{D}$. Define $\mathbf{r}_\mu := \mathbf{y} + \mu(\mathbf{z} - \mathbf{y})$ for all $\mu \in [0, 1]$. Since $\mathcal{D}$ is convex, the line segment $\{\mathbf{r}_\mu : \mu \in [0, 1]\}$ is contained in $\mathcal{D}$ and clearly there are numbers $0 = \mu_0 < \mu_1 < \mu_2 < \ldots < \mu_K = 1$ and $\nu_1, \nu_2, \ldots, \nu_K$ such that $\mathbf{r}_\mu \in \mathfrak{S}_{\nu_i}$ for all $\mu \in [\mu_{i-1}, \mu_i]$, $i = 1, 2, \ldots, K$. Now $\mathbf{r}_0 = \mathbf{y}$ and $\mathbf{r}_1 = \mathbf{z}$, and for every $i = 1, 2, \ldots, K$ we have $V(\mathbf{x}) = \nabla V_{\nu_i} \cdot (\mathbf{x} - \mathbf{x}_0^{\nu_i}) + V_{\mathbf{x}_0^{\nu_i}}$ for $\mathbf{x} \in \mathfrak{S}_{\nu_i} = \text{co}\left(\mathbf{x}_0^{\nu_i}, \mathbf{x}_1^{\nu_i}, \ldots, \mathbf{x}_n^{\nu_i}\right)$. Thus, by (16),

$$\begin{aligned}
|V(\mathbf{z}) - V(\mathbf{y})| &= \left|\sum_{i=1}^{K} [V(\mathbf{r}_i) - V(\mathbf{r}_{i-1})]\right| \leq \sum_{i=1}^{K} |\nabla V_{\nu_i} \cdot (\mathbf{r}_i - \mathbf{r}_{i-1})| \\
&\leq \sum_{i=1}^{K} \|\nabla V_{\nu_i}\|_1 \|\mathbf{r}_i - \mathbf{r}_{i-1}\|_\infty \leq \sum_{i=1}^{K} C(\mu_i - \mu_{i-1})\|\mathbf{z} - \mathbf{y}\|_\infty \\
&= (\mu_K - \mu_0)C\|\mathbf{z} - \mathbf{y}\|_\infty = C\|\mathbf{z} - \mathbf{y}\|_\infty. \qquad (21)
\end{aligned}$$

A direct consequence is that if $\mathbf{y}, \mathbf{z} \in \mathfrak{S}_\nu \subset \mathcal{O}$, then $\mathbf{g}(\mathbf{y}), \mathbf{g}(\mathbf{z}) \in \mathcal{D}$ and by (19)

$$|V(\mathbf{g}(\mathbf{z})) - V(\mathbf{g}(\mathbf{y}))| \leq C\|\mathbf{g}(\mathbf{z}) - \mathbf{g}(\mathbf{y})\|_\infty \leq CG_\nu\|\mathbf{z} - \mathbf{y}\|_\infty \leq CG_\nu h_\nu. \qquad (22)$$

We now show that $V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \leq -\alpha\|\mathbf{x}\|_Q$ for all $\mathbf{x} \in \mathcal{O}$. We first show this for all $\mathbf{x} \in \mathcal{O} \setminus \mathcal{F}^\circ$ and then for all $\mathbf{x} \in \mathcal{F}$.

**Case 1:** Let $\mathbf{x} \in \mathcal{O} \setminus \mathcal{F}^\circ$ be arbitrary. Then there is an $\mathfrak{S}_\nu = \text{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \subset \mathcal{O} \setminus \mathcal{F}^\circ$ such that $\mathbf{x} \in \mathfrak{S}_\nu$, which in turn implies that $\mathbf{x}$ can be written as a convex combination of the vertices of the simplex, $\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i$. But then by (22) and the constraints (17) we have

$$\begin{aligned}
V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) &= V(\mathbf{g}(\mathbf{x})) - \sum_{i=0}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i)) + \sum_{i=0}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i)) - \sum_{i=0}^{n} \lambda_i V(\mathbf{x}_i) \\
&= \sum_{i=0}^{n} \lambda_i \left[V(\mathbf{g}(\mathbf{x})) - V(\mathbf{g}(\mathbf{x}_i)) + V(\mathbf{g}(\mathbf{x}_i)) - V(\mathbf{x}_i)\right] \\
&\leq \sum_{i=0}^{n} \lambda_i [CG_\nu h_\nu + V(\mathbf{g}(\mathbf{x}_i)) - V(\mathbf{x}_i)] \qquad (23) \\
&\leq -\alpha \sum_{i=0}^{n} \lambda_i \|\mathbf{x}_i\|_Q \leq -\alpha\|\mathbf{x}\|_Q.
\end{aligned}$$

**Case 2:** We now come to the more involved case $\mathbf{x} \in \mathcal{F}$. Let $\mathbf{x} \in \mathcal{F}$ be arbitrary. Then there is a simplex $\mathfrak{S}_\nu = \text{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \subset \mathcal{F}$ such that $\mathbf{x} \in \mathfrak{S}_\nu$ and $\mathbf{x}$ can be written as a convex sum of its vertices, $\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i$. However, now $\mathbf{x}_0 = \mathbf{0}$, which

also implies $\mathbf{g}(\mathbf{x}_0) = \mathbf{0}$ and $V(\mathbf{g}(\mathbf{x}_0)) = 0$. Therefore

$$V(\mathbf{x}) = \sum_{i=0}^{n} \lambda_i V(\mathbf{x}_i) = \sum_{i=1}^{n} \lambda_i V(\mathbf{x}_i), \tag{24}$$

$$\sum_{i=0}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i) = \sum_{i=1}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i), \quad \text{and} \tag{25}$$

$$\sum_{i=0}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i)) = \sum_{i=1}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i)). \tag{26}$$

We extend $V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x})$ to three differences a), b), and c), namely

$$V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) = \underbrace{V(\mathbf{g}(\mathbf{x})) - V\left(\sum_{i=1}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i)\right)}_{a)} + \underbrace{V\left(\sum_{i=1}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i)\right) - \sum_{i=1}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i))}_{b)}$$

$$+ \underbrace{\sum_{i=1}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i)) - \sum_{i=1}^{n} \lambda_i V(\mathbf{x}_i)}_{c)}, \tag{27}$$

and then we find upper bounds for a), b), and c) separately.

**a)** By (25), (22), and Proposition 2.9 we get

$$\begin{aligned} \left| V(\mathbf{g}(\mathbf{x})) - V\left(\sum_{i=1}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i)\right) \right| &\le C \left\| \mathbf{g}(\mathbf{x}) - \sum_{i=1}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i) \right\|_\infty \\ &= C \left\| \mathbf{g}(\mathbf{x}) - \sum_{i=0}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i) \right\|_\infty \\ &\le C \sum_{i=0}^{n} \lambda_i B_\nu h_\nu \|\mathbf{x}_i - \mathbf{x}_0\|_2 \\ &= C B_\nu h_\nu \sum_{i=1}^{n} \lambda_i \|\mathbf{x}_i\|_2. \end{aligned} \tag{28}$$

**b)** Set $\mathbf{z}_i := \mathbf{g}(\mathbf{x}_i)$ for $i = 0, 1, \ldots, n$ and $\mathbf{z} = \sum_{i=0}^{n} \lambda_i \mathbf{z}_i = \sum_{i=1}^{n} \lambda_i \mathbf{z}_i$. We show that

$$V(\mathbf{z}) - \sum_{i=1}^{n} \lambda_i V(\mathbf{z}_i) \le V(\mathbf{z}) - V_P(\mathbf{z}) \le E_{\mathcal{F}} \sum_{i=1}^{n} \lambda_i. \tag{29}$$

A norm is a convex function, so $V_P$, cf. (5), is convex. Using (25) and (26) we get by Jensen's inequality that

$$V_P(\mathbf{z}) = V_P\left(\sum_{i=1}^{n} \lambda_i \mathbf{z}_i\right) = V_P\left(\sum_{i=0}^{n} \lambda_i \mathbf{z}_i\right) \le \sum_{i=0}^{n} \lambda_i V_P(\mathbf{z}_i) = \sum_{i=1}^{n} \lambda_i V_P(\mathbf{z}_i). \tag{30}$$

For $i = 1, 2, \ldots, n$ we have $\mathbf{z}_i = \mathbf{g}(\mathbf{x}_i) \in \mathfrak{S}_{\nu_i} = \operatorname{co}\left(\mathbf{y}_0^{\nu_i}, \mathbf{y}_1^{\nu_i}, \ldots, \mathbf{y}_n^{\nu_i}\right) \subset \mathcal{I}$ since $\mathbf{x}_i \in \mathcal{F}$. Thus we can write $\mathbf{z}_i$ as a convex combination of the vertices of $\mathfrak{S}_{\nu_i}$, $\mathbf{z}_i = \sum_{j=0}^{n} \gamma_j \mathbf{y}_j^{\nu_i}$,

and by the definition of $V$ on $\mathcal{I}$ (constraint (I)) and Jensen's inequality we get

$$V_P(\mathbf{z}_i) = V_P\left(\sum_{j=0}^{n} \gamma_j \mathbf{y}_j^{\nu_i}\right) \leq \sum_{j=0}^{n} \gamma_j \underbrace{V_P\left(\mathbf{y}_j^{\nu_i}\right)}_{=V\left(\mathbf{y}_j^{\nu_i}\right)} = V\left(\sum_{j=0}^{n} \gamma_j \mathbf{y}_j^{\nu_i}\right) = V(\mathbf{z}_i) \tag{31}$$

Together, (30) and (31) imply

$$\begin{aligned} V(\mathbf{z}) - \sum_{i=1}^{n} \lambda_i V(\mathbf{z}_i) &\leq V(\mathbf{z}) - V_P(\mathbf{z}) + \sum_{i=1}^{n} \lambda_i [V_P(\mathbf{z}_i) - V(\mathbf{z}_i)] \\ &\leq V(\mathbf{z}) - V_P(\mathbf{z}), \end{aligned}$$

i.e. the first inequality in (29) holds true.

To prove the second inequality in (29) we first show two auxiliary inequalities, (34) and (36). If $\mathbf{z} \in \mathcal{I} \setminus \mathcal{F}$, then we can use Proposition 2.9 to gain upper bounds on $V(\mathbf{z}) - V_P(\mathbf{z})$. The Hessian matrix of $V_P$ at $\mathbf{z}$ is given by

$$H(\mathbf{z}) = \frac{1}{\|\mathbf{z}\|_P} P - \frac{1}{\|\mathbf{z}\|_P^3} (P\mathbf{z})(P\mathbf{z})^T, \tag{32}$$

from which, with $H_{\max}$ from (7),

$$\|H(\mathbf{z})\|_2 \leq \frac{H_{\max}}{\|\mathbf{z}\|_2} \leq \frac{H_{\max}}{F}, \tag{33}$$

follows. There is an $\mathfrak{S}_\mu = \mathrm{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n) \subset \mathcal{I} \setminus \mathcal{F}^\circ$ such that $\mathbf{z} \in \mathfrak{S}_\mu$ and we can write $\mathbf{z}$ as a convex combination of the vertices of $\mathfrak{S}_\mu$, $\mathbf{z} = \sum_{j=0}^{n} \mu_j \mathbf{y}_j$. Hence, by Proposition 2.9, $\mathbf{z} \in \mathcal{I} \setminus \mathcal{F}$ implies

$$\begin{aligned} V(\mathbf{z}) - V_P(\mathbf{z}) &= V\left(\sum_{j=0}^{n} \mu_j \mathbf{y}_j\right) - V_P(\mathbf{z}) \\ &= \sum_{j=0}^{n} \mu_j V_P(\mathbf{y}_j) - V_P\left(\sum_{j=0}^{n} \mu_j \mathbf{y}_j\right) \\ &\leq \frac{H_{\max}}{F} (h_{\mathcal{I}\setminus\mathcal{F}})^2. \end{aligned} \tag{34}$$

If $\mathbf{z} \in \mathcal{F}$, then there is an $\mathfrak{S}_\mu = \mathrm{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n) \subset \mathcal{F}$ such that $\mathbf{z} \in \mathfrak{S}_\mu$. Define $\mathbf{u}_i := \mathbf{y}_i - \mathbf{y}_1$ for $i = 1, 2, \ldots, n$. We can write $\mathbf{z}$ as a convex combination of the vertices of $\mathfrak{S}_\mu$ and since $\mathbf{y}_0 = \mathbf{0}$ this now implies

$$\mathbf{z} = \sum_{i=0}^{n} \mu_i \mathbf{y}_i = \sum_{i=1}^{n} \mu_i \mathbf{y}_i = \sum_{i=1}^{n} \mu_i (\mathbf{y}_1 + \mathbf{u}_i). \tag{35}$$

Now

$$V(\mathbf{z}) = \sum_{i=1}^{n} \mu_i \|\mathbf{y}_1 + \mathbf{u}_i\|_P \leq \sum_{i=1}^{n} \mu_i \left(\|\mathbf{y}_1\|_P + \|\mathbf{u}_i\|_P\right) \leq \sum_{i=1}^{n} \mu_i \left(\|\mathbf{y}_1\|_P + h_{\partial\mathcal{F},P}\right)$$

and

$$V_P(\mathbf{z}) = \left\| \sum_{i=1}^{n} \mu_i (\mathbf{y}_1 + \mathbf{u}_i) \right\|_P \geq \left\| \sum_{i=1}^{n} \mu_i \mathbf{y}_1 \right\|_P - \left\| \sum_{i=1}^{n} \mu_i \mathbf{u}_i \right\|_P \geq \sum_{i=1}^{n} \mu_i \left(\|\mathbf{y}_1\|_P - h_{\partial\mathcal{F},P}\right)$$

Hence, $\mathbf{z} \in \mathcal{F}$ implies

$$V(\mathbf{z}) - V_P(\mathbf{z}) \leq 2h_{\partial\mathcal{F},P} \sum_{i=1}^{n} \mu_i \leq 2h_{\partial\mathcal{F},P}. \tag{36}$$

We now prove the second inequality in (29), considering two complementary cases: $\sum_{i=1}^{n} \lambda_i > G_{\mathcal{F}}^{-1}$ and $\sum_{i=1}^{n} \lambda_i \leq G_{\mathcal{F}}^{-1}$. If $\sum_{i=1}^{n} \lambda_i > G_{\mathcal{F}}^{-1}$, then by (34), (36), $\sum_{i=1}^{h} \mu_i \leq 1$, and the definition of $E_{\mathcal{F}}$ we have

$$V(\mathbf{z}) - V_P(\mathbf{z}) \leq \max\left\{\frac{H_{\max}}{F}(h_{\mathcal{I}\setminus\mathcal{F}})^2, 2h_{\partial\mathcal{F},P}\right\} < E_{\mathcal{F}} \sum_{i=1}^{n} \lambda_i \tag{37}$$

If $\sum_{i=1}^{n} \lambda_i \leq G_{\mathcal{F}}^{-1}$, it follows from

$$\|\mathbf{z}\|_\infty = \left\|\sum_{i=1}^{n} \lambda_i \mathbf{g}(\mathbf{x}_i)\right\|_\infty \leq \sum_{i=1}^{n} \lambda_i \|\mathbf{g}(\mathbf{x}_i)\|_\infty \leq \sum_{i=1}^{n} \lambda_i G_{\mathcal{F}} F \tag{38}$$

by (20) that $\|\mathbf{z}\|_\infty \leq F$, i.e. $\mathbf{z} \in \mathcal{F}$. Thus, we can write $\mathbf{z}$ as in formula (35). Note that the vertices $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ in that formula are not only in the boundary of $\mathcal{F} = [-F, F]^n$, a paraxial hypercube, but are also all points at the same side, i.e. there is an $n^* \in \{1, 2, \ldots, n\}$ such that $(\mathbf{y}_i)_{n^*} = F$ for all $i = 1, 2, \ldots, n$ or $(\mathbf{y}_i)_{n^*} = -F$ for all $i = 1, 2, \ldots, n$. Therefore,

$$\|\mathbf{z}\|_\infty = \left\|\sum_{i=1}^{n} \mu_i \mathbf{y}_i\right\|_\infty = \sum_{i=1}^{n} \mu_i \|\mathbf{y}_i\|_\infty = \sum_{i=1}^{n} \mu_i F,$$

which together with (38) implies

$$\sum_{i=1}^{n} \mu_i \leq G_{\mathcal{F}} \sum_{i=1}^{n} \lambda_i.$$

Hence, by (36) and the definition of $E_{\mathcal{F}}$ we get

$$V(\mathbf{z}) - V_P(\mathbf{z}) \leq 2h_{\partial\mathcal{F},P} \sum_{i=1}^{n} \mu_i \leq 2h_{\partial\mathcal{F},P} G_{\mathcal{F}} \sum_{i=1}^{n} \lambda_i \leq E_{\mathcal{F}} \sum_{i=1}^{n} \lambda_i.$$

This inequality and (37) prove the second inequality in (29).

**c)** The constraints (18) imply

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i V(\mathbf{g}(\mathbf{x}_i)) - \sum_{i=1}^{n} \lambda_i V(\mathbf{x}_i) &= \sum_{i=1}^{n} \lambda_i \left[V(\mathbf{g}(\mathbf{x}_i)) - V(\mathbf{x}_i)\right] \\ &\leq -\sum_{i=1}^{n} \lambda_i \left[CB_\nu h_\nu \|\mathbf{x}_i\|_2 + E_{\mathcal{F}} + \alpha \|\mathbf{x}_i\|_Q\right]. \end{aligned} \tag{39}$$

We now finish the proof by applying the results from a), b), and c), i.e. (28), (29),

and (39), to (27) and obtain

$$V(\mathbf{g}(\mathbf{x})) - V(\mathbf{x}) \leq \underbrace{CB_\nu h_\nu \sum_{i=1}^{n} \lambda_i \|\mathbf{x}_i\|_2}_{a)} + \underbrace{E_{\mathcal{F}} \sum_{i=1}^{n} \lambda_i}_{b)} \tag{40}$$

$$\underbrace{- \sum_{i=1}^{n} \lambda_i \left[ CB_\nu h_\nu \|\mathbf{x}_i\|_2 + E_{\mathcal{F}} + \alpha \|\mathbf{x}_i\|_Q \right]}_{c)}$$

$$\leq -\alpha \sum_{i=1}^{n} \lambda_i \|\mathbf{x}_i\|_Q \leq -\alpha \left\| \sum_{i=1}^{n} \lambda_i \mathbf{x}_i \right\|_Q = -\alpha \|\mathbf{x}\|_Q$$

□

*Remark 9.* One might be tempted to assume that the CPA approximation of a convex function is also convex. As this would imply that the term b) in (27) was negative, the factor $E_{\mathcal{F}}$ in the constraints (18) would not be necessary and the proof of Theorem 2.10 would be much shorter. However, in general this is not true as shown by the following counterexample:

Consider the convex function

$$P(x, y) \mapsto \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and triangles with the vertices $(0, 2), (-1, 1), (1, 1)$ and $(0, 0), (-1, 1), (1, 1)$. For the CPA approximation $\widetilde{P}$ of $P$ on these triangles we have $\widetilde{P}(0, 2) = P(0, 2) = 4$, $\widetilde{P}(0, 0) = P(0, 0) = 0$ but $\widetilde{P}(0, 1) = 0.5 \cdot P(-1, 1) + 0.5 \cdot P(1, 1) = 4$. Thus

$$2 = 0.5 \cdot \widetilde{P}(0, 2) + 0.5 \cdot \widetilde{P}(0, 0) < \widetilde{P}(0.5 \cdot 0 + 0.5 \cdot 0, 0.5 \cdot 2 + 0.5 \cdot 0) = \widetilde{P}(0, 1) = 4$$

and $\widetilde{P}$ is not convex.

*Remark 10.* A practical note for the implementation of the linear programming problem: Theorem 2.10 still holds true if the constants in (7), (8), (11), (12), (13), (14) and are replaced by upper bounds rather than the exact maxima. Similarly, (6) can be replaced by $0 < \alpha < \frac{1}{8} \sqrt{\lambda^Q_{\min} / \lambda^P_{\max}}$.

## 3. The Algorithm

In the next definition we present an algorithm that generates linear programming problems as in Definition 2.8 for the system (1). It starts with a fixed triangulation of a hypercube $\mathcal{D} \in \mathcal{N}$ and refines the triangulation whenever the linear programming problem does not possess a feasible solution. The refinement is such that eventually a linear programming problem is generated, which possesses a feasible solution, whenever the origin is an exponentially stable equilibrium of the system and $\mathcal{D}$ is in its basin of attraction. This is proved in Theorem 4.2 in the next section, the main contribution of this paper.

The main idea of the algorithm is to define a sequence of finer and finer grids, indexed by $K$. They become finer both near the origin, so a finer and smaller fan,

as well as outside. Hence, $\mathcal{O}$ and $\mathcal{D}$ will not depend on $K$, whereas $\mathcal{F}_K$ and $\mathcal{I}_K$ do depend on $K$.

For the algorithm we must first fix some parameters. Let $Q \in \mathrm{R}^{n\times n}$ be an arbitrary, positive definite matrix and let $P \in \mathrm{R}^{n\times n}$ be the unique solution to the discrete Lyapunov equation (4). We fix a real number $F_0 > 0$ and positive integers $N_{I,0}, N_{O,0}$, and $N_{D,0}$. Define

$$I_0 := N_{I,0}F_0, \qquad O_0 := N_{O,0}F_0, \qquad D_0 := N_{D,0}F_0,$$
$$\mathcal{F}_0 := [-F_0, F_0]^n, \quad \mathcal{I}_0 := [-I_0, I_0]^n, \quad \mathcal{O} := [-O_0, O_0]^n, \quad \mathcal{D} := [-D_0, D_0]^n.$$

The number $N_{I,0}$ must be chosen such that $N_{O,0} > N_{I,0} \geq 2$ and

$$N_{I,0} \geq n \cdot \max_{\substack{i,j=1,2,\ldots,n \\ \|\mathbf{z}\|_\infty \leq F_0}} \left| \frac{\partial g_i}{\partial x_j}(\mathbf{z}) \right|.$$

This last inequality implies

$$I_0 = F_0 N_{I,0} \geq \max_{\|\mathbf{x}\|_\infty \leq F_0} \|\mathbf{g}(\mathbf{x})\|_\infty,$$

cf. Remark 6.

The numbers $N_{O,0}$ and $N_{D,0}$ must be chosen such that $N_{D,0} > N_{O,0}$ and $\mathbf{g}(\mathcal{O}) \subset \mathcal{D}$, i.e.,

$$\max_{\|\mathbf{x}\|_\infty \leq O_0} \|\mathbf{g}(\mathbf{x})\|_\infty \leq D_0. \tag{41}$$

For all $K \in \mathrm{N}_0$ we define

$$F_K := 2^{-K}F_0,$$
$$N_{I,K} := N_{I,0}, \qquad I_K := N_{I,K}F_K,$$
$$N_{O,K} := 2^K N_{O,0}, \qquad O_K := N_{O,K}F_K = N_{O,0}F_0,$$
$$N_{D,K} := 2^K N_{D,0}, \qquad D_K := N_{D,K}F_K = N_{D,0}F_0,$$
$$\mathcal{F}_K := [-F_K, F_K]^n, \qquad \mathcal{I}_K := [-I_K, I_K]^n.$$

We fix constants $B$ and $G$ such that

$$B \geq n \cdot \max_{\substack{m,r,s=1,2,\ldots,n \\ \mathbf{z}\in\mathcal{D}}} \left| \frac{\partial^2 g_m}{\partial x_r \partial x_s}(\mathbf{z}) \right| \quad \text{and}$$

$$G \geq n \cdot \max_{\substack{i,j=1,2,\ldots,n \\ \mathbf{z}\in\mathcal{D}}} \left| \frac{\partial g_i}{\partial x_j}(\mathbf{z}) \right|.$$

Now, for any $K \in \mathrm{N}_0$ we can construct a linear programming problem as in Definition 2.8 by giving the following values to the parameters of the problem:

$$F := F_K, \quad N_I := N_{I,K}, \quad N_O := N_{O,K} \quad \text{and} \quad N_D := N_{D,K}.$$

Denote by $\mathfrak{L}_K$ such a linear programming problem using these parameter values, the triangulation $\mathcal{T}_K := \mathcal{T}^{std}_{K,F_K}$, and $B_\nu := B$ and $G_\nu := G$ for all simplices $\mathfrak{S}_\nu$ in the triangulation of $\mathcal{D}$ as defined in (3).

For $\mathfrak{L}_K$ the constants $I$, $O$, and $D$ in Definition 2.8 are given by $I := I_K$, $O := O_K = O_0$, and $D := D_K = D_0$. Note especially that $F := F_K$ and $I := I_K$ change with $K$, but $O$ and $D$ do not. Thus, (41) holds true with $O_0$ replaced by $O_K$ and $D_0$ replaced by $D_K$.

Further, for all $K \in \mathrm{N}_0$ we have $\mathbf{g}(\mathcal{F}_K) \subset \mathcal{I}_K$ because $F_K \leq F_0$ and therefore

$$\max_{\|\mathbf{x}\|_\infty \leq F_K} \|\mathbf{g}(\mathbf{x})\|_\infty \leq F_K \cdot n \cdot \max_{\substack{i,j=1,2,\ldots,n \\ \|\mathbf{z}\|_\infty \leq F_K}} \left| \frac{\partial g_i}{\partial x_j}(\mathbf{z}) \right| \leq F_K N_{I,0} = I_K.$$

Hence, the matrices $Q$ and $P$ and the parameters $F := F_K$, $N_I := N_{I,K}$, $N_O := N_{O,K}$, and $N_D := N_{D,K}$ are suitable for the linear programming problem in Definition 2.8, i.e. $\mathfrak{L}_K$ is properly defined.

The algorithm is as follows:

**Definition 3.1** (The algorithm)

1. Set $K = 0$.
2. Construct the linear programming problem $\mathfrak{L}_K$ as described above.
3. If the linear programming problem $\mathfrak{L}_K$ has a feasible solution, then use it to parameterize a CPA Lyapunov function $V : \mathcal{D} \to \mathrm{R}$ for the system (1) as in Theorem 2.10. If the linear programming problem $\mathfrak{L}_K$ does not have a feasible solution, then increase $K$ by one, i.e. $K \leftarrow K + 1$, and repeat step 2.

Note, that if better estimates for the $B_\nu$'s and $G_\nu$'s than the uniform bounds $B$ and $G$ in the algorithm are available, then these can be used.

*Remark 11.* Note that the scaling factor $\rho$ from item (3) in Definition 2.3 for the simplicial complex $\mathcal{T}_K = \mathcal{T}^{std}_{K,F_K}$ is $\rho = 2^{-K} F_K = 2^{-2K} F_0$.

The number of simplices in the simplicial fan at the origin grows exponentially. Indeed, it is not difficult to see that the simplicial fan of $\mathcal{T}_{K+1}$ contains $2^{n-1}$-times the number of simplices in the simplicial fan of $\mathcal{T}_K$.

## 4. Main result

First, we state a fundamental lemma, the results of which are used in the proof of Theorem 4.2, which is the main contribution of this paper. Lemma 4.1 ensures the existence of a certain Lyapunov function for the system (1) if the origin is an exponentially stable equilibrium. It states results similar to Theorem 5 in [16] for continuous time systems, adapted to discrete time systems.

LEMMA 4.1. *Consider the system (1) and assume that the origin is an exponentially stable equilibrium of the system with basin of attraction $\mathcal{A}$, which is an open set. Let $Q \in \mathrm{R}^{n\times n}$ be an arbitrary positive definite matrix, $A := D\mathbf{g}(\mathbf{0})$ be the Jacobi matrix of $\mathbf{g}$ at the origin, and $P \in \mathrm{R}^{n\times n}$ be the unique (positive definite) solution to the discrete Lyapunov equation $A^T P A - P = -Q$. Let $\mathcal{D} \in \mathcal{N}$ be a subset of $\mathcal{A}$. Then there exists a function $W : \mathcal{A} \to \mathrm{R}$ that satisfies the following conditions:*

*a) $W \in C^2(\mathcal{A} \setminus \{\mathbf{0}\}, \mathrm{R})$*

*b) There is a constant $C^* < +\infty$ such that*

$$\sup_{\mathbf{x}\in\mathcal{D}\setminus\{\mathbf{0}\}} \|\nabla W(\mathbf{x})\|_2 \leq C^*. \tag{42}$$

*c) Set $\varepsilon^* := \min_{\mathbf{x}\in\partial\mathcal{D}} \|\mathbf{x}\|_2$. For all $0 < \varepsilon < \varepsilon^*$ define*

$$A_\varepsilon := \max_{i,j=1,2,\ldots,n} \left\{ \left| \frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{x}) \right| : \mathbf{x} \in \mathcal{D} \setminus \mathcal{B}_\varepsilon \right\}. \tag{43}$$

*Then there is a constant $A < +\infty$ such that*

$$A_\varepsilon \leq \frac{A}{\varepsilon} \quad \textit{for all } 0 < \varepsilon < \varepsilon^*. \tag{44}$$

*d)*

$$W(\mathbf{x}) \geq \|\mathbf{x}\|_P \quad \textit{and} \quad W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) \leq -2\alpha\|\mathbf{x}\|_Q \tag{45}$$

*for all $\mathbf{x} \in \mathcal{D}$. Here $\alpha := 1/8 \cdot \sqrt{\lambda^Q_{\min}/\lambda^P_{\max}}$, i.e. the $\alpha$ from (6).*

*e) There is a constant $\delta > 0$ such that*

$$W(\mathbf{x}) = \|\mathbf{x}\|_P \quad \textit{for all } \mathbf{x} \in \mathcal{B}_\delta. \tag{46}$$

*Proof*. For completeness we show that $\mathcal{A}$ is open: Since the equilibrium at the origin is exponentially stable, there is an $\epsilon > 0$ such that $\mathcal{B}_\epsilon \subset \mathcal{A}$. Take an arbitrary $\mathbf{x} \in \mathcal{A}$. There is a $k \in \mathrm{N}$ such that $\mathbf{g}^{\circ k}(\mathbf{x}) \in \mathcal{B}_{\epsilon/2}$. By the continuity of $\mathbf{g}^{\circ k}$ there is a $\delta > 0$ such that for all $\mathbf{y} \in \mathbf{x} + \mathcal{B}_\delta$ we have $\mathbf{g}^{\circ k}(\mathbf{y}) \in \mathbf{g}^{\circ k}(\mathbf{x}) + \mathcal{B}_{\epsilon/2} \subset \mathcal{B}_\epsilon \subset \mathcal{A}$, i.e. $\mathbf{y} \in \mathcal{A}$.

The idea of how to construct the function $W$ is as follows: Locally, at the origin, $W$ is given by the formula (46) and away from the origin by the formula $W(\mathbf{x}) := \beta \sum_{k=0}^{+\infty} \|\mathbf{g}^{\circ k}(\mathbf{x})\|_Q$, $\beta > 0$ a constant. In between, $W$ is a smooth interpolation of these two. First we work this construction out and then we show that the constructed function fulfills the claimed properties a), b), c), d), and e).

**Definition of $W$:** Since $P$ is a solution to the discrete Lyapunov equation (4), it follows immediately that $\widetilde{V}_P(\mathbf{x}) = \|\mathbf{x}\|^2_P$ is a Lyapunov function for the linear system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, satisfying

$$\widetilde{V}_P(A\mathbf{x}) - \widetilde{V}_P(\mathbf{x}) = -\|\mathbf{x}\|^2_Q.$$

Since $\mathbf{g}$ is differentiable at the origin, the function $\boldsymbol{\psi}(\mathbf{x}) := (\mathbf{g}(\mathbf{x}) - A\mathbf{x})/\|\mathbf{x}\|_2$ fulfills $\lim_{\mathbf{x}\to\mathbf{0}} \boldsymbol{\psi}(\mathbf{x}) = \mathbf{0}$. Simple calculations give, with $\boldsymbol{\psi}^*(\mathbf{x}) := \mathbf{g}(\mathbf{x}) - A\mathbf{x} = \|\mathbf{x}\|_2\boldsymbol{\psi}(\mathbf{x})$, that

$$\begin{aligned}
&\widetilde{V}_P(\mathbf{g}(\mathbf{x})) - \widetilde{V}_P(\mathbf{x}) \\
&= [\mathbf{g}(\mathbf{x})]^T P[\mathbf{g}(\mathbf{x})] - \mathbf{x}^T P\mathbf{x} \\
&= [\boldsymbol{\psi}^*(\mathbf{x}) + A\mathbf{x}]^T P[\boldsymbol{\psi}^*(\mathbf{x}) + A\mathbf{x}] - \mathbf{x}^T P\mathbf{x} \\
&= \boldsymbol{\psi}^{*T}(\mathbf{x})P\boldsymbol{\psi}^*(\mathbf{x}) + \boldsymbol{\psi}^{*T}(\mathbf{x})PA\mathbf{x} + \mathbf{x}^T A^T P\boldsymbol{\psi}^*(\mathbf{x}) + \underbrace{\mathbf{x}^T A^T PA\mathbf{x} - \mathbf{x}^T P\mathbf{x}}_{=-\|\mathbf{x}\|^2_Q} \\
&\leq -\|\mathbf{x}\|^2_Q + \|\boldsymbol{\psi}^*(\mathbf{x})\|_2\|P\|_2\left(\|\boldsymbol{\psi}^*(\mathbf{x})\|_2 + 2\|A\|_2\|\mathbf{x}\|_2\right) \\
&= -\|\mathbf{x}\|^2_Q + \|\mathbf{x}\|^2_2 \cdot \|\boldsymbol{\psi}(\mathbf{x})\|_2\|P\|_2\left(\|\boldsymbol{\psi}(\mathbf{x})\|_2 + 2\|A\|_2\right)
\end{aligned} \tag{47}$$

and it follows that there is a $\delta^* > 0$ such that $\widetilde{V}_P(\mathbf{g}(\mathbf{x})) - \widetilde{V}_P(\mathbf{x}) \leq -\frac{1}{2}\|\mathbf{x}\|^2_Q$ for all $\mathbf{x} \in \mathcal{B}_{\delta^*}$. Hence, with $V_P(\mathbf{x}) = \sqrt{\widetilde{V}_P(\mathbf{x})} = \|\mathbf{x}\|_P$ we have, because $\widetilde{V}_P(\mathbf{g}(\mathbf{x})) < \widetilde{V}_P(\mathbf{x})$

and $\|\mathbf{x}\|_Q/\|\mathbf{x}\|_P \geq \sqrt{\lambda_{\min}^Q/\lambda_{\max}^P} = 8\alpha$ for all $\mathbf{x} \in \mathcal{B}_{\delta^*} \setminus \{\mathbf{0}\}$, that

$$V_P(\mathbf{g}(\mathbf{x})) - V_P(\mathbf{x}) = \frac{\widetilde{V}_P(\mathbf{g}(\mathbf{x})) - \widetilde{V}_P(\mathbf{x})}{\sqrt{\widetilde{V}_P(\mathbf{g}(\mathbf{x}))} + \sqrt{\widetilde{V}_P(\mathbf{x})}} \leq \frac{-\|\mathbf{x}\|_Q^2/2}{2\sqrt{\widetilde{V}_P(\mathbf{x})}} = \frac{-\|\mathbf{x}\|_Q^2}{4\|\mathbf{x}\|_P} \leq -2\alpha\|\mathbf{x}\|_Q$$

for all $\mathbf{x} \in \mathcal{B}_{\delta^*} \setminus \{\mathbf{0}\}$. Thus

$$V_P(\mathbf{g}(\mathbf{x})) - V_P(\mathbf{x}) \leq -2\alpha\|\mathbf{x}\|_Q \tag{48}$$

for all $\mathbf{x} \in \mathcal{B}_{\delta^*}$.

Consider the function $\widetilde{W} : \mathcal{A} \to \mathrm{R}$,

$$\widetilde{W}(\mathbf{x}) := \sum_{k=0}^{+\infty} \|\mathbf{g}^{\circ k}(\mathbf{x})\|_Q^2. \tag{49}$$

It follows from the exponential stability of the equilibrium that the series on the right-hand side is convergent and in the proof of Theorem 2.8 in [11] it is shown that $\mathbf{g} \in C^2(\mathrm{R}^n, \mathrm{R}^n)$ implies $\widetilde{W} \in C^2(\mathcal{A}, \mathrm{R})$. By the definition of $\widetilde{W}$ clearly

$$\widetilde{W}(\mathbf{x}) = \sum_{k=0}^{+\infty} \|\mathbf{g}^{\circ k}(\mathbf{x})\|_Q^2 = \|\mathbf{x}\|_Q^2 + \sum_{k=1}^{+\infty} \|\mathbf{g}^{\circ k}(\mathbf{x})\|_Q^2 \geq \|\mathbf{x}\|_Q^2 \tag{50}$$

and

$$\widetilde{W}(\mathbf{g}(\mathbf{x})) - \widetilde{W}(\mathbf{x}) = \sum_{k=0}^{+\infty} \left( \|\mathbf{g}^{\circ(k+1)}(\mathbf{x})\|_Q^2 - \|\mathbf{g}^{\circ k}(\mathbf{x})\|_Q^2 \right) = -\|\mathbf{x}\|_Q^2 \tag{51}$$

for all $\mathbf{x} \in \mathcal{A}$.

Now choose an $r > 0$ such that $\{\mathbf{x} \in \mathrm{R}^n : V_P(\mathbf{x}) \leq r\} \subset \mathcal{B}_{\delta^*}$ and define the sets

$$\mathcal{E}_1 := \{\mathbf{x} \in \mathrm{R}^n : V_P(\mathbf{x}) < r/2\} \text{ and}$$

$$\mathcal{E}_2 := \{\mathbf{x} \in \mathrm{R}^n : V_P(\mathbf{x}) > r\} \cap \mathcal{A}.$$

See Figure 2 for a schematic picture of the sets $\mathcal{E}_1$, $\mathcal{D} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$, and $\mathcal{E}_2 \cap \mathcal{D}$ that we will use in the rest of the proof.

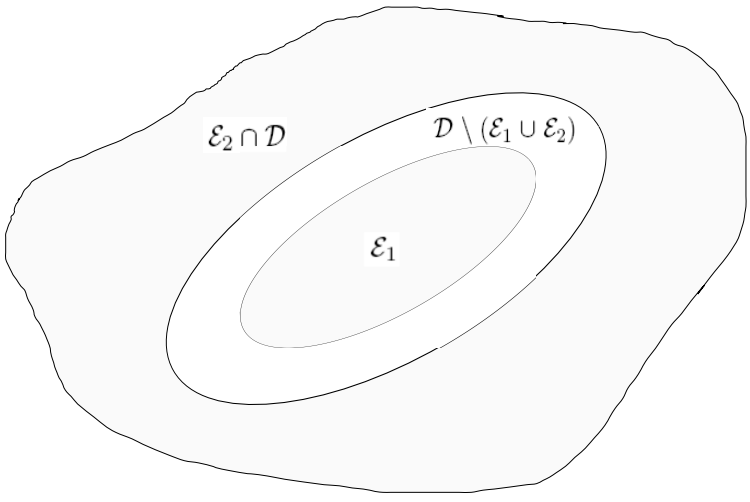



Figure 2. Schematic figure of the sets $\mathcal{E}_1$, $\mathcal{D} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$, and $\mathcal{E}_2 \cap \mathcal{D}$.

Let $\widetilde{\rho} \in C^\infty(\mathrm{R}, [0,1])$ be a non-decreasing function, such that $\widetilde{\rho}(x) = 0$ if $x < r/2$ and $\widetilde{\rho}(x) = 1$ if $x > r$. Such a function can be constructed by standard methods of

partitions of unity, cf. e.g. [40]. Then $\rho(\mathbf{x}) := \widetilde{\rho}(V_P(\mathbf{x}))$ fulfills $\rho \in C^2(\mathcal{A} \setminus \{\mathbf{0}\}, \mathrm{R})$, $0 \le \rho(\mathbf{x}) \le 1$ for all $\mathbf{x} \in \mathrm{R}^n$, $\rho(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{E}_1$, and $\rho(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathcal{E}_2$. Define

$$\beta := \max_{\mathbf{x} \in \mathcal{D} \setminus \mathcal{E}_1} \max \left\{ \frac{V_P(\mathbf{x})}{\widetilde{W}(\mathbf{x})}, \frac{2\alpha}{\|\mathbf{x}\|_Q}, \frac{4\alpha\|\mathbf{x}\|_Q + r}{2\widetilde{W}(\mathbf{x})} \right\} \quad \text{and} \quad \widetilde{W}_\beta(\mathbf{x}) := \beta \widetilde{W}(\mathbf{x}).$$

Note that this definition of $\beta$ and $\widetilde{W}_\beta$ implies for all $\mathbf{x} \in \mathcal{D} \setminus \mathcal{E}_1$ that

$$\widetilde{W}_\beta(\mathbf{x}) \ge V_P(\mathbf{x}) = \|\mathbf{x}\|_P, \tag{52}$$

$$\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x}) = -\beta\|\mathbf{x}\|_Q^2 \le -2\alpha\|\mathbf{x}\|_Q, \quad \text{and} \tag{53}$$

$$\frac{r}{2} - \widetilde{W}_\beta(\mathbf{x}) \le \frac{r}{2} - \frac{4\alpha\|\mathbf{x}\|_Q + r}{2} = -2\alpha\|\mathbf{x}\|_Q. \tag{54}$$

We define for all $\mathbf{x} \in \mathcal{A}$ the function $W$ through

$$W(\mathbf{x}) := \rho(\mathbf{x})\widetilde{W}_\beta(\mathbf{x}) + (1 - \rho(\mathbf{x}))V_P(\mathbf{x}). \tag{55}$$

We will now check that the function $W(\mathbf{x})$ satisfies the properties a)–e).

**a)** Because $\rho$, $V_P$, and $\widetilde{W}_\beta$ are in $C^2(\mathcal{A} \setminus \{\mathbf{0}\}, \mathrm{R})$ then so is $W$.

**b)** For every $\mathbf{x} \ne \mathbf{0}$ we have $\nabla V_P(\mathbf{x}) = P\mathbf{x}/\|\mathbf{x}\|_P$ so for every $\mathbf{x} \ne \mathbf{0}$

$$\|\nabla V_P(\mathbf{x})\|_2 = \frac{\|P\mathbf{x}\|_2}{\|\mathbf{x}\|_P} \le \frac{\lambda_{\max}^P}{\sqrt{\lambda_{\min}^P}} < +\infty.$$

Because $\nabla W$ is continuous on the compact set $\mathcal{D} \setminus \mathcal{E}_1$ and $W$ and $V_P$ coincide on $\mathcal{E}_1$

$$\sup_{\mathbf{x} \in \mathcal{D} \setminus \{\mathbf{0}\}} \|\nabla W(\mathbf{x})\|_2 = \max \left\{ \max_{\mathbf{x} \in \mathcal{D} \setminus \mathcal{E}_1} \|\nabla W(\mathbf{x})\|_2, \sup_{\mathbf{x} \in \mathcal{E}_1 \setminus \{\mathbf{0}\}} \|\nabla V_P(\mathbf{x})\|_2 \right\} < +\infty$$

and there is a constant $C^*$ such that (42) holds true.

**c)** Denote by $p_{\max}$ the maximum absolute value of the entities of $P$, i.e. $p_{\max} := \max_{i,j=1,2,\ldots,n} |p_{ij}|$. Define

$$A := \max \left\{ \varepsilon^* \cdot \max_{\substack{i,j=1,2,\ldots,n \\ \mathbf{x} \in \mathcal{D} \setminus \mathcal{E}_1}} \left| \frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{x}) \right|, \frac{1}{\sqrt{\lambda_{\min}^P}} \left( p_{\max} + \frac{(\lambda_{\max}^P)^2}{\lambda_{\min}^P} \right) \right\}.$$

For an arbitrary $\varepsilon$, $0 < \varepsilon < \varepsilon^*$, let $\mathbf{y} \in \mathcal{D} \setminus \mathcal{B}_\varepsilon$ and $i, j \in \{1, 2, \ldots, n\}$ be such that

$$A_\varepsilon = \left| \frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{y}) \right|.$$

To show (44), we distinguish between the two cases $\mathbf{y} \in \mathcal{D} \setminus \mathcal{E}_1$ and $\mathbf{y} \in \mathcal{E}_1$. In the first case, (44) clearly holds true because $\varepsilon^*/\varepsilon > 1$.

Now assume that $\mathbf{y} \in \mathcal{E}_1$. In this case $W(\mathbf{x})$ coincides with $V_P(\mathbf{x}) = \|\mathbf{x}\|_P$ in a neighbourhood of $\mathbf{y}$ and we have the formula (32) for its Hessian matrix. By definition, $A_\varepsilon$ is the maximum of the absolute values of the entities of the Hessian $H_W(\mathbf{x})$ for $\mathbf{x} \in \mathcal{D} \setminus \mathcal{B}_\varepsilon$ and because $\|\mathbf{y}\|_2 \ge \varepsilon$ we have

$$A_\varepsilon = \left| \frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{y}) \right| \le \frac{1}{\sqrt{\lambda_{\min}^P}} \left( \frac{p_{\max}}{\|\mathbf{y}\|_2} + \frac{(\lambda_{\max}^P)^2 \|\mathbf{y}\|_2^2}{\lambda_{\min}^P \|\mathbf{y}\|_2^3} \right) \le \frac{A}{\varepsilon}. \tag{56}$$

Hence, estimate (44) holds true for all $0 < \varepsilon < \varepsilon^*$.

**d)** For all $\mathbf{x} \in \mathcal{E}_1$ we have $W(\mathbf{x}) = V_P(\mathbf{x}) = \|\mathbf{x}\|_P$. For all $\mathbf{x} \in \mathcal{D} \setminus \mathcal{E}_1$ we have by (55) that $W$ is point-wise the convex combination of $\widetilde{W}_\beta$ and $V_P$. Hence, by (52) we have

$$W(\mathbf{x}) \geq \min\{\widetilde{W}_\beta(\mathbf{x}), V_P(\mathbf{x})\} \geq \|\mathbf{x}\|_P \quad \text{for all } \mathbf{x} \in \mathcal{D} \setminus \mathcal{E}_1$$

and the first estimate in (45) holds true.

To prove the second estimate in (45) we consider three complementary cases, $\mathbf{x} \in \mathcal{E}_1$, $\mathbf{x} \in \mathcal{D} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$, and $\mathbf{x} \in \mathcal{E}_2 \cap \mathcal{D}$, cf. Figure 2. The identity

$$\begin{aligned} &W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) \\ &\quad = \rho(\mathbf{g}(\mathbf{x}))\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) + (1 - \rho(\mathbf{g}(\mathbf{x})))V_P(\mathbf{g}(\mathbf{x})) - \rho(\mathbf{x})\widetilde{W}_\beta(\mathbf{x}) - (1 - \rho(\mathbf{x}))V_P(\mathbf{x}) \qquad (57) \\ &\quad = \rho(\mathbf{g}(\mathbf{x}))\left[\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x})\right] + (1 - \rho(\mathbf{g}(\mathbf{x})))\left[V_P(\mathbf{g}(\mathbf{x})) - V_P(\mathbf{x})\right] \\ &\qquad + \left[\rho(\mathbf{g}(\mathbf{x})) - \rho(\mathbf{x})\right]\left[\widetilde{W}_\beta(\mathbf{x}) - V_P(\mathbf{x})\right] \qquad (58) \end{aligned}$$

is useful for some of these cases. Further note that

$$\|\mathbf{g}(\mathbf{x})\|_P = V_P(\mathbf{g}(\mathbf{x})) \leq V_P(\mathbf{x}) = \|\mathbf{x}\|_P \quad \text{for all } \mathbf{x} \in \mathcal{D} \setminus \mathcal{E}_2$$

because $V_P$ is a Lyapunov function for the system (1) on $\mathcal{B}_{\delta^*} \supset \mathcal{D} \setminus \mathcal{E}_2$. This implies, because $\widetilde{\rho}$ is monotonically increasing,

$$\rho(\mathbf{x}) = \widetilde{\rho}(V_P(\mathbf{x})) \geq \widetilde{\rho}(V_P(\mathbf{g}(\mathbf{x}))) = \rho(\mathbf{g}(\mathbf{x})) \quad \text{for all } \mathbf{x} \in \mathcal{D} \setminus \mathcal{E}_2 \qquad (59)$$

as well as

$$\mathbf{x} \in \mathcal{E}_1 \Rightarrow \mathbf{g}(\mathbf{x}) \in \mathcal{E}_1 \quad \text{and} \quad \mathbf{x} \in \mathcal{D} \setminus \mathcal{E}_2 \Rightarrow \mathbf{g}(\mathbf{x}) \in \mathcal{D} \setminus \mathcal{E}_2. \qquad (60)$$

**Case 1:** Assume $\mathbf{x} \in \mathcal{E}_1$, then by (60) and the definition of $\rho$ we have $\rho(\mathbf{x}) = \rho(\mathbf{g}(\mathbf{x})) = 0$, and by (57) and (48) we get

$$W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) = V_P(\mathbf{g}(\mathbf{x})) - V_P(\mathbf{x}) \leq -2\alpha\|\mathbf{x}\|_Q. \qquad (61)$$

**Case 2:** Assume $\mathbf{x} \in \mathcal{D} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$. Then by (59) $\rho(\mathbf{g}(\mathbf{x})) - \rho(\mathbf{x}) \leq 0$ and by (52) $\widetilde{W}_\beta(\mathbf{x}) - V_P(\mathbf{x}) \geq 0$ so (58), (53), and (48) deliver

$$\begin{aligned} &W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) \\ &\quad \leq \rho(\mathbf{g}(\mathbf{x}))\left[\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x})\right] + (1 - \rho(\mathbf{g}(\mathbf{x})))\left[V_P(\mathbf{g}(\mathbf{x})) - V_P(\mathbf{x})\right] \\ &\quad \leq \max\left\{\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x}),\ V_P(\mathbf{g}(\mathbf{x})) - V_P(\mathbf{x})\right\} \leq -2\alpha\|\mathbf{x}\|_Q. \end{aligned}$$

**Case 3:** Assume that $\mathbf{x} \in \mathcal{E}_2 \cap \mathcal{D}$ until the end of this part of the proof. Here, we consider the three cases $\mathbf{g}(\mathbf{x}) \in \mathcal{E}_2 \cap \mathcal{D}$, $\mathbf{g}(\mathbf{x}) \in \mathcal{D} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$, and $\mathbf{g}(\mathbf{x}) \in \mathcal{E}_1$ separately.

- If $\mathbf{g}(\mathbf{x}) \in \mathcal{E}_2 \cap \mathcal{D}$, then $\rho(\mathbf{x}) = \rho(\mathbf{g}(\mathbf{x})) = 1$ and (57) and (53) imply
$$W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) = \widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x}) \leq -2\alpha\|\mathbf{x}\|_Q.$$
- If $\mathbf{g}(\mathbf{x}) \in \mathcal{D} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$, then we have $\rho(\mathbf{g}(\mathbf{x})) - \rho(\mathbf{x}) = \rho(\mathbf{g}(\mathbf{x})) - 1 \leq 0$ and by (52) $\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) \geq V_P(\mathbf{g}(\mathbf{x}))$. We can use this to simplify (58) and then use (53) to

estimate from above,

$$\begin{aligned} &W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) \\ &= \rho(\mathbf{g}(\mathbf{x}))\left[\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x})\right] + (1-\rho(\mathbf{g}(\mathbf{x})))\left[V_P(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x})\right] \\ &\le \rho(\mathbf{g}(\mathbf{x}))\left[\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x})\right] + (1-\rho(\mathbf{g}(\mathbf{x})))\left[\widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x})\right] \\ &= \widetilde{W}_\beta(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x}) \le -2\alpha\|\mathbf{x}\|_Q. \end{aligned}$$

- If $\mathbf{g}(\mathbf{x}) \in \mathcal{E}_1$ then $\rho(\mathbf{g}(\mathbf{x})) = 0$ and $\rho(\mathbf{x}) = 1$ and (57) simplifies to
$$W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) = V_P(\mathbf{g}(\mathbf{x})) - \widetilde{W}_\beta(\mathbf{x}).$$
Now $\mathbf{g}(\mathbf{x}) \in \mathcal{E}_1$ implies $V_P(\mathbf{g}(\mathbf{x})) = \|\mathbf{g}(\mathbf{x})\|_P < r/2$ and since $\mathbf{x} \in \mathcal{E}_2 \cap \mathcal{D}$, we have $W(\mathbf{x}) = \widetilde{W}_\beta(\mathbf{x})$. Thus, by (54)
$$W(\mathbf{g}(\mathbf{x})) - W(\mathbf{x}) < r/2 - \widetilde{W}_\beta(\mathbf{x}) \le -2\alpha\|\mathbf{x}\|_Q.$$

Thus, we have proved that the second estimate in (45) holds true.

**e)** By construction, $W(\mathbf{x}) = V_P(\mathbf{x}) = \|\mathbf{x}\|_P$ for all $\mathbf{x} \in \mathcal{E}_1$ and $\mathcal{E}_1$ is an open neighbourhood of the origin. Thus, for small enough $\delta > 0$ we have $\mathcal{B}_\delta \subset \mathcal{E}_1$ and (46) follows. □

*Remark 12.* The second order derivatives of $W$ will in general diverge at the origin, but at a known rate as stated in (44).

The next theorem, the main result of this paper, is valid for more general sequences $(\mathcal{T}_K)_{K\in\mathrm{N}_0}$ of triangulations, where $\mathcal{T}_{K+1}$ is constructed from $\mathcal{T}_K$ by scaling and tessellating its simplices, than for the sequence $(\mathcal{T}_K)_{K\in\mathrm{N}_0}$ in Definition 3.1. However, it is quite difficult to get hold of the exact conditions that must be fulfilled in a simple way so we restrict the theorem to this specific sequence.

THEOREM 4.2. *Consider the system (1) and assume that the origin is an exponentially stable equilibrium of the system with basin of attraction $\mathcal{A}$. Assume that $\mathcal{D}$ in Definition 3.1 is a subset of $\mathcal{A}$. Then, for every large enough $K \in \mathrm{N}_0$, the linear programming problem $\mathfrak{L}_K$ in Definition 3.1 possesses a feasible solution. Especially, the algorithm in the same definition succeeds in computing a CPA Lyapunov function for the system in a finite number of steps.*

*Proof.* We show that for all large enough $K \in \mathrm{N}_0$ the linear programming problem $\mathfrak{L}_K$ has a feasible solution. Let us first consider the matrices and constants that are used to initialize the linear programming problem $\mathfrak{L}_K$, $K \in \mathrm{N}_0$. The matrices $P$ and $Q$ and then the constants $\lambda^P_{\min}$, $\lambda^P_{\max}$, and $H_{\max}$, are all independent of $K$. So are the constants $B_\nu$ and $G_\nu$ because $D = D_K = D_0$ for all $K \in \mathrm{N}_0$. Indeed we set $B_\nu := B$ and $G_\nu := G$ in the algorithm for all $K \in \mathrm{N}_0$, which implies that $G_\mathcal{F}$ is also independent of $K \in \mathrm{N}_0$ (since $G$ is the same for all simplices). In contrast to this, the constants $h_\nu$, $h_{\mathcal{I}\setminus\mathcal{F}}$, $h_{\partial\mathcal{F},P}$ and $E_\mathcal{F}$ do depend on $K \in \mathrm{N}_0$.

For a particular $K \in \mathrm{N}_0$ we have for these constants in the linear programming problem $\mathfrak{L}_K$ that for an $\mathfrak{S}_\nu \in \mathcal{T}_K = \mathcal{T}^{std}_{K,F_K}$,

$$h_\nu := \max_{\mathbf{x},\mathbf{y}\in\mathfrak{S}_\nu} \|\mathbf{x}-\mathbf{y}\|_2 = \sqrt{n}\,2^{-2K}F_0 \ \text{ if } \mathfrak{S}_\nu \subset \mathcal{D}\setminus\mathcal{F}^\circ_K, \tag{62}$$

which implies

$$h_{\mathcal{I}\setminus\mathcal{F}} := \max_{\mathbf{x},\mathbf{y}\in\mathfrak{S}_\nu} \|\mathbf{x}-\mathbf{y}\|_2 = \sqrt{n}\,2^{-2K}F_0,$$

and

$$2^{-K}F_0 = F_K \le h_\nu \le \sqrt{n}F_K = \sqrt{n}\,2^{-K}F_0 \quad \text{if } \mathfrak{S}_\nu \subset \mathcal{F}_K.$$

Similarly

$$\begin{aligned} h_{\partial\mathcal{F},P} &:= \max\{\|\mathbf{x}-\mathbf{y}\|_P \,:\, \mathbf{x}\neq\mathbf{0} \text{ and } \mathbf{y}\neq\mathbf{0} \text{ vertices of } \mathfrak{S}_\nu \subset \mathcal{F}_K\} \\ &\le \sqrt{\lambda_{\max}^P(n-1)}\cdot 2^{-2K}F_0 \end{aligned} \tag{63}$$

and

$$\begin{aligned} E_{\mathcal{F}} &:= G_{\mathcal{F}}\cdot\max\left\{H_{\max}\cdot(h_{\mathcal{I}\setminus\mathcal{F}})^2/F_K, 2h_{\partial\mathcal{F},P}\right\} \\ &\le 2^{-2K}F_0G_{\mathcal{F}}\max\left\{H_{\max}n2^{-K}, 2\sqrt{\lambda_{\max}^P(n-1)}\right\} \end{aligned} \tag{64}$$

in $\mathfrak{L}_K$.

Set $V_{\mathbf{x}_i} = W(\mathbf{x}_i)$ for all vertices $\mathbf{x}_i$ of all simplices $\mathfrak{S}$ of the triangulation $\mathcal{T}_K$, where $W$ is the function from Lemma 4.1 for the system. Further, set the variable $C$ equal to $nC^*$, where $C^*$ is the constant from Lemma 4.1. We show that the linear constraints (I)-(IV) in Definition 2.8 are fulfilled for $\mathfrak{L}_K$, whenever $K\in\mathrm{N}_0$ is large enough.

For all $K$ so large that $\mathcal{I}_K\subset\mathcal{B}_\delta$, the constraints (I) are fulfilled for $\mathfrak{L}_K$ by (46). For all $K\in\mathrm{N}_0$, the constraints (II) for $\mathfrak{L}_K$ are fulfilled by (45). By the Mean Value Theorem and (42) we have $|(\nabla V_\nu)_i|\le C^*$ independent of $i$ and $\nu$ and therefore the constraints (III) are fulfilled for $\mathfrak{L}_K$. We come to the constraints (IV).

Let $\mathbf{x}_i\neq\mathbf{0}$ be an arbitrary vertex of an arbitrary simplex $\mathfrak{S}_\nu\in\mathcal{T}_K$, $\mathfrak{S}_\nu\subset\mathcal{O}_K = \mathcal{O}_0$. Then $\mathbf{g}(\mathbf{x}_i)\in\mathfrak{S}_\mu$ for some simplex $\mathfrak{S}_\mu = \mathrm{co}(\mathbf{y}_0,\mathbf{y}_1,\dots,\mathbf{y}_n)\in\mathcal{T}_K$ and we have $\mathbf{g}(\mathbf{x}_i) = \sum_{j=0}^n \mu_j\mathbf{y}_j$. We have assigned $V_{\mathbf{x}} = W(\mathbf{x})$ for all vertices $\mathbf{x}$ of all simplices $\mathfrak{S}$ of the triangulation $\mathcal{T}_K$. Hence,

$$\begin{aligned} \sum_{j=0}^n \mu_j V_{\mathbf{y}_j} - V_{\mathbf{x}_i} &= \sum_{j=0}^n \mu_j W(\mathbf{y}_j) - W(\mathbf{x}_i) \\ &= \sum_{j=0}^n \mu_j W(\mathbf{y}_j) - W\left(\sum_{j=0}^n \mu_j\mathbf{y}_j\right) + \underbrace{W(\mathbf{g}(\mathbf{x}_i)) - W(\mathbf{x}_i)}_{\le -2\alpha\|\mathbf{x}_i\|_Q \text{ by (45)}} \end{aligned}$$

If $\mathfrak{S}_\mu\subset\mathcal{D}\setminus\mathcal{F}_K^\circ$, then we can use Proposition 2.9, (44) with $\epsilon = F_K$ and (62) to get

$$\left|\sum_{j=0}^n \mu_j W(\mathbf{y}_j) - W\left(\sum_{j=0}^n \mu_j\mathbf{y}_j\right)\right| \le nA_\epsilon h_\nu^2 \le n\frac{A}{F_K}h_\nu^2 = n^2AF_0 2^{-3K}. \tag{65}$$

Thus,

$$\sum_{j=0}^n \mu_j V_{\mathbf{y}_j} - V_{\mathbf{x}_i} + CG_\nu h_\nu \le n^2AF_0 2^{-3K} - 2\alpha\|\mathbf{x}_i\|_Q + CG\sqrt{n}\,2^{-2K}F_0$$

and the constraints (17) are fulfilled if

$$n^2AF_0 2^{-3K} - 2\alpha\|\mathbf{x}_i\|_Q + CG\sqrt{n}\,2^{-2K}F_0 \le -\alpha\|\mathbf{x}_i\|_Q.$$

Because

$$\alpha\|\mathbf{x}_i\|_Q \geq \alpha F_K\sqrt{\lambda^Q_{\min}} = \alpha 2^{-K}F_0\sqrt{\lambda^Q_{\min}}$$

and

$$n^2AF_0 2^{-3K} + CG\sqrt{n}\,2^{-2K}F_0 \leq \alpha 2^{-K}F_0\sqrt{\lambda^Q_{\min}}$$

holds true for all large enough $K \in \mathrm{N}_0$, we get

$$n^2AF_0 2^{-3K} + CG\sqrt{n}\,2^{-2K}F_0 \leq \alpha\|\mathbf{x}_i\|_Q \tag{66}$$

and the constraints (17) are fulfilled for all large enough $K \in \mathrm{N}_0$.

If $\mathfrak{S}_\mu \subset \mathcal{F}_K$ and $K$ is so large that $\mathcal{F}_K \subset \mathcal{B}_\delta$, then we have $W(\mathbf{y}_j) = \|\mathbf{y}_j\|_P$ for $j = 0, 1, \ldots, n$ and we can use the estimate (36) in the proof of Theorem 2.10 to get

$$\sum_{j=0}^{n}\mu_j W(\mathbf{y}_j) - W\left(\sum_{j=0}^{n}\mu_j\mathbf{y}_j\right) \leq 2h_{\partial\mathcal{F},P} \leq F_0 2^{-2K+1}\sqrt{\lambda^P_{\max}(n-1)}, \tag{67}$$

using (63). Thus, by (64) and $h_\nu, \|\mathbf{x}_i\|_2 \leq \sqrt{n}F_K$ we have

$$\begin{aligned}
&\sum_{j=0}^{n}\mu_j V_{\mathbf{y}_j} - V_{\mathbf{x}_i} + B_\nu C h_\nu\|\mathbf{x}_i\|_2 + E_{\mathcal{F}}\\
&\leq -2\alpha\|\mathbf{x}_i\|_Q + F_0 2^{-2K+1}\sqrt{\lambda^P_{\max}(n-1)} + BCn2^{-2K}F_0^2\\
&\quad + 2^{-2K}F_0 G_{\mathcal{F}}\max\left\{H_{\max}n2^{-K}, 2\sqrt{\lambda^P_{\max}(n-1)}\right\}.
\end{aligned}$$

Since $\|\mathbf{x}_i\|_Q \geq F_K\sqrt{\lambda^Q_{\min}} = 2^{-K}F_0\sqrt{\lambda^Q_{\min}}$ we get, similarly to (66), that the constraints (18) are fulfilled if

$$\begin{aligned}
&F_0 2^{-2K+1}\sqrt{\lambda^P_{\max}(n-1)} + BCn2^{-2K}F_0^2 + 2^{-2K}F_0 G_{\mathcal{F}}\max\left\{H_{\max}n2^{-K}, 2\sqrt{\lambda^P_{\max}(n-1)}\right\}\\
&\qquad \leq \alpha 2^{-K}F_0\sqrt{\lambda^Q_{\min}},
\end{aligned}$$

which again is clearly the case for all large enough $K$. □

## 5. Example

As a proof of concept, we compute a CPA Lyapunov function by the methods described in this paper as an example. We consider the system

$$x_{k+1} = \frac{1}{2}x_k + x_k^2 - y_k^2, \quad y_{k+1} = -\frac{1}{2}y_k + x_k^2 \tag{68}$$

from [11]. That is, the system (1) with

$$\mathbf{g}(\mathbf{x}) = \mathbf{g}(x,y) = \begin{pmatrix} 0.5x + x^2 - y^2 \\ -0.5y + x^2 \end{pmatrix}.$$

With

$$x^\nu := \max_{(x,y)\in\mathfrak{S}_\nu}|x|, \qquad y^\nu := \max_{(x,y)\in\mathfrak{S}_\nu}|y|$$

we can assign

$$G_\nu := 2\cdot\max\{0.5 + 2x^\nu, 2y^\nu\} \quad \text{and} \quad B_\nu := 2\cdot 2 = 4$$

for all $\mathfrak{S}_\nu \in \mathcal{T}$ in the linear programming problem from Definition 2.8. The Jacobian matrix of $\mathbf{g}$ at the origin is given by

$$A := D\mathbf{g}(\mathbf{0}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

We set $Q := I$, i.e. the identity matrix, which results in $P := 4/3 \cdot I$ being the solution to the discrete Lyapunov equation (4). We take

$$\max_{\mathfrak{S}_\nu \in \mathcal{T}} \|\nabla V_\nu\|_\infty$$

as the objective function of our linear programming problem and we minimize it. This objective function has the advantage that it, to our experience, usually leads to the level sets of the Lyapunov function being rather equally distributed.

We solve the linear programming problem from Definition 2.8, constructed for the system (68) with the triangulation $\mathcal{T}^{\text{std}}_{K,F}$, where the parameters are $K = 4$, $F := 0.033$, $N_I := 2$, $N_O := 10$, and $N_D := 12$. For these parameters the linear programming problem has a feasible solution, which was computed using the Gnu Linear Programming Kit (http://www.gnu.org/software/glpk/) from Andrew Makhorin. The computed CPA Lyapunov function is depicted in Figure 4. As described in Definition 2.5, a simplicial fan is used to triangulate $\mathcal{F} = [-0.033, 0.033]^2$. This simplicial fan is depicted in Figure 5. The domain of the computed CPA Lyapunov function is $\mathcal{D} = [-N_D \cdot F, N_D \cdot F]^2 = [-0.396, 0.396]^2$. The largest connected component of a sublevel set compact in $\mathcal{O} = [-N_I \cdot F, N_I \cdot F]^2 = [-0.33, 0.33]^2$ is assured to be in the basin of attraction of the equilibrium at the origin, cf. Remark 2. This set is depicted in Figure 3.

Let us compare these result with the quadratic Lyapunov function $\widetilde{V}_P(\mathbf{x}) := \mathbf{x}^T P \mathbf{x}$, obtained by solving the discrete Lyapunov equation. By equation (47), $\widetilde{V}_P(\mathbf{g}(\mathbf{x})) - \widetilde{V}_P(\mathbf{x}) < 0$ for all $\mathbf{x}$ such that

$$\|\boldsymbol{\psi}(\mathbf{x})\|_2 \|P\|_2 \left(\|\boldsymbol{\psi}(\mathbf{x})\|_2 + 2\|A\|_2\right) = \frac{4}{3} \cdot \|\boldsymbol{\psi}(\mathbf{x})\|_2 \left(\|\boldsymbol{\psi}(\mathbf{x})\|_2 + 1\right) < 1$$

where $\boldsymbol{\psi}(\mathbf{x}) = (\mathbf{g}(\mathbf{x}) - A\mathbf{x})/\|\mathbf{x}\|_2$; note that $\|\mathbf{x}\|_2 = \|\mathbf{x}\|_Q$. By using the general estimate derived directly above inequality (4.6) in [17], we get for all $\|\mathbf{x}\|_2 = r > 0$ that

$$\|\boldsymbol{\psi}(\mathbf{x})\|_2 \leq \frac{r^2}{2\|\mathbf{x}\|_2} \sqrt{\sum_{i=1}^{2} \left( \sum_{k,j=1}^{2} \max_{\boldsymbol{\xi} \in [-r,r]^2} \left| \frac{\partial^2 g_i}{\partial x_j \partial x_k}(\boldsymbol{\xi}) \right| \right)^2} = r\sqrt{5}$$

and $\widetilde{V}_P$ is a Lyapunov function for the system in the set $\{\mathbf{x} : \|\mathbf{x}\|_2 < \sqrt{5}/10\} \approx \mathcal{B}_{0.224}$.

In Figure 3 we compare this estimate of the basin of attraction with the inner approximation delivered by the CPA Lyapunov function from above. Note also that there is an (unstable) equilibrium at $(0.59, 0.23)$ which must lie outside the basin of attraction.

For further comparison we solved the linear programming problem from Definition 2.8 for the same system with the parameters $K = 5$, $F = 0.1$, $N_I = 2$, $N_O = 4$, and $N_D = 6$. Moreover, we do not include the simplicial fan in the linear programming problem, i.e. we exclude (18) with $\mathcal{F} = [-0.1, 0.1]^2$ from the constraints (IV). Note,

that in this case the sublevel set in Figure 3 is a forward invariant set with the property that for any $\boldsymbol{\xi}$ in the sublevel set, there exists a strictly increasing sequence $(t_k)_{k\in\mathrm{N}}$ of natural numbers, such that $\mathbf{g}^{\circ t_k}(\boldsymbol{\xi}) \in \mathcal{F} = [-0.1, 0.1]^2$ for all $k \in \mathrm{N}$. Since $\mathcal{F} = [-0.1, 0.1]^2$ is a subset of the basin of attraction, as shown by the quadratic Lyapunov function, we can also conclude that the sublevel set is a subset of the basin of attraction.

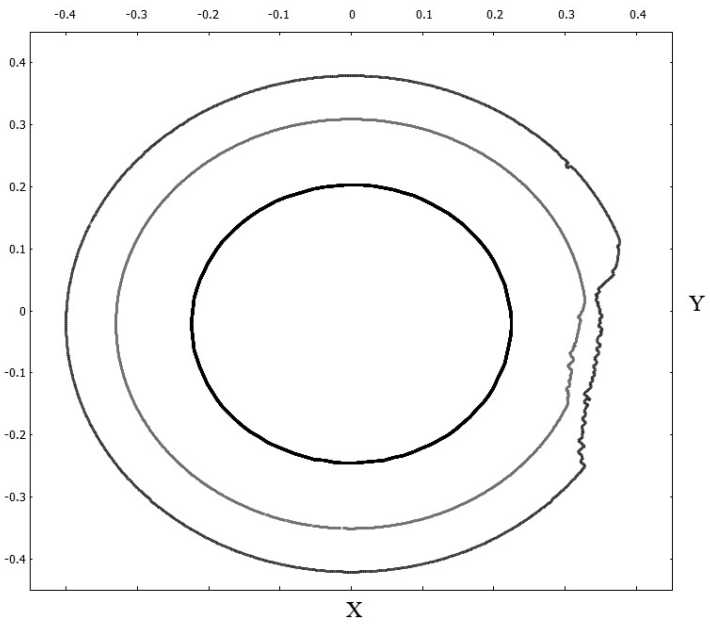



Figure 3. The figure shows three subsets of the basin of attraction. The smallest one is obtained by the quadratic Lyapunov function, derived from the discrete Lyapunov equation, the middle one is obtained by the CPA Lyapunov function with the simplicial fan at the origin, and the largest one is obtained by the CPA Lyapunov function excluding the set $\mathcal{F} = [-0.1, 0.1]^2$.

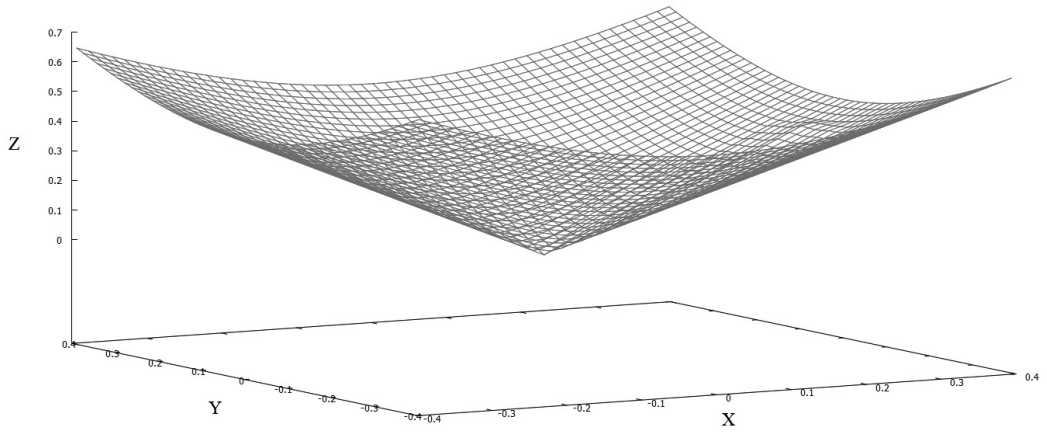



Figure 4. The CPA Lyapunov function without the fan computed for the system (68). The CPA Lyapunov function computed with the fan looks very similar but is defined on a smaller domain.

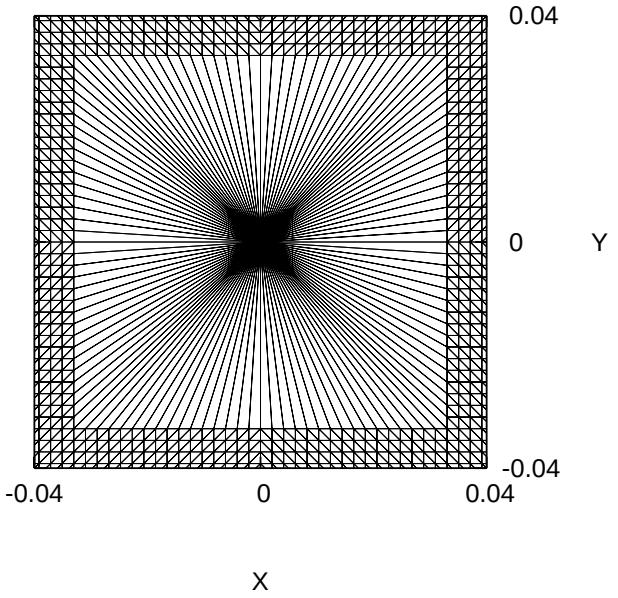



Figure 5. The simplicial fan and its closest neighbourhood of the simplicial complex $\mathcal{T}^{\text{std}}_{4,0.033}$.

## 6. Conclusion and Future Directions

In this paper, we fully adapted the CPA method to compute Lyapunov functions to autonomous discrete time systems. In Definition 2.8 we presented a linear programming problem, of which a feasible solution parameterizes a CPA Lyapunov function for the system in question. In Definition 3.1 we offered an algorithm that generates linear programming problems as in Definition 2.8 for ever more refined triangulations of a hypercube $\mathcal{D}$ containing the origin. In Theorem 4.2 we proved, that if the system at hand has an exponentially stable equilibrium at the origin and $\mathcal{D}$ is a subset of its region of attraction, then the algorithm succeeds in a finite number of steps in computing a CPA Lyapunov function for the system. Finally, in Section 5, we applied the method to compute a CPA Lyapunov function for a nonlinear system.

The CPA method for continuous time systems has been extended to compute CPA Lyapunov functions for switched systems [19] and differential inclusions [2, 3]. It seems very promising for further research in this direction to combine the theory on the stability of difference inclusions and smooth Lyapunov functions given in [24–28] with the theory developed in this paper to design an algorithm to compute CPA Lyapunov functions for exponentially stable difference inclusions.

Further, the choice of an objective function in the linear programming problem, optimal in some sense, remains an open problem. The one we choose for our example in this paper usually leads to Lyapunov functions with rather good properties, since it eliminates Lyapunov functions with extremely large local gradients. Different objective functions should be systematically studied in the future.

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